Maximizing the number of H-colorings of graphs with a fixed minimum degree

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Abstract

For graphs G and H, an H-coloring of G is an adjacency-preserving map from the vertex set of G to the vertex set of H. The number of H-colorings of G is denoted $\hom(G,H)$. Given a fixed graph H and family of graphs $\mathcal{G}$, what is the maximum value of $\hom(G,H)$ over all $G \in \mathcal{G}$?

For the family of n-vertex d-regular graphs, it has been conjectured that

$$\hom(G,H) \le \max_{G^*} \hom(G^*,H)^{\frac{n}{|V(G^*)|}},$$

where the maximum is taken over all d-regular graphs G^* with at most $\kappa(d)$ vertices. This has been verified for various classes of H, but remains open in general.

We consider the related family of n-vertex graphs with minimum degree at least δ. For fixed δ and H, we show that

$$\hom(G,H) \le \max_{G^*} \hom(G^*,H)^{\frac{n}{|V(G^*)|}}$$

where the maximum is taken over all graphs G^* with minimum degree δ on at most $\kappa(\delta, H)$ vertices and the graph $G^* = K_{\delta,n-\delta}$. For fixed δ, we also find new conditions on H for which $\hom(G,H) \le \hom(K_{\delta,n-\delta},H)$ for all n-vertex graphs G with minimum degree at least δ when n is sufficiently large.

1 Introduction and Statement of Results

Let $G = (V(G), E(G))$ be a finite simple graph, and let $H = (V(H), E(H))$ be a finite graph that may have loops but does not have multi-edges. An *H-coloring of G*, or *homomorphism from G to H*, is an adjacency-preserving map $f : V(G) \to V(H)$, that is, a map satisfying $f(v)f(w) \in E(H)$ whenever $vw \in E(G)$. We let $\mathrm{Hom}(G,H)$ denote the set of all H-colorings of G, and let $\hom(G,H) = |\mathrm{Hom}(G,H)|$, i.e., $\hom(G,H)$ is the number of H-colorings of G.

The notion of H-coloring is a generalization of some important concepts in graph theory. For example, when $H = H_{\text{ind}} =$  , the H-colorings of G correspond to *independent sets* (or *stable sets*) in G via the vertices mapped to the unlooped vertex of H_{ind}. And when $H = K_q$, the complete graph on q vertices, the H-colorings of G correspond to *proper q-colorings* of

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the vertices of G. Motivated by the latter example, it can be useful to think of the vertices of the graph H as the allowable colors to use on the vertices of G, and the edges of the graph H encoding the allowable color pairs that can appear on the endpoints of an edge in G. H-colorings also have a natural interpretation as hard-constraint spin models from statistical physics (see e.g. [1] and have connections to graph limits, property testing, and quasi-randomness (see e.g. [9]).

Given a family of graphs $\mathcal{G}$ and a fixed graph H, a natural extremal question is to determine the maximum and minimum values of $\hom(G, H)$ over all $G \in \mathcal{G}$. One family that is particularly relevant to the present work is the family of all n-vertex d-regular graphs. Kahn considered all *bipartite* graphs G in this family and $H = H_{\text{ind}} =$  , and showed that

$$\hom(G, H_{\text{ind}}) \leq \hom(K_{d,d}, H_{\text{ind}})^{\frac{n}{2d}}.$$

In a generalization of Kahn's work, Galvin and Tetali [7] proved that

$$\hom(G, H) \leq \hom(K_{d,d}, H)^{\frac{n}{2d}} \tag{1}$$

holds for *all* H and all n-vertex d-regular bipartite graphs. The question then was asked about what holds when considering larger collections of graphs G in the family.

Recently, Sah, Sawhney, Stoner and Zhao [12] proved that (1) is true for all H and all *triangle-free graphs* G in this family, and they furthermore showed that the triangle-free assumption is needed. In particular, they illustrated that if G contains a triangle, then there exists some graph H so that (1) is false. When considering $H = K_q$ (i.e. proper colorings), they also proved that (1) holds over all graphs in the family (i.e. over *all* n-vertex d-regular graphs G); other graphs H for which (1) holds true over all n-vertex d-regular graphs, including $H = H_{\text{ind}}$, can be found in e.g. [13, 15, 16].

