Extremal colorings and independent sets

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Abstract

We consider several extremal problems of maximizing the number of colorings and independent sets in some graph families with fixed chromatic number and order.

First, we address the problem of maximizing the number of colorings in the family of connected graphs with chromatic number k and order n where $k \geq 4$. It was conjectured that extremal graphs are those which have clique number k and size $\binom{k}{2} + n - k$. We affirm this conjecture for 4-chromatic claw-free graphs and for all k-chromatic line graphs with $k \geq 4$. We also reduce this extremal problem to a finite family of graphs when restricted to claw-free graphs.

Secondly, we determine the maximum number of independent sets of each size in the family of n-vertex k-chromatic graphs (respectively connected n-vertex k-chromatic graphs and n-vertex k-chromatic graphs with c components). We show that the unique extremal graph is $K_k \cup E_{n-k}$, $K_1 \vee (K_{k-1} \cup E_{n-k})$ and $(K_1 \vee (K_{k-1} \cup E_{n-k-c+1})) \cup E_{c-1}$ respectively.

1 Introduction and Statement of Results

Let $G = (V(G), E(G))$ be a finite simple graph. For an integer $x \geq 1$, a *proper x-coloring of G*, or simply *x-coloring of G*, is a function $f : V(G) \to \{1, \ldots, x\}$ so that $f(v_1) \neq f(v_2)$ for every $v_1v_2 \in E(G)$. We let $\pi(G, x)$ denote the chromatic polynomial of G and so for positive integers x, $\pi(G, x)$ is simply the number of x-colorings of G. The *chromatic number* of G, denoted $\chi(G)$, is the smallest positive integer so that $\pi(G, x) \neq 0$, and we say that G is *k-chromatic* if $\chi(G) = k$. A graph G is called *critical* if $\chi(G - v) < \chi(G)$ for every vertex v of G. A k-chromatic critical graph is called *k-critical*. It is easy to see that if G is a k-critical graph then $\delta(G) \geq k - 1$ and k-critical graphs are 2-connected.

Much recent work has investigated the question of maximizing the number of x-colorings over various families of graphs, including n-vertex m-edge graphs [19, 20], n-vertex 2-connected graphs [10], connected graphs with fixed minimum degree [8, 17], bipartite regular graphs [16], and regular graphs [6, 14, 15].

One family that we focus on in this note is the family of n-vertex k-chromatic graphs. In this family Tomescu [25] showed that the disjoint union of the complete graph K_k with the empty graph E_{n-k} uniquely maximizes $\pi(G, x)$ for all $x \geq k$. When restricting to the set of *connected* n-vertex k-chromatic graphs (which we denote by $\mathcal{C}_k(n)$), the problem of determining the maximum value of $\pi(G, x)$ for $G \in \mathcal{C}_k(n)$ seems to be much more difficult. The answer is trivial for $k = 2$, where the extremal graphs are trees (when $x \geq 3$), and is known for $k = 3$ (see [24, 26]). For $k \geq 4$, we have the following conjecture [7, 23]. Let $\mathcal{C}^*_k(n)$ be the set containing all n-vertex graphs obtained from a k-clique by recursively attaching leaves.

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Conjecture 1.1. *Let $k \geq 4$ and $G \in \mathcal{C}_k(n)$. Then for every integer $x \geq k$ we have*

$$\pi(G,x) \leq (x)_{\downarrow k}\,(x-1)^{n-k}$$

with equality if and only if $G \in \mathcal{C}_k^*(n)$.

A complete answer to Conjecture 1.1 is not yet known, although it has been verified for $k \geq 4$ and x large [2] and graphs with the additional constraint of having independence number at most 2 [11] (equivalently, graphs that are complements of triangle-free graphs). For $k = 4$ the conjecture is reduced to understanding a finite number of graphs [12] and is also known to hold when the graphs are required to be planar [26].

It is not difficult to see that Conjecture 1.1 holds for graphs G with $\chi(G) = \omega(G)$. Therefore, when studying this problem we only need to consider graphs whose chromatic number is different from the clique number. An important family of such graphs is the family of claw-free graphs. So, in this paper we first consider the graphs in $\mathcal{C}_k(n)$ which are additionally claw-free. For $k = 4$ we obtain a result for all n.

Theorem 1.2. *Let G be a connected n-vertex claw-free 4-chromatic graph. Then for every integer $x \geq 4$ we have*

$$\pi(G,x) \leq (x)_{\downarrow 4}(x-1)^{n-4}$$

with equality if and only if $G \in \mathcal{C}_4^*(n)$.

For general k we have the following result which reduces the problem to a finite family of graphs.

