The typical structure of H-colorings of the Hamming cube

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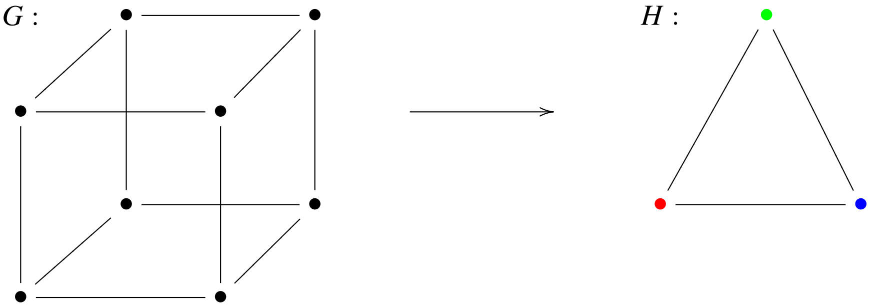

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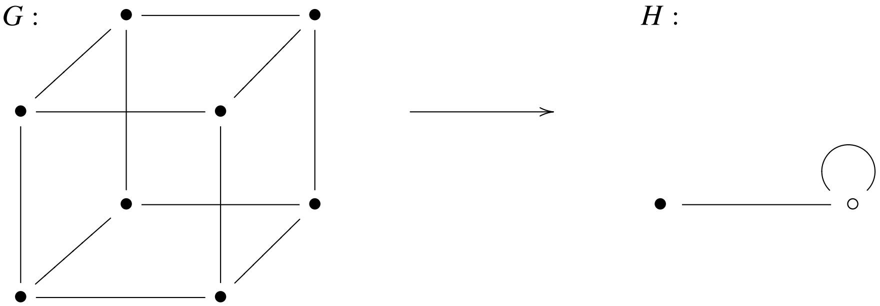

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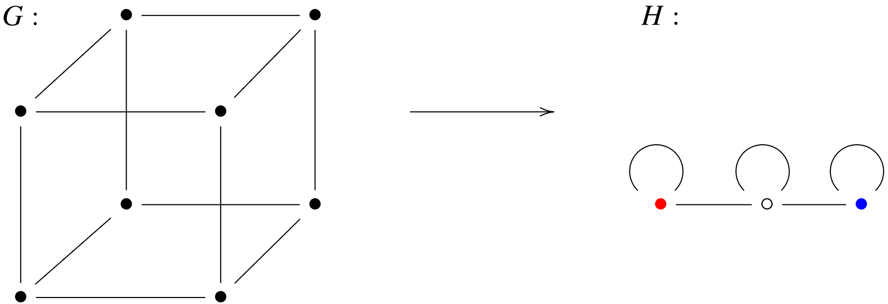

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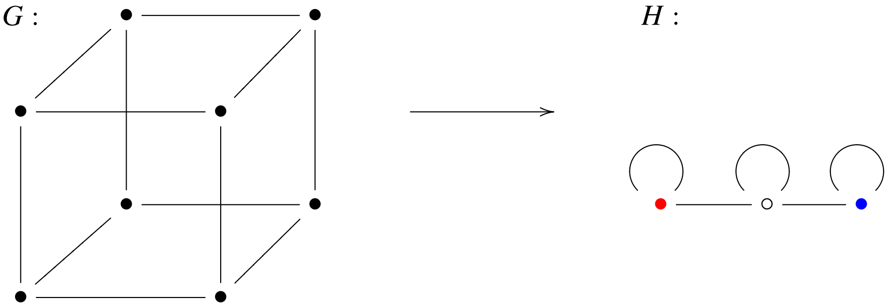

- H-colorings generalize: proper q-colorings, independent sets, the Widom-Rowlinson model.
- We'll discuss proper q-colorings in this talk.

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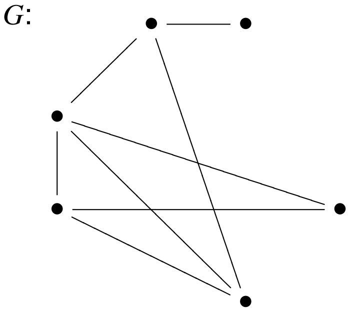
Let G be a graph, $q \in \mathbb{Z}_+$, and suppose we select a proper q-coloring of G at random. Natural question: What does it look like?