When $H =$  , the number of H-colorings is maximized by a graph with the largest number of components, and so for this particular H and all n-vertex d-regular G where $d+1$ divides n we have

$$\hom(G, H) \leq \hom(K_{d+1}, H)^{\frac{n}{d+1}}, \tag{2}$$

where equality is achieved by G consisting of $\frac{n}{d+1}$ disjoint copies of K_{d+1}. Other H have been shown to satisfy (2) for all n-vertex d-regular graphs G [2, 13]. Further, Sernau [13] produced graphs H for which neither $\hom(K_{d,d}, H)^{\frac{n}{2d}}$ nor $\hom(K_{d+1}, H)^{\frac{n}{d+1}}$ is the maximizing value of $\hom(G, H)$ over all n-vertex d-regular G. It is unknown if there is a finite list of graphs so that for any H and any n-vertex d-regular G we have $\hom(G, H) \leq \hom(G^*, H)^{\frac{n}{|V(G^*)|}}$ for some graph G^* on the list. The following conjecture is an equivalent formulation of Conjecture 2.9 in [17].

Conjecture 1.1. *Fix $d \geq 1$. Then there is a constant $\kappa = \kappa(d)$ such that for any n-vertex d-regular graph G and any H we have*

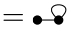
$$\hom(G, H) \leq \max_{G^*} \hom(G^*, H)^{\frac{n}{|V(G^*)|}},$$

where the maximum is taken over all d-regular graphs G^* with at most κ vertices.

Conjecture 1.1 is known to be true for $d = 1$ (trivial) and $d = 2$ [4]. For further results and questions, see the survey [17] and the references therein. We note that to date the maximizing value of $\hom(G, H)$ over all n-vertex d-regular graphs G for a particular H has been of the form $\hom(G, H)^{\frac{n}{|V(G)|}}$ where we can take $d + 1 \leq |V(G)| \leq 2d$, and so it would be interesting to either show $\kappa = 2d$ or find an example of an H where the maximum value comes only from a graph G^* with $|V(G^*)| > 2d$.

By relaxing the condition that requires all degrees to be equal, we arrive at the family of interest in this note.

Notation. Let $\mathcal{G}(n, \delta)$ denote the set of all n-vertex graphs with minimum degree at least δ.

Since edges in G give restrictions on the possible colors on the endpoints of the edge, it is natural to think that a regular (or close to regular) graph G would maximize $\hom(G, H)$ for any H and all $G \in \mathcal{G}(n, \delta)$, but this turns out not to be the case. For the graph $H = H_{\text{ind}} =$  and all $G \in \mathcal{G}(n, \delta)$ with $n \geq 2\delta$, Cutler and Radcliffe [3] showed that

$$\hom(G, H_{\text{ind}}) \leq \hom(K_{\delta, n-\delta}, H_{\text{ind}}).$$

The graph $K_{\delta,n-\delta}$ turns out to maximize the number of H-colorings for a large class of H over all $G \in \mathcal{G}(n, \delta)$ with n large [4], and also when considering the subfamily of *connected* n-vertex graphs G with minimum degree $\delta \geq 3$ with n large [10]. However, there are also examples of H for which

$$\hom(G, H) \leq \hom(K_{\delta+1}, H)^{\frac{n}{\delta+1}}$$

(use e.g. $H =$ ) or

$$\hom(G, H) \leq \hom(K_{\delta,\delta}, H)^{\frac{n}{2\delta}}$$

(use e.g. $H =$ ) for all $G \in \mathcal{G}(n, \delta)$. Furthermore, Guggiari and Scott [10] built on the ideas of Sernau [13] to produce examples of H for which $\hom(K_{\delta,n-\delta}, H)$, $\hom(K_{\delta+1}, H)^{\frac{n}{\delta+1}}$, and $\hom(K_{\delta,\delta}, H)^{\frac{n}{2\delta}}$ are not the maximizing value of $\hom(G, H)$ over all $G \in \mathcal{G}(n, \delta)$. We remark that their examples produce a different graph $G^* \in \mathcal{G}(|V(G^*)|, \delta)$, with $\delta + 1 < |V(G^*)| < 2\delta$, so that

$$\hom(G, H) \leq \hom(G^*, H)^{\frac{n}{|V(G^*)|}}$$

for all $G \in \mathcal{G}(n, \delta)$. This leads to the corresponding conjecture for n-vertex graphs with minimum degree at least δ.

Conjecture 1.2. *Fix $\delta \geq 1$. Then there a constant $\kappa = \kappa(\delta)$ such that for all $G \in \mathcal{G}(n, \delta)$ and all H we have*

$$\hom(G, H) \leq \max_{G^*} \hom(G^*, H)^{\frac{n}{|V(G^*)|}},$$

where the maximum is taken over all graphs G^* with minimum degree δ on at most κ vertices and the graph $G^* = K_{\delta,n-\delta}$.