Theorem 1.3. *For every $k \geq 4$, there exists a finite family $\mathcal{F}$ of k-chromatic claw-free graphs such that if every graph G in $\mathcal{F}$ satisfies $\pi(G,x) \leq (x)_{\downarrow k}(x-1)^{|V(G)|-k}$ then so does every connected k-chromatic claw-free graph.*

We also consider line graphs or, equivalently, edge colorings of graphs. A *proper x-edge-coloring of G*, or simply *x-edge-coloring of G* is a function $f :\ E(G) \to \{1,\ldots,x\}$ so that $f(e_1) \neq f(e_2)$ for all distinct edges e_1 and e_2 in $E(G)$ that share an endvertex. The *chromatic index of G*, denoted $\chi'(G)$, is the smallest integer x for which G has an x-edge-coloring. The *line graph $L(G)$ of G* is the graph whose vertices represent the edges of G (i.e. $V(L(G)) = E(G)$) and ef is an edge of $L(G)$ if and only if e and f are adjacent edges of G. Observe that $\chi'(G) = \chi(L(G))$ for every graph G. A graph G is called a *line graph* if there exists a graph H such that $G = L(H)$. Not every graph is a line graph and line graphs form a subfamily of claw-free graphs. We find the n-vertex k-chromatic line graphs that maximize the number of proper x-colorings for all n and k.

Theorem 1.4. *Let G be a connected n-vertex k-chromatic line graph with $k \geq 4$. Then for every integer $x \geq k$ we have*

$$\pi(G,x) \leq (x)_{\downarrow k}(x-1)^{n-k}$$

with equality if and only if G is obtained from a K_k by attaching paths of sizes $n_1,\ldots,n_k$ to the k vertices where $0 \leq n_i \leq n-4$ for $i = 1,\ldots,k$.

Let $k \geq 3$. A tree is called *k-starlike* if it has exactly one vertex with degree k and all other vertices have degree at most 2. An immediate consequence of the above theorem is the following extremal result for proper edge-colorings.

Corollary 1.5. *For every integer $x \geq k \geq 4$, k-starlike trees maximize the number of x-edge-colorings in the family of connected n-edge k-edge-chromatic graphs.*

We also consider independent sets in this paper. A set $I \subseteq V(G)$ is an *independent set* (or *stable set*) if $v_1, v_2 \in V(G)$ implies that $v_1v_2 \notin E(G)$. The *size* of an independent set I is $|I|$. Let $i(G)$ denote the number of independent sets of G and $i_t(G)$ denote the number of independent sets of size t in G. The quantity $i(G)$ has also been referred to as the *Fibonacci number* of G [22] (as these values for the path P_n are Fibonacci numbers), or in the field of molecular chemistry, the *Merrifield-Simmons index* of G [21]. Notice that each color class of a proper coloring of G is an independent set.

There has also been a large amount of work on investigating which graphs maximize $i(G)$ and $i_t(G)$ in various families of graphs, we refer the reader to two surveys and the references therein [4, 29] for a summary of some of the results and conjectures in this area.

In [18] the n-vertex graph with clique number ω containing the maximum number of independent sets of each fixed size is found, along with the characterization of uniqueness. Also, it was shown that the Turan graph $T_{n,k}$ is the unique n-vertex k-chromatic graph with the minimum number of independent sets [28] and the minimum number of independent sets of each size t [18] (where it is implicit that $2 \leq t \leq \lceil \frac{n}{k} \rceil$). We next find the n-vertex k-chromatic graph that has the maximum number of independent sets of each size. We remark that when $t = 0$ and $t = 1$, all n-vertex graphs have the same number of independent sets of size t, and for an n-vertex k-chromatic graph G we have $\alpha(G) \leq n - k + 1$ and hence $i_t(G) = 0$ for $t \geq n - k + 2$.

Theorem 1.6. *Let G be an n-vertex k-chromatic graph. Then we have*

$$i_t(G) \leq \binom{n-k}{t} + k\binom{n-k}{t-1}$$

For $2 \leq t \leq n - k + 1$ we have equality if and only if $G = K_k \cup E_{n-k}$.

We remark that the graph $K_k \cup E_{n-k}$ also uniquely maximizes the number of proper colorings in this family. Theorem 1.6 immediately gives the following.

Corollary 1.7. *Let G be an n-vertex k-chromatic graph. Then we have*

$$i(G) \leq i(K_k \cup E_{n-k}) = (k + 1)2^{n-k}$$

with equality if and only if $G = K_k \cup E_{n-k}$.

As with proper colorings, we now consider the *connected* n-vertex k-chromatic graph that has the most number of independent sets. Results on independent sets of size t in graphs with $\omega(G) = k$ appear in [18] (while not explicitly stated in Theorem 1.8 of [18], the maximizing results are for *connected* graphs with clique number k).

Observe that an independent set of size 2 induces an edge in the complement of the graph. Therefore, maximizing the number of independent sets of size 2 is equivalent to minimizing the number of edges. And the latter problem was already solved.

Theorem 1.8 ([23])**.** *Let G be a connected n-vertex k-chromatic graph. Then we have*

$$i_2(G) \leq \binom{n-k}{2} + (k - 1)(n - k).$$

Furthermore, for $k = 3$, extremal graphs are unicyclic graphs with an odd cycle, while for $k \neq 3$, extremal graphs belong to $\mathcal{C}_k^*(n)$.

