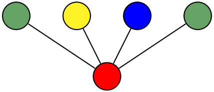


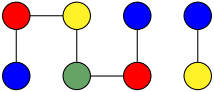
Extremal H-colorings of trees

John Engbers* David Galvin

Department of Mathematics, Statistics and Computer Science
Marquette University

2014 MathFest — Portland, OR

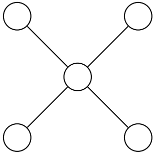
August 8, 2014



An extremal question

Graph homomorphism (H-coloring):

G :

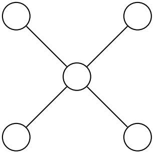


$H = H_{\text{ind}}$:

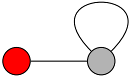
An extremal question

Graph homomorphism (H-coloring):

G :

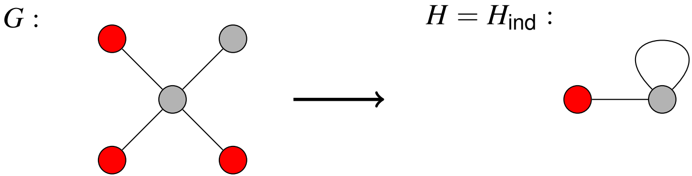


$H = H_{\text{ind}}$:



An extremal question

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

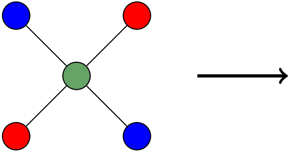


Examples: independent sets,

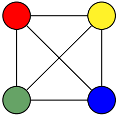
An extremal question

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

G :



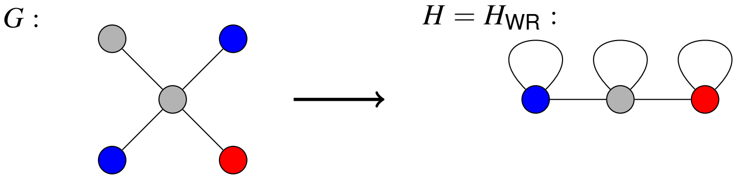
$H = K_q$:



Examples: independent sets, proper q-colorings,

An extremal question

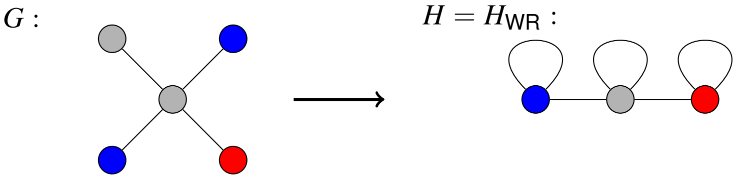
Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.



Examples: independent sets, proper q-colorings, Widom-Rowlinson

An extremal question

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.

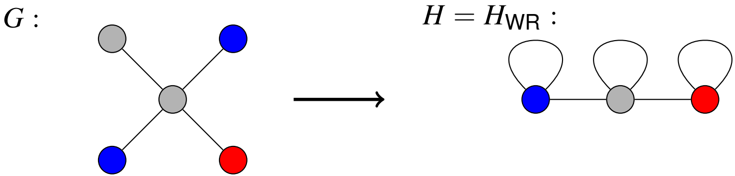


Examples: independent sets, proper q-colorings, Widom-Rowlinson

Notation: $\hom(G, H)$ = number of H-colorings of G.

An extremal question

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.



Examples: independent sets, proper q-colorings, Widom-Rowlinson

Notation: $\hom(G, H)$ = number of H-colorings of G.

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes/minimizes $\hom(G, H)$?

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov)

- $\mathcal{G}$ = n-vertex m-edge graphs Few H

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G,H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali)

- $\mathcal{G}$ = n-vertex m-edge graphs Few H
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs All H

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin)

- $\mathcal{G}$ = n-vertex m-edge graphs *Few H*
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs All H
- $\mathcal{G}$ = n-vertex d-regular ~~bipartite~~ graphs *Lots of H*

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.)

- $\mathcal{G}$ = n-vertex m-edge graphs *Few H*
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs All H
- $\mathcal{G}$ = n-vertex d-regular ~~bipartite~~ graphs *Lots of H*
- $\mathcal{G}$ = n-vertex graphs with minimum degree δ *Several H*

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.)

- $\mathcal{G}$ = n-vertex m-edge graphs Few H ← Open questions
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs All H
- $\mathcal{G}$ = n-vertex d-regular ~~bipartite~~ graphs Lots of H ← Open questions
- $\mathcal{G}$ = n-vertex graphs with minimum degree δ Several H ← Open questions

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G,H)$?