Conjecture 1.2 is known to be true when $\delta = 1$ [4]. To discuss further related results, we introduce the following convention that we will use here and throughout the remainder of this note.

Convention. Given a graph H, we let Δ denote the maximum degree of a vertex in H, where by convention a loop adds one to the degree of a vertex (i.e. $d_H(v) = |N[v]|$).

When $\delta = 2$, it is known [4] that if H satisfies $\hom(C_3, H) \geq \Delta^3$ or $\hom(C_4, H) \geq \Delta^4$, then $\hom(G, H) \leq \max\{\hom(C_3, H)^{\frac{n}{3}}, \hom(C_4, H)^{\frac{n}{4}}\}$ for all $G \in \mathcal{G}(n, \delta)$, and otherwise $\hom(G, H) \leq \hom(K_{2,n-2}, H)$ for all $G \in \mathcal{G}(n, \delta)$ when $n \geq c_H$, with c_H some constant that depends on H. This does not quite resolve Conjecture 1.2 in the case $\delta = 2$, as the constant c_H in the latter case depends on H.

Further related results appear in [10], where they consider fixed H and δ large (depending on H) and n large relative to H and δ; and also fixed δ and H large (depending on δ) with n large relative to H and δ. For all δ and H that are considered in these families, the inequality of Conjecture 1.2 holds.

In this paper we aim to study fixed δ and fixed H. Our first result is the following.

Theorem 1.3. *Let H and $\delta \geq 1$ be fixed. Then there is a constant $\kappa = \kappa(\delta, H)$ such that for all $G \in \mathcal{G}(n, \delta)$ we have*

$$\hom(G, H) \leq \max_{G^*} \hom(G^*, H)^{\frac{n}{|V(G^*)|}}$$

where the maximum is taken over all graphs G^* with minimum degree δ on at most $\kappa(\delta, H)$ vertices and the graph $G^* = K_{\delta,n-\delta}$.

Theorem 1.3 makes progress but does not fully resolve Conjecture 1.2, since the constant depends on both δ and H. The proof utilizes the result for connected graphs in $\mathcal{G}(n, \delta)$ of Guggiari and Scott [10] along with analytic techniques.

To date, the maximizing value of $\hom(G, H)$ over all $G \in \mathcal{G}(n, \delta)$ for a particular H has been either $\hom(K_{\delta,n-\delta}, H)$ or $\hom(G^*, H)^{\frac{n}{|V(G^*)|}}$ where $\delta + 1 \leq |V(G^*)| \leq 2\delta$, and so it would again be interesting to either show $\kappa = 2\delta$ or find a particular H whose maximum value comes only from a graph $G^* \neq K_{\delta,n-\delta}$ with $|V(G^*)| > 2\delta$.

A second way of approaching the problem of maximizing $\hom(G, H)$ over all $G \in \mathcal{G}(n, \delta)$ is to find conditions on H for which $G = K_{\delta,n-\delta}$ is the maximizing graph. This approach of finding classes of H mirrors the work done in the family of n-vertex d-regular graphs (see e.g. [2, 6, 13, 15, 16] or the survey [17]). Conjecture 1.2, if true, would give a necessary and sufficient condition on H so that $G = K_{\delta,n-\delta}$ would produce the maximizing value of $\hom(G, H)$ over all $G \in \mathcal{G}(n, \delta)$. In particular, it would imply that if H makes $\hom(G', H)$ not too large for all "small" graphs G' with minimum degree δ, then $G = K_{\delta,n-\delta}$ would maximize the value $\hom(G, H)$ over all $G \in \mathcal{G}(n, \delta)$.

Along these lines, we aim to consider H that make $\hom(G', H)$ not too large for some small graph G'. To our knowledge, the best current result in this direction is the following; recall that Δ is the maximum degree of a vertex in H.

Theorem 1.4 ([4])**.** *Fix $\delta \geq 1$. Suppose H satisfies $\hom(K_2, H)^{\frac{1}{2}} < \Delta$. Then for sufficiently large n and all $G \in \mathcal{G}(n, \delta)$ we have*

$$\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H),$$

with equality if and only if $G = K_{\delta,n-\delta}$.

We mention here that all of the results for the family $\mathcal{G}(n, \delta)$ where $G = K_{\delta,n-\delta}$ gives the maximizing value, aside from trivial H or $H = H_{\text{ind}}$, are results where n is assumed to be large enough depending on δ and H.