When $k = 2$, the connected 2-chromatic graphs that maximize the number of independent sets of size t are trees (as deleting edges from G cannot decrease the number of independent sets of size t). The maximization (and minimization) of the number of independent sets of size t in trees was solved for all t by Wingard [27]; see also [18]. We generalize this to all k.

Theorem 1.9. *Let $k \geq 2$ and let G be a connected k-chromatic graph of order n. Then*

$$i_t(G) \leq \binom{n-k}{t} + (k-1)\binom{n-k}{t-1} + \binom{0}{t-1}.$$

For $3 \leq t \leq n-k+1$ we have equality if and only if $G = K_1 \vee (K_{k-1} \cup E_{n-k})$.

This gives the following corollary, whose proof is included in Section 3.

Corollary 1.10. *Let G be a connected k-chromatic graph with n vertices. Then*

$$i(G) \leq k2^{n-k} + 1$$

with equality if and only if $G = K_1 \vee (K_{k-1} \cup E_{n-k})$.

We can refine the extremal graphs based on the number of components.

Theorem 1.11. *Let G be an n-vertex k-chromatic graph with c components. Then we have*

$$i_t(G) \leq \binom{n-k}{t} + (k-1)\binom{n-k}{t-1} + \binom{c-1}{t-1}.$$

For $3 \leq t \leq n-k+1$ we have equality if and only if $G = (K_1 \vee (K_{k-1} \cup E_{n-k-c+1})) \cup E_{c-1}$.

Corollary 1.12. *Let G be an n-vertex k-chromatic graph with c components. Then we have*

$$i(G) \leq k2^{n-k} + 2^{c-1}$$

with equality if and only if $G = (K_1 \vee (K_{k-1} \cup E_{n-k-c+1})) \cup E_{c-1}$.

In the rest of the paper we provide the proofs of these results. In Section 2, we study extremal colorings and give proofs of Theorems 1.2, 1.3 and 1.4. Section 3 deals with independent sets and we give proofs for Theorems 1.6, 1.9 and 1.11. Lastly, in Section 4 we pose several questions on extremal colorings and independent sets.

2 Extremal Colorings

In this section, we present the proofs of the results about proper colorings. We begin by recalling a few results that will be frequently used in our proofs. We let $\omega(G)$ denote the size of the largest clique in G. The first result bounds the number of colorings of a k-chromatic graph which contains a clique of size k.

Proposition 2.1 ([11])**.** *Let $G \in \mathcal{C}_k(n)$ and $\omega(G) = k$. Then for all integers $x \geq k$ we have*

$$\pi(G, x) \leq (x)_{\downarrow k}\,(x-1)^{n-k}$$

with equality if and only if $G \in \mathcal{C}^*_k(n)$.

The next result allows us to focus on subgraphs that have nice properties.

Proposition 2.2 ([12])**.** *If H is a connected subgraph of a connected graph G, then for all $x \in \mathbb{N}$ we have*

$$\pi(G, x) \leq \pi(H, x)(x-1)^{|V(G)|-|V(H)|}.$$

2.1 4-chromatic claw-free graphs

In this section, we present a number of results which, at the end, are used to prove Theorem 1.2. To begin, let $\mathcal{F}_{n,t}$ be the family of graphs G with $n+t$ vertices and $n+2t$ edges such that G contains an induced odd cycle C_n and t triangles such that every triangle overlaps the cycle C_n in an edge and no two triangles share a common edge. It is easy to see that if $G \in \mathcal{F}_{n,t}$, then

$$\pi(G,x) = (x-2)^t(x-1)((x-1)^{n-1}-1).$$

Let G and H be two graphs with clique number at least r. We let $G \oplus_r H$ denote a graph which is obtained from G and H by gluing them at an r-clique.

Proposition 2.3. *Let G be an n-vertex 4-critical claw-free graph with $\omega(G) \neq 4$. Then for every integer $x \geq 4$ we have*

$$\pi(G,x) < (x)_{\downarrow 4}(x-1)^{n-4}.$$

Proof of Proposition 2.3. We shall consider two cases.
Case 1: G contains an odd hole.

Let C_r be an odd hole with vertices $v_1, \ldots, v_r$ in standard order.

Subcase 1: There exist a vertex $u \notin V(C_r)$ such that u is adjacent to three consecutive vertices of C_r.

If u is adjacent to all vertices of C_r, then G has a subgraph $H \cong K_1 \vee C_r$ whose chromatic polynomial is $x\,\pi(C_r, x-1) = x((x-2)^r - (x-2))$. By Proposition 2.2, it suffices to show that $x\,((x-2)^r-(x-2))\,(x-1)^{n-r-1} < (x)_{\downarrow 4}\,(x-1)^{n-4}$ which is equivalent to $(x-2)^{r-1}-1 < (x-3)\,(x-1)^{r-2}$. Since $r \geq 5$, to prove the latter it would be sufficient to show that $(x-2)^4 < (x-3)\,(x-1)^3$. Calculations show that $(x-3)\,(x-1)^3 - (x-2)^4$ has a positive leading coefficient and its largest real root is 3.191.... Hence we are done.