Theorem (Cutler-Radcliffe, Loh-Pikhurko-Sudakov; Kahn, Galvin-Tetali; Zhao, Galvin; E.; (E., Galvin) Sidorenko)

- $\mathcal{G}$ = n-vertex m-edge graphs *Few H←Open questions*
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs All H
- $\mathcal{G}$ = n-vertex d-regular ~~bipartite~~ graphs *Lots of H←Open questions*
- $\mathcal{G}$ = n-vertex graphs with minimum degree δ *Several H←Open questions*
- $\mathcal{G}$ = n-vertex trees *All H*

Trees

Question

Fix H. Which n-vertex tree T maximizes $\hom(T, H)$?

Trees

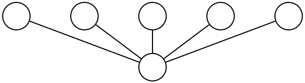
Question

Fix H. Which n-vertex tree T maximizes $\hom(T,H)$?

Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T,H) \leq \hom(K_{1,n-1},H).$$



However.....(New to me, May 2014)

Trees

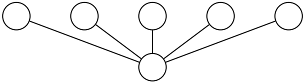
Question

Fix H. Which n-vertex tree T maximizes $\hom(T, H)$?

Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$



However.....(New to me, May 2014)

Theorem (Sidorenko 1994)

Fix H. For any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$

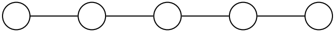
What next?

The star $K_{1,n-1}$ maximizes # of H-colorings in trees. What minimizes?

Conjecture

Fix H. For any n-vertex tree T,

$$\hom(P_n, H) \leq \hom(T, H).$$



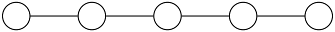
What next?

The star $K_{1,n-1}$ maximizes # of H-colorings in trees. What minimizes?

Conjecture

Fix H. For any n-vertex tree T,

$$\hom(P_n, H) \leq \hom(T, H).$$



FALSE! (Even for $n = 7$)

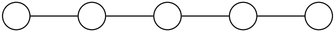
What next?

The star $K_{1,n-1}$ maximizes # of H-colorings in trees. What minimizes?

Conjecture

Fix H. For any n-vertex tree T,

$$\hom(P_n, H) \leq \hom(T, H).$$

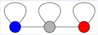
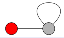


FALSE! (Even for $n = 7$)

Theorem (E., Galvin 2014)

For a certain class of H, for any n-vertex tree T we have

$$\hom(P_n, H) \leq \hom(T, H).$$

(This class includes the Widom-Rowlinson graph H_{WR}  *and the independent set graph H_{ind}*  *)*

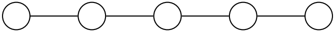
What next?

The star $K_{1,n-1}$ maximizes # of H-colorings in trees. What minimizes?

Conjecture

Fix H. For any n-vertex tree T,

$$\hom(P_n, H) \leq \hom(T, H).$$

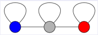
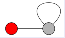


FALSE! (Even for $n = 7$) $\leftarrow$Open question — what H is it true for?

Theorem (E., Galvin 2014)

For a certain class of H, for any n-vertex tree T we have

$$\hom(P_n, H) \leq \hom(T, H).$$

(This class includes the Widom-Rowlinson graph H_{WR}  *and the independent set graph H_{ind}*  *)*

What next?

Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$

See what else your proof can do!

What next?

Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$

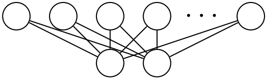
See what else your proof can do!

Theorem (E., Galvin 2014)

Fix a non-regular H. For n large and any 2-connected graph G,

$$\hom(G, H) \leq \hom(K_{2,n-2}, H).$$

$K_{2,n-2}$:



What next?

Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$

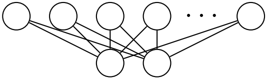
See what else your proof can do!

Theorem (E., Galvin 2014)

Fix a non-regular H. For n large and any 2-connected graph G,

$$\hom(G, H) \leq \hom(K_{2,n-2}, H).$$

$K_{2,n-2}$:



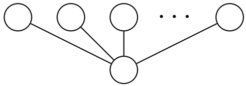
Question: Does $K_{k,n-k}$ maximize the H-colorings among all k-connected graphs (for most H)?