Our second result enlarges the class of H in Theorem 1.4, and does so by conditioning on the number of H-colorings of a graph whose order is a function of δ. This improves on the size being the fixed constant $|V(K_2)| = 2$ from Theorem 1.4.

Theorem 1.5. *Fix $\delta \geq 1$. Suppose H satisfies $\hom(K_{1,\delta}, H)^{\frac{1}{\delta+1}} < \Delta$. Then for sufficiently large n and all $G \in \mathcal{G}(n, \delta)$ we have*

$$\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H)$$

with equality if and only if $G = K_{\delta,n-\delta}$.

The proof of Theorem 1.5 uses a stability argument: first, we show that any graph G that has a large number of disjoint copies of $K_{1,\delta}$ cannot maximize the value $\hom(G, H)$. So the graphs G that can maximize $\hom(G, H)$ must have few disjoint copies of $K_{1,\delta}$, and in this way are structurally similar to $K_{\delta,n-\delta}$. Those latter graphs G are then analyzed based on the presence of those structures.

To see why Theorem 1.5 enlarges the class of H from Theorem 1.4, first notice that $\hom(K_2, H) = \sum_{v \in V(H)} d(v)$. Assuming that H satisfies $\sum_{v \in V(H)} d(v) < \Delta^2$, we have

$$\sum_{v \in V(H)} d(v) < \Delta^2 \implies \Delta^{\delta-1} \sum_{v \in V(H)} d(v) < \Delta^{\delta+1},$$

and therefore

$$\hom(K_{1,\delta}, H) = \sum_{v \in V(H)} d(v)^\delta < \Delta^{\delta+1},$$

and so H satisfies the condition in Theorem 1.5. Furthermore, with $H = P_3$ being a path on 3 vertices, we have $\sum_{v \in V(H)} d(v) = 4 = \Delta^2$, while for any $\delta > 1$ we have $\sum_{v \in V(H)} d(v)^\delta = 2 + 2^\delta < 2^{\delta+1} = \Delta^{\delta+1}$. Therefore the class of H from Theorem 1.5 is strictly larger than the class of H from Theorem 1.4.

The rest of this paper is laid out as follows. In Section 2, we prove Theorem 1.3. Section 3 begins with a few introductory remarks and observations before proving Theorem 1.5. We then close with some related questions in Section 4.

2 Proof of Theorem 1.3

Fix H with maximum degree Δ. Notice that we can assume that H has no isolated vertices. For $\delta = 1$ and $\delta = 2$, the result holds from [4], so fix $\delta \geq 3$.

Let $G \in \mathcal{G}(n, \delta)$. By Corollary 1.2 in [10] there exists a constant $\kappa(\delta, H) =: N$ such that for $n \geq N$ the n-vertex *connected* graph with minimum degree at least δ that maximizes the number of H-colorings is $K_{\delta,n-\delta}$.

Suppose that G has components $G_1, \ldots, G_r$ with $|V(G_i)| = n_i$ for $i = 1, \ldots, r$. Then

$$\hom(G, H) = \prod_i \hom(G_i, H) = \prod_i \hom(G_i, H)^{\frac{n_i}{n_i}},$$

and if $n_i \geq N$ we have $\hom(G_i, H) \leq \hom(K_{\delta,n_i-\delta}, H)$. So this means

$$\hom(G, H) \leq \prod_{i: n_i < N} \hom(G_i, H)^{\frac{n_i}{n_i}} \cdot \prod_{i: n_i \geq N} \hom(K_{\delta,n_i-\delta}, H)^{\frac{n_i}{n_i}}.$$

We next compare the values of $\hom(K_{\delta,n_i-\delta},H)^{\frac{1}{n_i}}$ for those $n_i \geq N$. Let Z denote the vertices in the size δ partition class of $K_{\delta,n_i-\delta}$. By first coloring Z, the number of H-colorings of $K_{\delta,n_i-\delta}$ is given by

$$\hom(K_{\delta,n_i-\delta},H) = \sum_{(v_1,\ldots,v_\delta)\subseteq V(H)^\delta} |N(v_1)\cap\cdots\cap N(v_\delta)|^{n_i-\delta} = \sum_{d=1}^{\Delta} c_d\cdot d^{n_i-\delta}$$

for some constants $c_d \geq 0$; namely, c_d is the number of vectors containing δ elements of $V(H)$ that have exactly d common neighbors. Since n_i is the variable in our expression, we let $x \geq \delta+1$ be a real number and consider the expression

$$\left(\sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta}\right)^{\frac{1}{x}}. \tag{3}$$

We want the maximum value of (3) for $N \leq x \leq n$.