Now suppose that there is a vertex v_j of C_r which is not adjacent to u. We may assume that u is adjacent to v_1, v_2, v_3 and that $j \geq 4$. Since G is 4-critical, $\delta(G) \geq 3$. So v_j has a neighbor u' which is not in $V(C_r)$. As G is claw-free, u' must be adjacent to v_{j-1} or v_{j+1} (v_1 if $r=5$); denote one such neighbor by v. Let H be the subgraph of G with vertex set $V(C_r) \cup \{u, u'\}$ and edge set $E(C_r) \cup \{uv_1, uv_2, uv_3, u'v_j, u'v\}$. Observe that $H + v_1v_3 \cong K_4 \oplus_2 H_{r-1,1}$ and $H/v_1v_3 \cong K_3 \oplus_1 H_{r-2,1}$ where $H_{p,q}$ denotes a graph in the family $\mathcal{F}_{p,q}$. It is easy to see that $\pi(K_4 \oplus_2 H_{r-1,1}, x) = (x-1)(x-2)^2(x-3)((x-1)^{r-2}-1)$ and $\pi(K_3 \oplus_1 H_{r-2,1}, x) = (x-1)^2(x-2)^2((x-1)^{r-3}-1)$. So by the edge addition-contraction formula,

$$\pi(H,x) = \pi(H+v_1v_3, x) + \pi(H/v_1v_3, x) < (x-1)^{r-1}(x-2)^3.$$

Now by Proposition 2.2, it suffices to show that $(x-2)^2 \leq x(x-3)$, which clearly holds for $x \geq 4$.

Subcase 2: There is no vertex in G which is adjacent to three consecutive vertices of C_r.

Since $\delta(G) \geq 3$ and C_r is an induced subgraph, every vertex of C_r has a neighbor outside of C_r. Let u_1 be a vertex such that $v_1u_1 \in E(G)$ and $u_1 \notin V(C_r)$. Since G is claw-free, either $u_1v_2 \in E(G)$ or $u_1v_r \in E(G)$. Without loss, we assume that $u_1v_2 \in E(G)$. By the assumption, u_1 cannot be adjacent to v_3. So there exists a vertex u_3 such that $u_3 \neq u_1$ and $u_3v_3 \in E(G)$. Since G is claw-free, either $u_3v_2 \in E(G)$ or $u_3v_4 \in E(G)$. So we shall consider two cases again.

First assume that $u_3v_2 \in E(G)$. If there exists a vertex $v_j \in V(C_r)$ with $j \geq 4$ such that v_j has a neighbor, say w_j, which is not in $\{u_1, u_3\}$, then w_j would be adjacent to a neighbor of v_j in C_r, as G is claw free. So we would have a subgraph H of G which belongs to the family $\mathcal{F}_{r,3}$ and $\pi(H,x) = (x-2)^3(x-1)((x-1)^{r-1}-1)$. Now by Proposition 2.2 it suffices to show that $(x-2)^3((x-1)^{r-1}-1)(x-1)^{n-r-2}$ is less than $(x)_{\downarrow 4}(x-1)^{n-4}$ which follows from $(x-2)^2 \leq x(x-3)$

for $x \geq 4$ and $(x-1)^{r-1}-1 < (x-1)^{r-1}$. Now we may assume that $N_G(v_j)\setminus V(C_r) \subseteq \{u_1, u_3\}$. Since there is no vertex adjacent to three consecutive vertices of C_r, we get $u_1v_r, u_1v_3, u_3v_4, u_3v_1 \notin E(G)$. Also, $u_1v_4, u_3v_r \in E(G)$ by the assumptions. Then we must have $r \geq 7$, since if $r = 5$ then the vertices u_1, v_3, v_4, v_5 would induce a claw. As G is claw-free, u_3v_{r-1} and u_1v_5 are in $E(G)$. Now, we have four edge disjoint triangles with vertex sets $\{u_1, v_1, v_2\}$, $\{u_3, v_2, v_3\}$, $\{u_3, v_r, v_{r-1}\}$ and $\{u_1, v_4, v_5\}$. Let H be a minimal subgraph containing these four triangles. It is easy to see that $\pi(H, x) = x(x-1)^4(x-2)^4 < (x)_{\downarrow 4}(x-1)^5$ and the result follows by Proposition 2.2.