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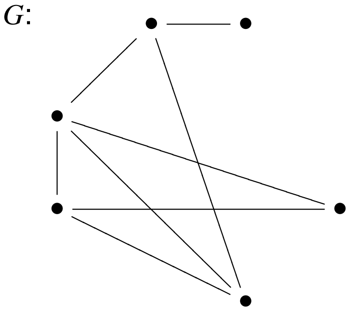

Slight problem: G might not have a proper q-coloring.

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Let's restrict our graphs G to be regular, bipartite.

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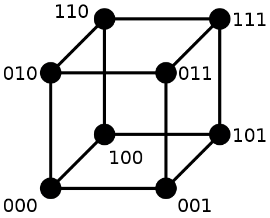
  - bipartition classes E and O



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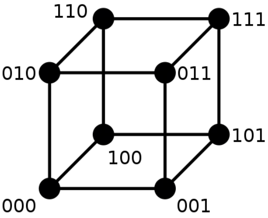

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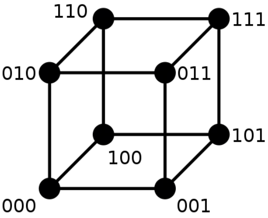

- Symmetry: $E(\text{\# of vertices colored with fixed color}) = N/q$

Results

Theorem (E., Galvin 2010)

Given an N-vertex, d-regular bipartite graph G and a uniformly chosen q-coloring of G, a.a.s. (as $d \to \infty$) each color appears on about

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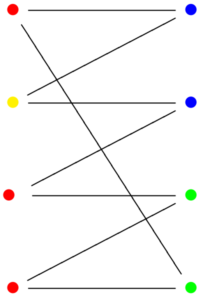
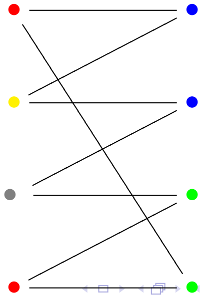
Where are these numbers coming from?

4-coloring:



5-coloring:



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What drives these? Number of components, expansion

Hamming Cube

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Theorem (E., Galvin 2010)

For a uniformly chosen q-colorings on $\{0,1\}^d$ (with $N = 2^d$), we have

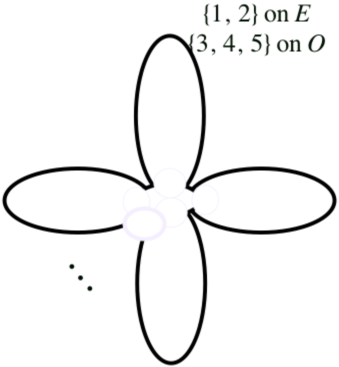
1. *for q even, each color appears on about N/q vertices,*
2. *for q odd, $(q+1)/2$ colors appear on about $N/(q+1)$ vertices and the remaining $(q-1)/2$ colors appear on about $N/(q-1)$ vertices.*

Additionally, each color appears almost exclusively on one partition class of $\{0,1\}^d$.

Corollaries

Corollary

The space of 5-colorings of $\{0,1\}^d$ breaks up into 20 large classes based on the dominant colors on one partition of $\{0,1\}^d$, plus a small extra class.



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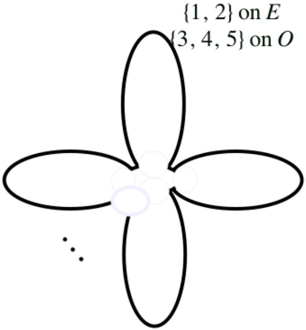
  - If w is in the same partition class as v, conditional probability vector for v is $(2/5, 3/20, 3/20, 3/20, 3/20)$,
  - If w is in the other partition class from v, conditional probability vector for v is $(0, 1/4, 1/4, 1/4, 1/4)$.



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Generalizes a proof of Kahn (2001) on homomorphisms from $\{0,1\}^d$ to a doubly infinite path.

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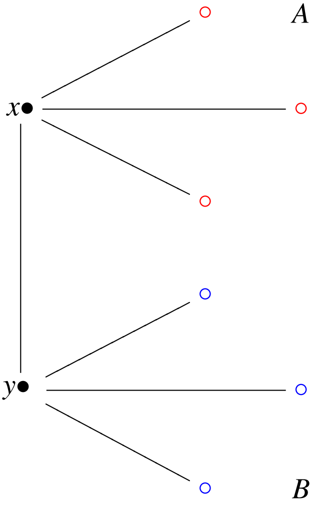

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- $N(x) \setminus \{y\}$ are colored using set of colors A
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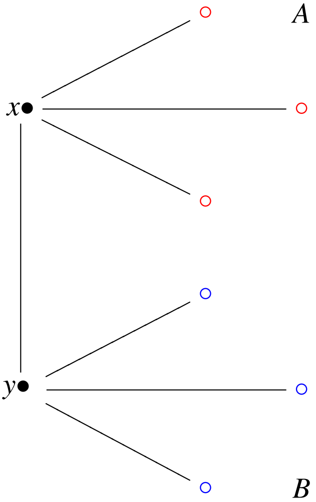

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Edge is *ideal* if A and B are disjoint, use all available colors, and as equal in size as possible.