Let $a = a(x,H,\delta) \in \mathbb{R}$ be such that

$$\sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta} = a^x.$$

Note that for any $\varepsilon > 0$ we have

$$\sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta+\varepsilon} \leq \Delta^\varepsilon \sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta} = \Delta^\varepsilon a^x \tag{4}$$

with strict inequality if $c_i \neq 0$ for some $i < \Delta$. We consider the relative values of a and Δ to maximize the expression in (3).

Case 1: Suppose first that $a > \Delta$. Then (4) gives

$$\left(\sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta+\varepsilon}\right)^{\frac{1}{x+\varepsilon}} \leq (\Delta^\varepsilon a^x)^{\frac{1}{x+\varepsilon}} < a = \left(\sum_{d=1}^{\Delta} c_d\cdot d^{x-\delta}\right)^{\frac{1}{x}}, \tag{5}$$

and so $x = N$ is the smallest value of x and thus gives the maximum value of the expression in (3).

Case 2: If $a = \Delta$ and $c_i = 0$ for each $i < \Delta$, then $c_\Delta = \Delta^\delta$ and every vector in $V(H)^\delta$ whose elements have at least one common neighbor must have exactly Δ common neighbors. By considering $(x,x,\ldots,x) \in V(H)^\delta$ for each $x \in V(H)$, this implies that each $x \in V(H)$ has degree Δ (here we use that H has no isolated vertices). Further, if y and z are neighbors of x, then y and z have Δ common neighbors as well as degree Δ, and so $N(y) = N(z)$.

If x does not have a loop, then this produces $K_{\Delta,\Delta}$ in H, and so $c_\Delta \geq 2\Delta^\delta$ which is a contradiction. So therefore x has a loop, and so all vertices in H have loops. Also, if x and y are neighbors of x, then y has the same neighborhood as x. It now follows that H must contain the completely looped graph on Δ vertices. Since $c_\Delta = \Delta^\delta$, we have that H is exactly the completely looped graph on Δ vertices. In this case, $\hom(G,H) = \Delta^n$ for all n-vertex graphs G, and the result is clear in this case.

Case 3: If $a = \Delta$ and $c_i > 0$ for some $i < \Delta$, then from (4) we have

$$\left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta+\varepsilon}\right)^{\frac{1}{x+\varepsilon}} < a = \left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta}\right)^{\frac{1}{x}} \tag{6}$$

and so $x = N$ again gives the maximum value of the expression in (3).

Case 4: Finally, if $a < \Delta$ then $\Delta^\varepsilon a^x < \Delta^{x+\varepsilon}$, so by (4) we have

$$\left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta+\varepsilon}\right)^{\frac{1}{x+\varepsilon}} \le (\Delta^\varepsilon a^x)^{\frac{1}{x+\varepsilon}} < \Delta. \tag{7}$$

In this last case, we still need to identify the maximum value of the expression in (3) for $N \le x \le n$. We use the following lemma, whose proof we delay until after finishing the proof of Theorem 1.3.

Lemma 2.1. *The function* $f : [\delta+1, \infty) \to \mathbb{R}$ *defined by*

$$f(x) = \left(\sum_{(v_1,\ldots,v_\delta)\in V(H)^\delta} |N(v_1)\cap\cdots\cap N(v_\delta)|^{x-\delta}\right)^{\frac{1}{x}} = \left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta}\right)^{\frac{1}{x}}$$

has at most one local maximum or local minimum.

Since the function with output

$$\left(\sum_{(v_1,\ldots,v_\delta)\in V(H)^\delta} |N(v_1)\cap\cdots\cap N(v_\delta)|^{x-\delta}\right)^{\frac{1}{x}} = \left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta}\right)^{\frac{1}{x}}$$

tends to Δ as $x \to \infty$, it follows from (5), (6), and (7) that it is either decreasing to Δ on $x \ge N$, increasing to Δ on $x \ge N$, or decreasing on $N \le x < x_0$ and increasing to Δ on $x_0 < x$. Therefore the maximum value of the expression in (3) occurs on the endpoints of the interval $N \le x \le n$.