Now let us assume that $u_3v_4 \in E(G)$ (and so $u_3v_2 \notin E(G)$). As in the previous case, we may assume that $N_G(v_j) \setminus V(C_r) \subseteq \{u_1, u_3\}$. Again it must be that $r \geq 7$, as $r = 5$ implies that the neighbor of v_5 is adjacent to three consecutive vertices of C_5. Furthermore, by the assumptions we have u_1v_r, $u_3v_5 \notin E(G)$ and u_3v_r, $u_1v_5 \in E(G)$. Now, since G is claw free either $u_3v_{r-1} \in E(G)$ or $u_3v_1 \in E(G)$ (otherwise $\{u_3, v_{r-1}, v_r, v_1\}$ would induce a claw). If $u_3v_{r-1} \in E(G)$ (resp. $u_3v_1 \in E(G)$) then G has a subgraph H_1 (resp. H_2) show in Figure 1. In each case, it is easy to check that $\pi(H_i, x) < (x)_{\downarrow 4}(x-1)^{|V(H)|-4}$ holds and we are done by Proposition 2.2.

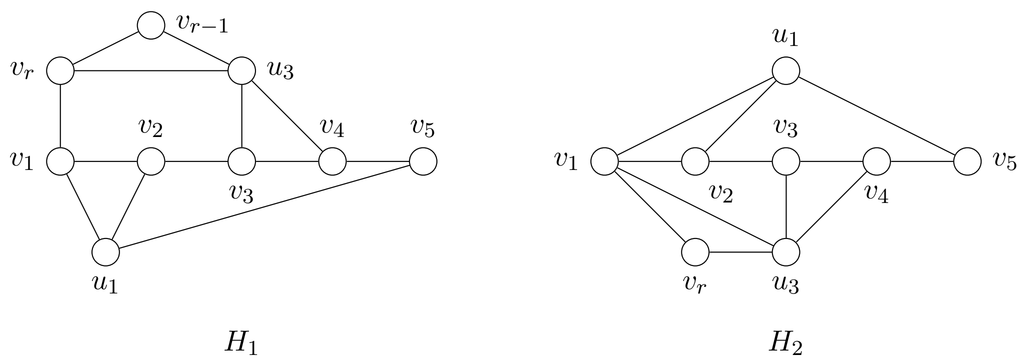

Figure 1: The graphs H_1 and H_2.

Case 2: G does not contain an odd hole.

By assumption, G is not a perfect graph. So by the strong perfect graph theorem [3], G must contain an odd anti-hole. The graph G cannot contain an anti-hole of order 5 because $C_5 \cong \overline{C}_5$. Also, G cannot contain an odd anti-hole of order larger than 7 because otherwise it would contain a K_4. Hence, G must contain a $\overline{C}_7$. Calculations show that

$$\pi(\overline{C}_7, x) = x\,(x-1)\,(x-2)\,(x-3)\left(x^3 - 8\,x^2 + 25\,x - 29\right).$$

Hence, by Proposition 2.2, it suffices to show that $(x-1)^3 - (x^3 - 8x^2 + 25x - 29) \geq 0$. But $(x-1)^3 - (x^3 - 8x^2 + 25x - 29)$ is a quadratic with positive leading coefficient and no real roots. Thus, the result follows. $\square$

Proof of Theorem 1.2. Suppose that G is a 4-chromatic claw-free graph. If $\omega(G) = 4$, then by Proposition 2.1 we have $\pi(G, x) \leq (x)_{\downarrow 4}(x-1)^{n-4}$ with equality if and only if $G \in \mathcal{C}_4^*(n)$. If $\omega(G) < 4$, then we first find a subgraph G' of G that is 4-critical and claw-free by removing some vertices from G. Propositions 2.2 and 2.3 then show that $\pi(G, x) < (x)_{\downarrow 4}(x-1)^{n-4}$, which finishes the proof of Theorem 1.2. $\square$

2.2 Claw-free graphs of large order

We next prove Theorem 1.3. To do so we use the following result, which provides a large number of disjoint triangles in G.

Theorem 2.4 ([13])**.** *If G is an n-vertex claw-free graph, then G contains at least* $\left(\frac{\delta(G)-2}{\delta(G)+1}\right)\frac{n}{3}$ *vertex disjoint triangles.*

Theorem 2.4 allows us to analyze critical graphs.

Proposition 2.5. *Let G be an n-vertex k-critical claw-free graph where $k \geq 4$ and*

$$n > \frac{3k}{k-3} \log\left(\frac{(k-2)!}{(k-1)^{k-2}}\right) \frac{1}{\log\left(\frac{k-2}{k-1}\right)}.$$

Then for every integer $x \geq k$ we have $\pi(G,x) < (x)_{\downarrow k}(x-1)^{n-k}$.