Idea of proof

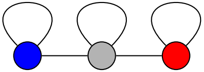
Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T, H) \leq \hom(K_{1,n-1}, H).$$



H_{WR} :



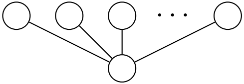
Idea: Stability

Idea of proof

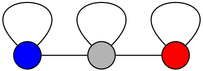
Theorem (E., Galvin 2014)

Fix H. For n large and any n-vertex tree T,

$$\hom(T,H) \le \hom(K_{1,n-1},H).$$



H_{WR} :



Idea: Stability

Step 0. Note $\hom(K_{1,n-1},H_{\mathsf{WR}}) \ge 3^{n-1}$

Step 1. Extremal tree must be structurally close to $K_{1,n-1}$

Step 2. Small blemishes added to star can't be extremal

Eject!

Idea of proof

Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Idea of proof

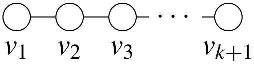
Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can't contain a long path.

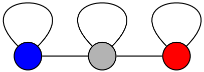
Idea of proof

Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can't contain a long path.



H_{WR} :

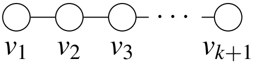


$$A(H) = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

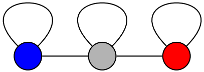
Idea of proof

Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can't contain a long path.



H_{WR} :



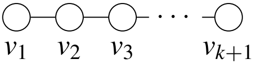
$$A(H) = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- $(A^k)_{(ij)}$ = # colorings of P_{k+1} with endpoints colored i, j

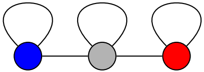
Idea of proof

Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can't contain a long path.



H_{WR} :



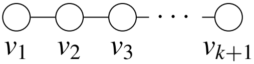
$$A(H) = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- $(A^k)_{(ij)}$ = # colorings of P_{k+1} with endpoints colored i, j
- Perron-Frobenius: largest eigenvalue is $\lambda < 3$ (H not regular)

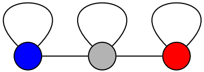
Idea of proof

Step 0. $\hom(K_{1,n-1}, H) \geq 3^{n-1}$

Step 1. Show that an extremal tree can't contain a long path.



H_{WR} :



$$A(H) = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- $(A^k)_{(ij)}$ = # colorings of P_{k+1} with endpoints colored i, j
- Perron-Frobenius: largest eigenvalue is $\lambda < 3$ (H not regular)
- $\hom(T, H) \leq c\lambda^k 3^{n-k} < 3^{n-1}$ (for constant k, uses $n > c_H$) Eject!

Idea of proof

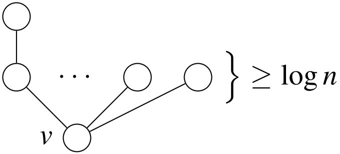
Step 2: Any non-star with no long path has fewer H-colorings

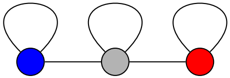
Idea of proof

Step 2: Any non-star with no long path has fewer H-colorings

No k-path implies there is a vertex with at least $\log n$ neighbors.

If not a star:



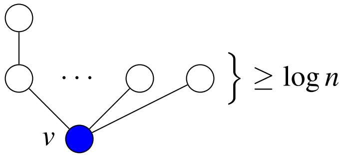


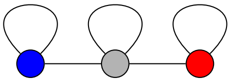
Idea of proof

Step 2: Any non-star with no long path has fewer H-colorings

No k-path implies there is a vertex with at least $\log n$ neighbors.

If not a star:





Number of colorings where:

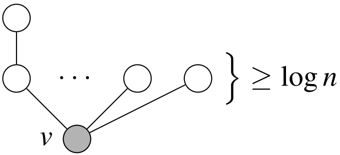
- v has color w; $d(w) < 3 \implies < c2^{\log n}3^{n - \log n - 1} \leq cn^{\frac{-1}{3}}3^n = o(1)3^n$

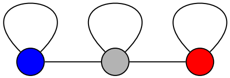
Idea of proof

Step 2: Any non-star with no long path has fewer H-colorings

No k-path implies there is a vertex with at least $\log n$ neighbors.

If not a star:





Number of colorings where:

- v has color w; $d(w) < 3 \implies < c2^{\log n}3^{n-\log n-1} \leq cn^{\frac{-1}{3}}3^n = o(1)3^n$
- v has color w; $d(w) = 3$: constant in leading term dampened if not $K_{1,n-1}$

Thank you!

Slides available on my homepage:

`http://www.mscs.mu.edu/~engbers/`