Finally, we identify the maximum value of $\hom(G_i, H)^{1/n_i}$ over the graphs G_i where either $G_i = K_{\delta,n-\delta}$ or G_i satisfies $G_i \in \mathcal{G}(n_i, \delta)$ with $n_i \le N$. Let G^* denote the graph that produces the maximum value. Then

$$\begin{aligned}
\hom(G,H) &\le \prod_{i:n_i<N} \hom(G_i,H)^{\frac{n_i}{n_i}} \cdot \prod_{i:n_i\ge N} \hom(K_{\delta,n_i-\delta},H)^{\frac{n_i}{n_i}} \\
&\le \prod_{i:n_i<N} \hom(G^*,H)^{\frac{n_i}{|V(G^*)|}} \cdot \prod_{i:n_i\ge N} \hom(G^*,H)^{\frac{n_i}{|V(G^*)|}} \\
&= \hom(G^*,H)^{\frac{n}{|V(G^*)|}}
\end{aligned}$$

where G^* is either $K_{\delta,n-\delta}$, or G^* has at most $N = \kappa(\delta, H)$ vertices. This completes the proof of Theorem 1.3. □

We now return to Lemma 2.1. To prove this, we will use the following proposition about L^p norms, which is itself a special case of Lemma 1.11.5 in [14].

Proposition 2.2 ([14]). *Define a measure* μ *on* $V(H)^\delta$ *by*

$$\mu((v_1,\dots,v_\delta)) = \begin{cases} |N(v_1)\cap\cdots\cap N(v_\delta)|^{-\delta} & \text{if } N(v_1)\cap\cdots\cap N(v_\delta)\neq\varnothing \\ 0 & \text{if } N(v_1)\cap\cdots\cap N(v_\delta)=\varnothing, \end{cases}$$

and let $g : V(H)^\delta \to \mathbb{R}$ *be defined by* $g((v_1,\dots,v_\delta)) = |N(v_1)\cap\cdots\cap N(v_\delta)|$. *Then the function defined by*

$$\frac{1}{x} \mapsto ||g||_{L^x(V(H))} = \left(\sum_{(v_1,\dots,v_\delta)\in V(H)^\delta} |N(v_1)\cap\cdots\cap N(v_\delta)|^{x-\delta}\right)^{\frac{1}{x}}$$

is log convex for $x \geq \delta+1$.

We prove Lemma 2.1 as a consequence of Proposition 2.2.

Proof of Lemma 2.1: Recall that a log convex function has at most one local maximum or local minimum. The composition of the reciprocal map and the map given in Proposition 2.2 is the function defined by

$$x \mapsto \left(\sum_{(v_1,\dots,v_\delta)\in V(H)^\delta} |N(v_1)\cap\cdots\cap N(v_\delta)|^{x-\delta}\right)^{\frac{1}{x}} = \left(\sum_{d=1}^{\Delta} c_d \cdot d^{x-\delta}\right)^{\frac{1}{x}}.$$

Since the reciprocal map is strictly monotone on $x \geq \delta+1$ and therefore preserves local extremal values, proof of the lemma is complete. □

3 Proof of Theorem 1.5

Before starting the proof of Theorem 1.5, we state an important lemma from [4], followed by a few other remarks. Recall that Δ refers to the maximum degree of a vertex in H.

Lemma 3.1 ([4]). *Suppose* H *does not contain the completely looped graph on* Δ *vertices or* $K_{\Delta,\Delta}$ *as a component. Then for any two vertices* i, j *of* H *and for* $k \geq 4$ *there are at most* $(\Delta^2-1)\Delta^{k-4}$ H*-colorings of* P_k *that map the initial vertex of that path to* i *and the terminal vertex to* j.

We will often build our colorings in stages by coloring some vertices and extending this coloring to the remaining vertices. The conclusion of Lemma 3.1 holds by our assumptions on H, and it will be frequently used to give an upper bound of Δ^2-1 on the number of ways of extending a coloring to the vertices of an edge that has previously colored neighbors. When we reach a single vertex that has a previously colored neighbor, we will often give an upper bound of Δ on the number of ways of extending a coloring to this single vertex.

We next provide a simple lower bound on $\hom(K_{\delta,n-\delta},H)$. Suppose that $S(\delta,H)$ is the set of vectors in $V(H)^\delta$ so that the entries of the vector have Δ common neighbors, and let $s(\delta,H) = |S(\delta,H)|$. Let Z denote the set of vertices in the size δ partition class in $K_{\delta,n-\delta}$.

By coloring the vertices of Z with a fixed element of $S(\delta, H)$ and then coloring each vertex in $V(K_{\delta,n-\delta}) \setminus Z$ independently with one of the Δ common neighbors, we have

$$\hom(K_{\delta,n-\delta}, H) \geq s(\delta, H)\Delta^{n-\delta}. \tag{8}$$

Now we move on to the proof of Theorem 1.5.