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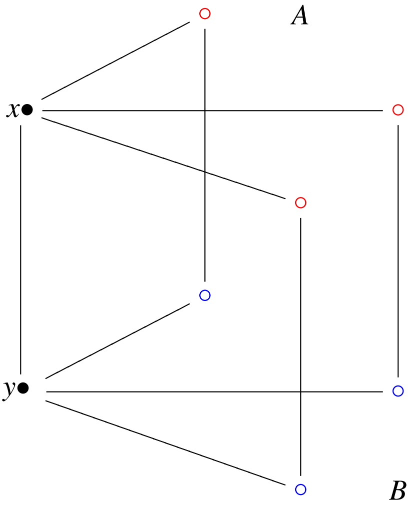
Local structure of $\{0,1\}^d$:



Rough idea is to maximize count for an A, B:

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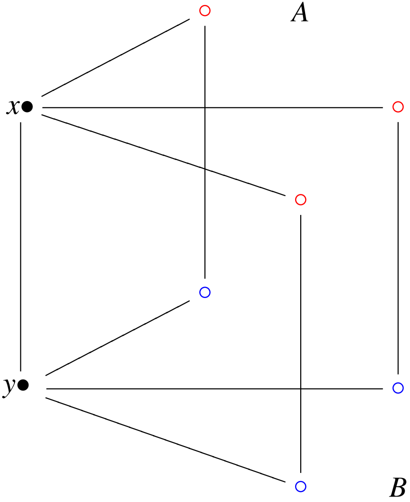

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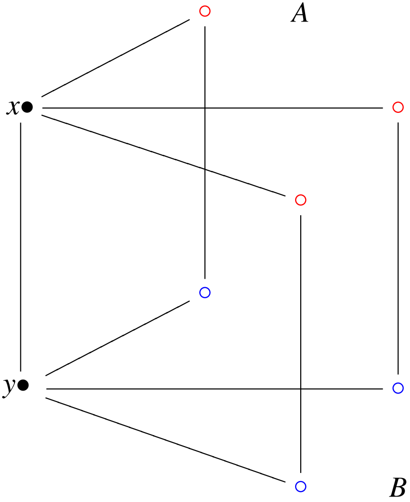

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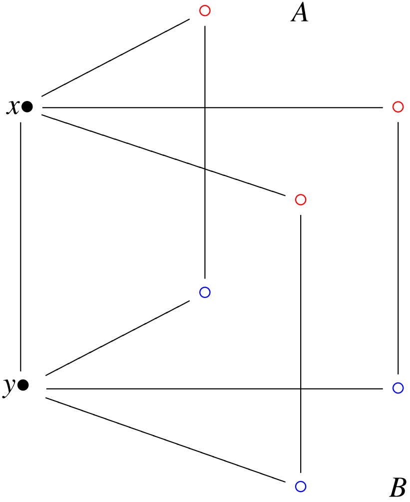

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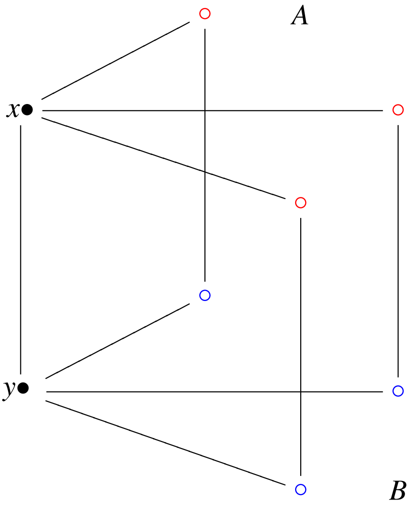

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- Can't have too many 'non-ideal' (A,B)

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- X = uniformly chosen q-coloring of $\{0,1\}^d$.

$$\log |(q/2)^{2^d}| \leq \log |\# \text{ of } q - \text{colorings}| = H(X)$$

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- Results for $\mathbb{Z}^d$?
  Want: $M \to \infty$, d fixed.
  Now: M fixed, $d \to \infty$.
  Can do: $M = c \log d$

End

Thank You!