Entropy and Counting

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Outline

Entropy

Definition

The *entropy* of a discrete random variable $\mathbf{X}$ is

$$H(\mathbf{X}) = \sum_x p(x) \log_2 \frac{1}{p(x)},$$

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- All random variables will be discrete, and $\log = \log_2$.

Example

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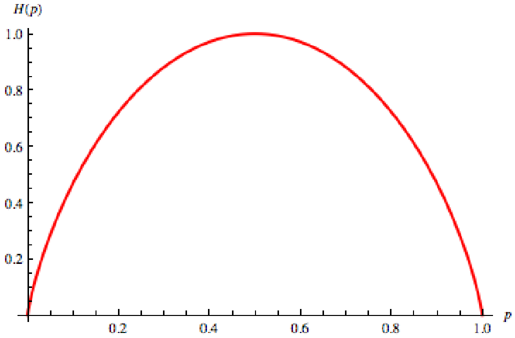

Note: $H(p) := H(\mathbf{X})$

Basic Properties

- If Q is an event, we define $H(\mathbf{X}|Q) = \sum p(x|Q) \log \frac{1}{p(x|Q)}$.

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The *conditional entropy* of $\mathbf{X}$ given $\mathbf{Y}$ is

$$H(\mathbf{X}|\mathbf{Y}) = E[H(\mathbf{X}|\{\mathbf{Y} = y\})] = \sum_y p(y) \sum_x p(x|y) \log \frac{1}{p(x|y)}.$$

Basic Properties

- (Chain Rule)
  $H(\mathbf{X}_1, \ldots, \mathbf{X}_n) = H(\mathbf{X}_1) + H(\mathbf{X}_2|\mathbf{X}_1) + \cdots + H(\mathbf{X}_n|\mathbf{X}_{n-1}, \ldots, \mathbf{X}_1)$

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- (Uniform Bound) By Jensen's inequality (as $\sum_x p(x) = 1$ and $\log$ is concave), we have

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- (Subadditivity) $H(\mathbf{X}_1, \ldots, \mathbf{X}_n) \leq \sum H(\mathbf{X}_i)$

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For a random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $A \subset [n] = \{1, 2, \ldots, n\}$, let $\mathbf{X}_A := (\mathbf{X}_i : i \in A)$.

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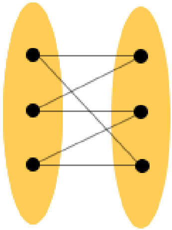
Lemma (Shearer's Lemma)

Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be a random vector and $\mathcal{A}$ a collection of subsets (possibly with repeats) of $[n]$, with each element of $[n]$ contained in at least t members of $\mathcal{A}$. Then

$$H(\mathbf{X}) \leq \frac{1}{t} \sum_{A \in \mathcal{A}} H(\mathbf{X}_A).$$

Brégman's Theorem

Suppose that we have a bipartite, N-vertex, d-regular graph G.
How many perfect matchings are there?



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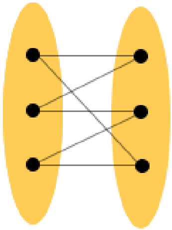

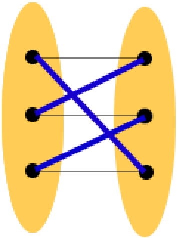

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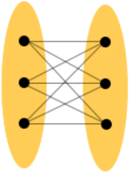

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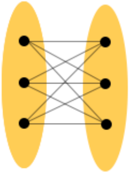

Remark 2: The theorem can be interpreted as a theorem about permanents in $\{0,1\}$-matrices. It can also easily be generalized beyond the d-regular condition.

Proof of Brégman's Theorem

Proof: (Radhakrishnan) Choose σ from $\mathcal{M}$ uniformly, so $H(\sigma) = \log(|\mathcal{M}|)$. Label the vertices on the left as $1, 2, \ldots, N/2$; so $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N/2))$.

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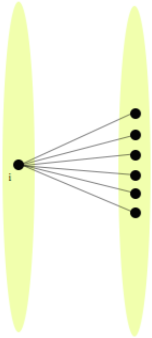

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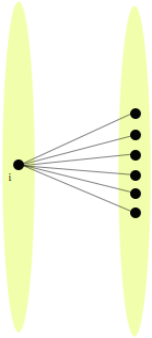

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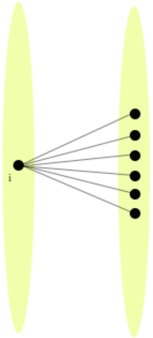

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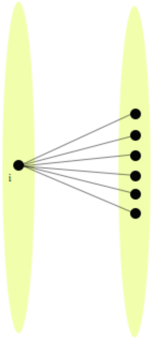

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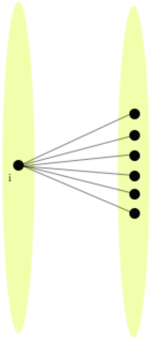

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For fixed σ, $P_\tau(|N_i(\sigma,\tau)| = j) = \frac{1}{d}$ for $j = 1, \ldots, d$.

$$\implies P_{\sigma,\tau}(|N_i(\sigma,\tau)| = j) = \frac{1}{d}.$$

Proof of Brégman's Theorem

Putting all of this together, we have:

$$\log|\mathcal{M}| = H(\sigma) = \sum_{i=1}^{N/2} H(\sigma(\tau(i)) \mid \sigma(\tau(1)), \ldots, \sigma(\tau(i-1)))$$

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&= \sum_{i=1}^{N/2} \frac{1}{d} \log d! \\
&= \log(d!)^{N/2d}
\end{aligned}
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Questions

Question: Is this result true if we remove the word 'perfect'?