Proof of Proposition 2.5. Since G is k-critical, $\delta(G) \geq k-1$. By Theorem 2.4, G contains at least $\frac{(k-3)n}{3k}$ vertex disjoint triangles. Let H be a minimal connected spanning subgraph containing these triangles. So H is a block graph and it is easy to see that

$$\pi(H,x) = x(x-2)^t(x-1)^{n-t-1}$$

where $t = \frac{(k-3)n}{3k}$. It suffices to show that, for every $x \geq k$,

$$x(x-2)^t(x-1)^{n-t-1} < (x)_{\downarrow k}(x-1)^{n-k}$$

which is equivalent to

$$t > \log\left(\frac{(x-2)_{\downarrow k-2}}{(x-1)^{k-2}}\right) \frac{1}{\log\left(\frac{x-2}{x-1}\right)}.$$

Now the latter follows as $\log\left(\frac{(x-2)_{\downarrow k-2}}{(x-1)^{k-2}}\right) \frac{1}{\log\left(\frac{x-2}{x-1}\right)}$ is a decreasing function on $[k,\infty)$ and by the assumption on n. $\square$

Proof of Theorem 1.3. The result follows from Propositions 2.5 and 2.2, as every k-chromatic claw-free graph contains a k-critical subgraph which is claw-free. $\square$

2.3 Line Graphs

In this section we prove Theorem 1.4. We begin with a classic result for edge-colorings.

Theorem 2.6 (Vizing's Theorem)**.** *For every graph H, either $\chi'(H) = \Delta(H)$ or $\chi'(H) = \Delta(H)+1$.*

A graph G is called *chromatic index critical* if G is connected, $\chi'(G) = \Delta(G)+1$ and $\chi'(G-e) < \chi'(G)$ for every edge e of G.

Theorem 2.7 (Vizing's Adjacency Lemma)**.** *Let H be a chromatic index critical graph. If v and w are two adjacent vertices of H with $\deg_H(v) = \Delta(H)$, then w is adjacent to at least two vertices of degree $\Delta(H)$.*

Lemma 2.8. *Let G be an n-vertex k-critical line graph with $\omega(G) = k-1$ and $k \geq 4$. Then for every integer $x \geq k$ we have*

$$\pi(G,x) < (x)_{\downarrow k}(x-1)^{n-k}.$$

Proof. Suppose that G is the line graph of H, i.e. $L(H) = G$. It is clear that $\chi(G) = \chi'(H)$ and $\omega(G) = \Delta(H)$. Since G is connected we may assume that H is also connected (by ignoring isolated vertices, if any). Since G is a critical graph, the graph H is a chromatic index critical graph. First let us show that G contains at least two edge disjoint $(k-1)$-cliques. Let v and w be two adjacent vertices of H with $\deg_H(v) = k-1$. By Vizing's adjacency lemma, the vertex w is adjacent to at least two vertices of degree $k-1$. Let w' be a neighbor of w in H such that $\deg_H(w') = k-1$ and $w' \neq v$. Let E_v (resp. $E_{w'}$) be the set of edges of H which are incident to the vertex v (resp. w'). Observe that $|E_v \cap E_{w'}| \leq 1$. Let G_1 and G_2 be subgraphs of G induced by the vertices of G which represent the edges in E_v and $E_{w'}$ respectively. It is clear that $G_1 \cong G_2 \cong K_{k-1}$ and G_1 and G_2 are edge disjoint, as $|E_v \cap E_{w'}| \leq 1$.

Case 1: $k \geq 6$.

Let G' be a connected minimal spanning subgraph of G which contains G_1 and G_2. Exactly two blocks of G' are $k-1$-cliques and all the rest of the blocks are edges. Therefore,

$$\pi(G', x) = (x)_{\downarrow k-1}\,(x-1)_{\downarrow k-2}\,(x-1)^{n-2k+3}. \tag{1}$$

Since $\pi(G, x) \leq \pi(G', x)$, it suffices to show that $\pi(G', x) < (x)_{\downarrow k}(x-1)^{n-k}$ which is equivalent to

$$(x-2)(x-3)\cdots(x-k+3)(x-k+2) < (x-k+1)(x-1)^{k-4}.$$

Subcase 1: $k \geq 7$. It is clear that $(x-i) < (x-1)$ for $i = 2, \ldots k-4$. So we only need to show that

$$(x-k+3)(x-k+2) \leq (x-k+1)(x-1)$$

holds for $x \geq k$. Observe that

$$(x-k+1)(x-1) - (x-k+3)(x-k+2) = (k-5)x - k^2 + 6k - 7.$$

Since $x \geq k$, we get $(k-5)x - k^2 + 6k - 7 \geq (k-5)k - k^2 + 6k - 7 = k - 7 \geq 0$.

Subcase 2: $k = 6$. In this case it suffices to show that $(x-2)(x-3)(x-4) < (x-5)(x-1)^2$ holds for $x \geq 6$. Calculations show that the polynomial $(x-5)(x-1)^2 - (x-2)(x-3)(x-4)$ has a positive leading coefficient and its largest real root is 5.88.... Thus the result follows.

Case 2: $k = 5$.

Consider a vertex $u \notin V(G_1) \cup V(G_2)$. Since $\deg(u) \geq 4$ and G is claw-free, there exist at least two triangles T_1 and T_2 containing the vertex u. Now it is straightforward to check that the number of x-colorings of a minimal connected spanning subgraph containing G_1, G_2, T_1 and T_2, combined with Proposition 2.2, gives an upper bound strictly less than $(x)_{\downarrow 5}(x-1)^{n-5}$.

Case 3: $k = 4$.

