Reversible Peg Solitaire on Graphs

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Peg Solitaire

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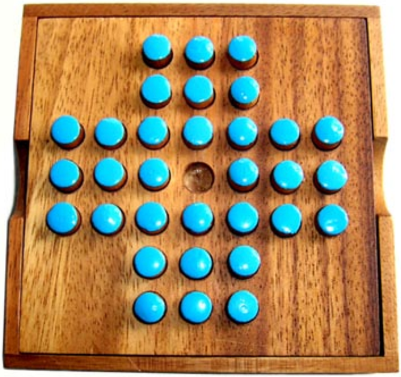
A common single-player game played around the world:



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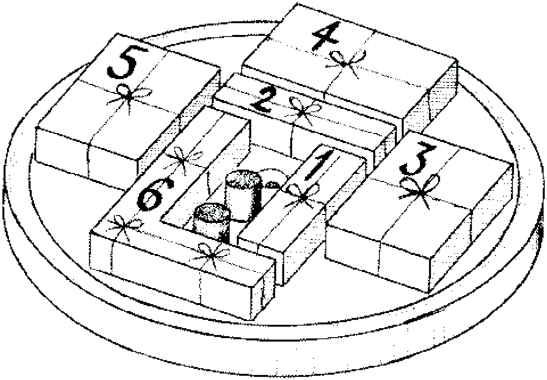
Peg Solitaire

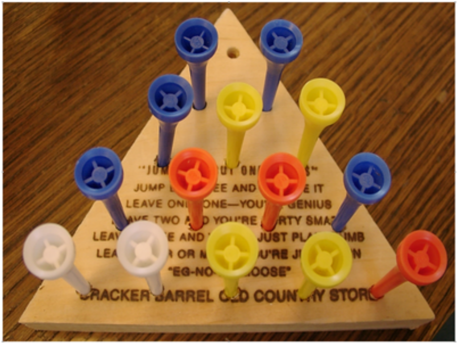
To solve peg solitaire:



Think in terms of 'packaged' moves.

Eg-No-Ra-Moose

A variation on the theme is found at Cracker Barrel restaurants.



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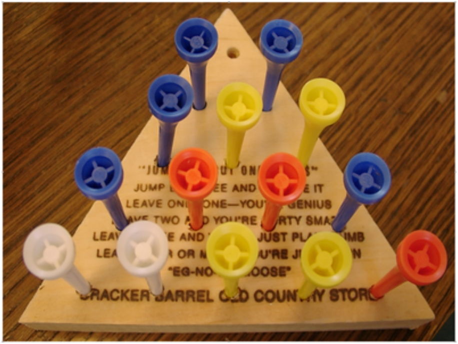
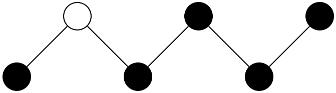


“Leave only one - you’re genius...leave four or more’n you’re just plain ‘eg-no-ra-moose’.”

Peg Solitaire on Graphs

Now: Play Peg Solitaire on a connected graph (lose geometry)



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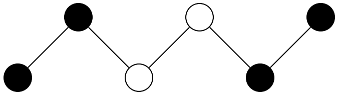

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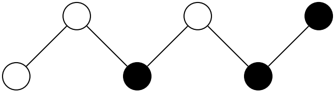

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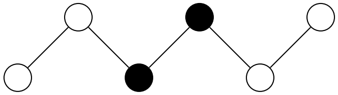

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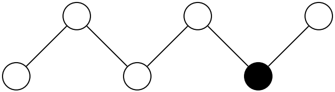

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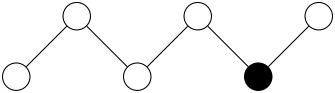

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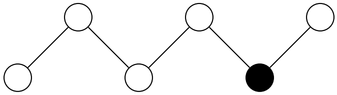

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Question: Which graphs are solvable in peg solitaire? [Beeler Hoilman, 2011]

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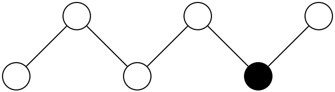

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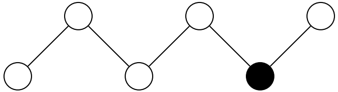

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Construct Solvable Graphs: [Beeler, Gray, Hoilman 2012]
Start with one peg, one hole. Reverse the game; adding pegs/holes.

Reverse moves

"The game called Solitaire pleases me much. I take it in reverse order. That is to say that instead of making a configuration according to the rules of the game, which is to jump to an empty place and remove the piece over which one has jumped, I thought it was better to reconstruct what had been demolished, by filling an empty hole over which one has leaped." — Leibniz[1].

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Conjecture

P_n and C_n are not solvable if n is not divisible by 2 or 3.

(Confirmed computationally for $n \leq 25$)

Idea of Proof

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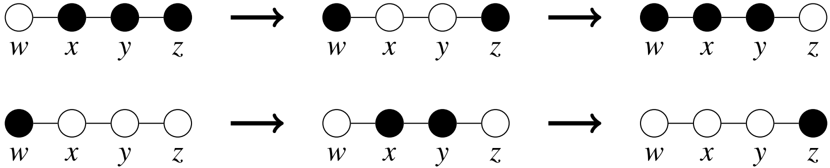

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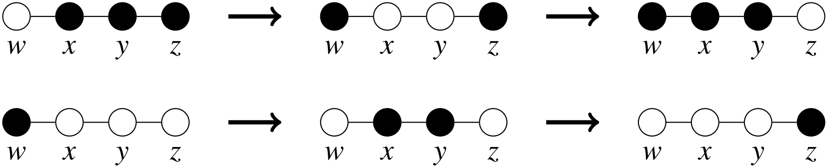
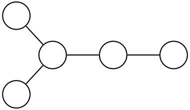


Gadget: Claw with subdivided edge.



Idea of Proof

Lemma

Columns: states obtained by jumps and unjumps within our gadget.

Class A	Class B	Class C	Class D	Class E	Class F
a	*c*	*abd*	*ce*	*abcde*	
b	*ab*				
d	*ad*				
e	*ae*				
ac	*bd*				
bc	*be*				
cd	*de*				
abe	*abc*				
ade	*acd*				
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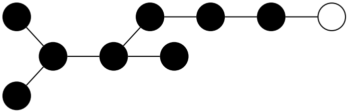

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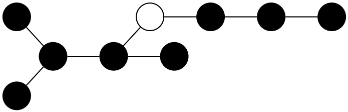

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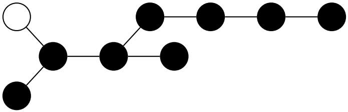

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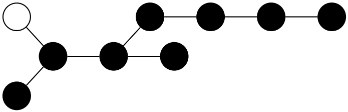

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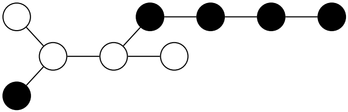

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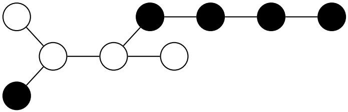

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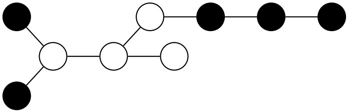

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Thank You

Slides available on my webpage:

`http://www.mscs.mu.edu/~engbers/`