Conjecture

In an N-vertex, d-regular bipartite graph G, let $\mathcal{M}_{tot}(G)$ be the set of all possible matchings of G. Then

$$|\mathcal{M}_{tot}(G)| \leq |\mathcal{M}_{tot}(K_{d,d})|^{N/2d} = \left(\sum_{i=0}^{d} \binom{d}{i}^2 i!\right)^{N/2d}.$$

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Conjecture (Friedland)

In a N-vertex, d-regular bipartite graph G, let $\mathcal{M}_t(G)$ be the set of all matchings of size t, $t \in \{0, 1, ..., N/2\}$ in G. Then

$$|\mathcal{M}_t(G)| \leq |\mathcal{M}_t(\frac{N}{2d}K_{d,d})|.$$

Graph Homomorphisms

Definition

Given graphs G and H (H possibly with loops), a function $f: V(G) \to V(H)$ is a *graph homomorphism* if $x \sim y$ implies $f(x) \sim f(y)$ for all $x, y \in V(G)$. Denote by $Hom(G, H)$ the set of all graph homomorphisms from G to H.

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Example:

G two H's:

Results

Let $\mathcal{I}(G)$ denote the set of all independent sets in a graph G.

Theorem (Kahn)

For any N-vertex, d-regular bipartite graph G,

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This result can be extended to the following:

Theorem (Galvin, Tetali)

For any N-vertex, d-regular bipartite graph G and any H (possibly with loops),

$$|Hom(G,H)| \leq |Hom(K_{d,d},H)|^{N/2d}.$$

Proof

Choose $\mathbf{f}$ uniformly from $Hom(G,H)$, so

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We've localized!

Proof

$$\log|Hom(G,H)| \leq \frac{1}{d}\sum_{v\in E}[H(\mathbf{N}_v) + dH(\mathbf{f}_v|\mathbf{N}_v)]$$

Proof

$$\log |Hom(G,H)| \leq \frac{1}{d} \sum_{v \in E} [H(\mathbf{N}_v) + dH(\mathbf{f}_v | \mathbf{N}_v)]$$

From the definitions, the uniform bound, and an application of Jensen's formula, we have:

$$H(\mathbf{N}_v) + dH(\mathbf{f}_v | \mathbf{N}_v) \leq \log |Hom(K_{d,d}, H)|$$

which completes the proof.

Related questions

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For any N-vertex, d-regular graph G and any H (possibly with loops),

$$|Hom(G,H)| \leq |Hom(K_{d,d},H)|^{N/2d}$$

This conjecture is FALSE! See H being two disjoint loops and $G = K_3$.

Related questions

Theorem (Zhao, 2009)

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An interesting question is: For what H's does this extension to general d-regular graphs hold?

Hard-Core Distribution

We now put a probability distribution on the set of all independent sets of G.

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For a finite graph G and $\lambda > 0$, the *hard-core distribution* with *activity* λ on $\mathcal{I}(G)$ is given by

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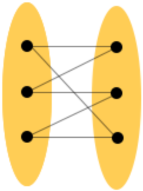
- Note: $\lambda = 1$ gives the uniform distribution on $\mathcal{I}(G)$.
- We'll restrict our G to be N-vertex, d-regular, and bipartite.

Hard-Core Distribution

Now, how do you get a lot of independent sets in G?



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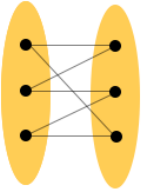

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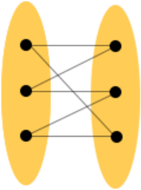

- Let $\alpha_\lambda = \dfrac{\lambda}{2(1+\lambda)}$.
- if $\mathbf{I}$ is an independent set chosen according to p_λ , let $p(v) := P(v \in \mathbf{I})$, and $\bar{p} = \sum_v p(v)$ $(= E[|\mathbf{I}|]/N)$.

Theorem

Theorem (Kahn)

Fix $\lambda > 0$, and let $\mathbf{I}$ be chosen according to p_λ on G. Then

$$\bar{p} \approx \alpha_\lambda$$

and, furthermore, most independent sets have size close to $\alpha_\lambda N$.

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- Example: $\lambda = 1$ is the uniform case, where $\alpha_\lambda = 1/4$.
- Entropy allows us to count independent sets of a fixed size.

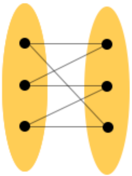
Extension

Theorem (E., Galvin)

Given any N-vertex, d-regular bipartite G and a random (uniform) q coloring of G, the fraction of vertices with any given color doesn't differ far from

a) $1/q$ (q even)

b) being in $[1/(q+1), 1/(q-1)]$ (q odd).



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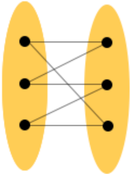

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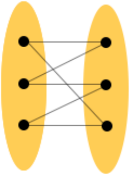

- Why the even/odd difference?
- Can the odd case be improved?

Extension

This idea can be extended to a weighted version:

Theorem (E., Galvin)

Given a fixed H and weights $\Lambda = \{\lambda_h\}_{h \in V(H)}$ on $V(H)$, and any N-vertex, d-regular bipartite graph G with some technical conditions, the number of vertices mapping to a fixed vertex of H is close to an ideal value.

Thanks

Thank you!