Since line graphs are claw-free, the result follows from Proposition 2.3. $\square$

Proof of Theorem 1.4. Suppose that G is a n-vertex k-chromatic connected line graph. If $\omega(G) = k$, then the inequality follows from Theorem 2.1. Since line graphs are claw-free, the only graphs in $\mathcal{C}_k^*(n)$ (which are the only graphs that can achieve equality) are those with pendant paths attached to the vertices of the k-clique. This gives the equality statement in Theorem 1.4.

Now suppose that $\omega(G) < k$. We can delete vertices of G until we reach a k-critical line graph G' which is a subgraph of G. Then the result follows from Lemma 2.8 and Proposition 2.2. $\square$

3 Independent sets

In this section, we prove the results relating to independent sets.

3.1 Fixed chromatic number

Proof of Theorem 1.6. We proceed by induction on n for all k with $1 \le k \le n$. The result is clear when $n = k$ or $k = 1$. So suppose that $k > 1$ and $n > k$.

Suppose that there exists a vertex v such that $G - v$ has chromatic number k. The number of independent sets of G with size t which do not contain v is equal to the number of independent sets of $G - v$ with size t. Then by induction on number of vertices, the graph $G - v$ has at most $\binom{n-1-k}{t} + k\binom{n-1-k}{t-1}$ independent sets with size t. The number of independent sets of G of size t that include v is at most the number of independent sets of size $t-1$ in $G - v$. Again by induction $i_{t-1}(G-v) \le \binom{n-1-k}{t-1} + k\binom{n-1-k}{t-2}$. (Note that this bound still holds when $t = 2$.) Therefore,

$$\begin{aligned} i_t(G) &\le i_t(G-v) + i_{t-1}(G-v) \\ &\le \binom{n-1-k}{t} + k\binom{n-1-k}{t-1} + \binom{n-1-k}{t-1} + k\binom{n-1-k}{t-2} \\ &= \binom{n-k}{t} + k\binom{n-k}{t-1} \end{aligned}$$

where the last equality follows from Pascal's identities $\binom{n-k}{t} = \binom{n-k-1}{t} + \binom{n-k-1}{t-1}$ and $\binom{n-k}{t-1} = \binom{n-1-k}{t-1} + \binom{n-1-k}{t-2}$.

Suppose then that v is a vertex such that $\chi(G-v) = k-1$. Then we know that $d(v) \ge k-1$. As before, an upper bound on the number of independent sets of size t that do not include v, by induction, is $\binom{n-1-(k-1)}{t} + (k-1)\binom{n-1-(k-1)}{t-1}$. The number of independent sets of size t that include v is at most $\binom{n-k}{t-1}$ (the number of $t-1$ sets in the at most $n-k$ remaining vertices). In all cases for t, summing the two bounds gives the desired upper bound.

□

As stated in the Introduction, the translation to the total count of all independent sets is trivial.

3.2 Connected with fixed chromatic number

In this section, we focus on the connected graphs with fixed chromatic number.

Proof of Theorem 1.9. Notice that for $t = 0$ and $t = 1$ the inequality in Theorem 1.9 is true as the value of $i_t(G)$ is constant over all n-vertex graphs. For $t = 2$ the inequality in Theorem 1.9 is true by Lemma 1.8.

We proceed by induction on the number of vertices. The result is clear if $n = k$ or $n = k+1$, so we may assume that $n \ge k+2$. Furthermore, by the remarks in the previous paragraph, we assume that $t \ge 3$. Let v be a vertex of G such that $G - v$ is connected. Observe that $i_t(G) = i_t(G-v) + i_{t-1}(G - v - N_G(v)) \le i_t(G-v) + i_{t-1}(G-v)$ and for $3 \le t \le n-k+1$ the equality can be achieved only if $N_G(v)$ has no vertex which belongs to an independent set of size $t-1$ of $G - v$. We consider two cases.

First suppose that $G-v$ is k-chromatic. By induction, $i_t(G-v) \le \binom{n-1-k}{t} + (k-1)\binom{n-1-k}{t-1} + \binom{0}{t-1}$ and $i_{t-1}(G-v) \le \binom{n-1-k}{t-1} + (k-1)\binom{n-1-k}{t-2} + \binom{0}{t-2}$, and both inequalties can be equalities at the same time only if $G - v = K_1 \vee (K_{k-1} \cup E_{n-1-k})$. Adding the right sides of these inequalities and using Pascal's identitity gives the desired inequality. In the extremal case, let v' be the dominating vertex of $G - v$. So v' is the only vertex of $G - v$ which cannot belong to any independent set of size $t \ge 2$. Therefore, v must be adjacent to v' only and $G = K_1 \vee (K_{k-1} \cup E_{n-k})$ in the extremal case.

Now suppose that $G-v$ is $k-1$ chromatic. By induction, we have that $i_t(G-v) \le \binom{n-1-(k-1)}{t} + (k-2)\binom{n-1-(k-1)}{t-1} + \binom{0}{t-1}$. Note that v has at least $k-1$ neighbors as $G-v$ is $k-1$ chromatic.