Proof of Theorem 1.5. Fix $\delta \geq 1$ and $G \in \mathcal{G}(n, \delta)$. Let H satisfy $\hom(K_{1,\delta}, H)^{\frac{1}{\delta+1}} < \Delta$, or equivalently

$$\sum_{v\in V(H)} d(v)^\delta < \Delta^{\delta+1}. \tag{9}$$

Since $(x, x, \ldots, x) \in S(\delta, H)$ for an $x \in V(H)$ with $d(x) = \Delta$, then from (8) we have

$$\hom(K_{\delta,n-\delta}, H) \geq s(\delta, H)\Delta^{n-\delta} \geq \Delta^{n-\delta}.$$

Let B be the union of the vertices of a maximum number of (vertex) disjoint copies of $K_{1,\delta}$ in G. Let $A = V(G) \setminus B$. Note that the maximality of B implies that (a) there are no vertices in A with δ neighbors in A, so each vertex in A has a neighbor in B, and (b) each component of G has some vertices in B.

We now indicate our coloring scheme that we will use to produce an upper bound on $\hom(G, H)$. We will first color the vertices in B independently, followed by the vertices in A. There are at most $\left(\sum_{v\in V(H)} d(v)^\delta\right)^{|B|/(\delta+1)}$ possible colorings of B, and since all components of G have some vertices in B, this implies

$$\hom(G, H) \leq \left(\sum_{v\in V(H)} d(v)^\delta\right)^{|B|/(\delta+1)} \Delta^{n-|B|} = \left(\frac{\sum_{v\in V(H)} d(v)^\delta}{\Delta^{\delta+1}}\right)^{|B|/(\delta+1)} \Delta^n.$$

By our assumptions on H, if $|B| > \delta(\delta+1)\log(\Delta)/\log\left(\frac{\Delta^{\delta+1}}{\sum_{v\in V(H)} d(v)^\delta}\right)$ then this upper bound is smaller than $\Delta^{n-\delta}$ and so we have $\hom(G, H) < \hom(K_{\delta,n-\delta}, H)$.

So now suppose that $|B| \leq \delta(\delta+1)\log(\Delta)/\log\left(\frac{\Delta^{\delta+1}}{\sum_{v\in V(H)} d(v)^\delta}\right)$. Once B has been colored, two vertices of A joined by an edge can be colored in at most $\Delta^2 - 1$ ways, which is a consequence of Lemma 3.1 and the fact that all components of G contain vertices of B. So if A contains a matching of size m, then

$$\hom(G, H) \leq \left(\sum_{v\in V(H)} d(v)^\delta\right)^{|B|/(\delta+1)} (\Delta^2 - 1)^m \Delta^{n-|B|-2m}.$$

which is smaller than $\Delta^{n-\delta}$ whenever $m > \delta\log(\Delta)/\log(\Delta^2/(\Delta^2-1))$. Therefore we can assume that $|B|$ and m are both smaller than a constant that depends on δ and H.

We add the endpoints of a maximum matching in A to the set B, and so this augmented set B contains a constant (depending on δ and H) number of vertices. The maximality of the matching implies that each vertex in $V(G) \setminus B$ has all of its (at least δ) neighbors in the set B. By the pigeonhole principle there exists a set Z of size δ so that Z is contained in the neighborhood of at least $(n-|B|)/\binom{|B|}{\delta} \geq cn$ vertices of $V(G) \setminus B$ for some constant $c = c(\delta, H)$. Therefore we assume G contains the (not necessarily induced) subgraph $K_{\delta,cn}$.

We next indicate how we will color the components of G based on whether they contain the subgraph $K_{\delta,cn}$ or not. For any component that does not contain the subgraph $K_{\delta,cn}$, we color any vertex and δ of its neighbors, and then greedily color the rest of that component. So an upper bound on the number of colorings in another component that has x vertices is

$$(\Delta^{\delta+1}-1)\Delta^{x-\delta-1}, \tag{10}$$

which is strictly smaller than Δ^x.

For the component containing $K_{\delta,cn}$, we again color Z and then the rest of that component. In this case, by utilizing (10) on any other components, the number of colorings of G that do not use an element of $S(\delta, H)$ on Z is at most

$$|V(H)|^\delta(\Delta-1)^{cn}\Delta^{n-\delta-cn}.$$

For those colorings that use an element of $S(\delta, H)$ on Z, we then color the rest of the vertices and have at most Δ choices of a color on each of those remaining vertices.

If G has more than one component, then using the upper bound from (10) in one such component that does not contain $K_{\delta,cn}$ we have

$$\begin{aligned}\hom(G,H) &\leq s(\delta,H)(\Delta^{\delta+1}-1)\Delta^{n-2\delta-1} + |V(H)|^\delta(\Delta-1)^{cn}\Delta^{n-\delta-cn}\\ &\leq s(\delta,H)\Delta^{n-\delta} - s(\delta,H)\Delta^{n-2\delta-1} + |V(H)|^\delta e^{-cn/\Delta}\Delta^{n-\delta}.\end{aligned}$$

Likewise, if there is an edge in the component containing $K_{\delta,cn}$ that does not have an endpoint in Z, then from Lemma 3.1 we have

$$\begin{aligned}\hom(G,H) &\leq s(\delta,H)(\Delta^{2}-1)\Delta^{n-\delta-2} + |V(H)|^\delta(\Delta-1)^{cn}\Delta^{n-\delta-cn}\\ &\leq s(\delta,H)\Delta^{n-\delta} - s(\delta,H)\Delta^{n-\delta-2} + |V(H)|^\delta e^{-cn/\Delta}\Delta^{n-\delta}.\end{aligned}$$

In either case, for large enough n we have $|V(H)|^\delta e^{-cn/\Delta} < 1/\Delta^{\delta+1}$, which implies $\hom(G,H) < s(\delta,H)\Delta^{n-\delta} \leq \hom(K_{\delta,n-\delta},H)$. So, if $\hom(G,H) \geq \hom(K_{\delta,n-\delta},H)$, the graph G must be connected and contain $K_{\delta,n-\delta}$ plus potentially some edges inside Z, the size δ partition class.

We now argue that for such a G that satisfy $\hom(G,H) \geq \hom(K_{\delta,n-\delta},H)$, there are no edges between two vertices in Z, which we do by showing that adding such an edge e will strictly decrease the number of H-colorings of $K_{\delta,n-\delta}$. If ij is an edge in H, then we can color Z with i and $V(K_{\delta,n-\delta}) \setminus Z$ with j. So if i is unlooped, this coloring of $K_{\delta,n-\delta}$ is not an H-coloring with the edge e added. If instead all vertices of H are looped, then as H is not the completely looped graph (by assumption on H) there will be non-adjacent vertices j_1 and j_2 with a common neighbor i in H. Then we map Z to j_1 and j_2 and $V(K_{\delta,n-\delta})$ to i. If the endpoints of the added edge e have colors j_1 and j_2, then again this is not an H-coloring with the edge e added. This shows that if $G \neq K_{\delta,n-\delta}$, then $\hom(G,H) < \hom(K_{\delta,n-\delta},H)$, which completes the proof. □

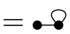
4 Concluding Remarks

In this section we mention a few interesting questions related to the contents of this article beyond Conjectures 1.1 and 1.2.

Consider the family of graphs with fixed minimum degree at least δ and a maximum degree at most D. When $D = \delta$ this is the family of δ-regular graphs, and when $D = n - 1$ this is the family $\mathcal{G}(n, \delta)$. If D is smaller than $n - \delta$, then the graph $K_{\delta,n-\delta}$ is not in this family.

Question 4.1. *Fix H, $\delta > 1$, and let $D \geq \delta$. What is the maximum value of* $\hom(G, H)$ *over all n-vertex graphs G with minimum degree at least δ and maximum degree at most D?*

For all H, $\delta = 1$, and all values D, the maximizing value of $\hom(G, H)$ is either $\hom(K_2, H)^{\frac{n}{2}}$ or $\hom(K_{1,D}, H)^{\frac{n}{D+1}}$ [4].

If H is such that a regular graph $G \in \mathcal{G}(n, \delta)$ gives the maximizing value of $\hom(G, H)$, then this graph G will still maximize $\hom(G, H)$ for all graphs in this new family. But many H have $K_{\delta,n-\delta} \in \mathcal{G}(n, \delta)$ as the graph that maximizes the value of $\hom(G, H)$, and for these H it is not obvious what the maximizing value of $\hom(G, H)$ would be when $D < n - \delta$. One appealing special case of Question 4.1 is when $H =$  , where we recall that $K_{\delta,n-\delta}$ has the most number of independent sets among n-vertex graphs with minimum degree at least δ.

Question 4.2. *Fix $\delta > 1$ and let $D \geq \delta$. Which n-vertex graph with minimum degree at least δ and maximum degree at most D has the most number of independent sets?*

A bound based on the product of the degrees of the endpoints of edges in G can be found in [11].

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