Choosing any $t-1$ vertices from the $n-k$ remaining vertices gives $i_{t-1}(G-v-N_G(v)) \le \binom{n-k}{t-1}$. Summing these bounds gives the inequality. In the extremal case, let v' be the dominating vertex of $G-v$, and note that v must have exactly $k-1$ neighbors. The fact that $G-v$ is $k-1$ chromatic while G is k-chromatic implies that $G = K_1 \vee (K_{k-1} \cup E_{n-k})$. $\square$

We now prove Corollary 1.10.

Proof of Corollary 1.10. If $n = k$, then $G = K_n$ and the result holds. If $n = k+1$, then $\alpha(G) \le 2$ and so the characterization of equality follows from Lemma 1.8 (note that when $k = 3$, the only unicyclic graph with an odd cycle is $K_1 \vee (K_2 \cup E_1)$, and for other k the only graph in $\mathcal{C}_k^*$ is $K_1 \vee (K_{k-1} \cup E_1)$). For $n \ge k+2$, again notice that $K_1 \vee (K_{k-1} \cup E_{n-k}) \in \mathcal{C}_k^*(n)$, and so the result follows from Lemma 1.8 and Theorem 1.9. $\square$

The next results interpolate between the results for fixed chromatic number and those for connected graphs with fixed chromatic number in that they also fix a number of components.

Proof of Theorem 1.11. Since removing edges does not decrease the number of independent sets, we may assume that $c-1$ components are each trees. For a forest on a fixed number of vertices and edges, the disjoint union of a star and isolated vertices maximizes the number of independent sets of any fixed size [5, Theorem 2.2]. Notice that if G is the disjoint union of G_1 and G_2, then $i_t(G) = \sum_k i_k(G_1) i_{t-k}(G_2)$. This implies that we may assume that our graph G has a component which is k-chromatic, a component that is a (possibly trivial, i.e. 1-vertex) star, and $c-2$ isolated vertices.

Let the the k-chromatic component and the star have $x := n-c+2$ total vertices. We now show that to maximizes the number of independent sets of size t, the star is an isolated vertex and the k-chromatic connected graph is $K_1 \vee (K_{k-1} \cup E_{x-k-1})$.

Suppose the star has a vertices and so the k-chromatic component has $x-a$ vertices. Then

$$i_t(K_{1,a-1}) = \binom{a-1}{t} + \binom{0}{t-1}$$

and

$$i_t(K_1 \vee (K_{k-1} \cup E_{x-a-k})) = \binom{x-a-k}{t} + (k-1)\binom{x-a-k}{t-1} + \binom{0}{t-1}.$$

Now, for any fixed t, we have

$$\sum_{i=0}^{t} \left(\binom{x-a-k}{i} + (k-1)\binom{x-a-k}{i-1} + \binom{0}{i-1} \right) \left(\binom{a-1}{t-i} + \binom{0}{t-i-1} \right)$$

which simplifies to

$$\binom{x-k-1}{t} + \binom{x-a-k}{t-1} + (k-1)\left(\binom{x-k-1}{t-1} + \binom{x-a-k}{t-2} \right) + \binom{a-1}{t-1} + \binom{0}{t-2}.$$

Comparing this to $K_1 \vee (K_{k-1} \cup E_{n-c+1-k}) \cup E_1$, which has

$$\binom{x-k-1}{t} + \binom{x-k-1}{t-1} + (k-1)\left(\binom{x-k-1}{t-1} + \binom{x-k-1}{t-2} \right) + \binom{0}{t-1} + \binom{0}{t-2}$$

independent sets of size t and using $\binom{y}{k} + \binom{z}{k} \le \binom{y+z}{k}$ for all values of k, we see that the maximizing graph in this family consists of $c-1$ isolated vertices and a maximizing connected k-chromatic graph on $n-c+1$ vertices.

Now we consider the cases of equality. Fpr $c < n-k$, the k-chromatic component has size at least $k+2$, and so the characterization of equality follows from the equality characterization in Theorem 1.9. For $c = n-k$, the k-chromatic component has size $k+1$, and so the characterization of equality comes from Lemma 1.8. But here $n = k+1$, and so for any k the k-chromatic component is $K_1 \vee (K_{k-1} \cup E_1)$. Finally, for $c = n-k+1$, the graph must be $K_k \cup E_{n-k}$. □

We next give the proof of Corollary 1.12.

Proof of Corollary 1.12. The inequality is clear, as the extremal graphs for $3 \le t \le n-k+1$ also have equality in the upper bound for $t = 0, 1, 2$. When $n > k+1$ (and so $n-k+1 \ge 3$) equality follows from the characterization of equality in Theorem 1.11. When $n = k$ we have $c = 1$ and $G = K_k$. When $n = k+1$ and $c = 1$ the result follows from Corollary 1.10. When $n = k+1$ and $c = 2$ then $G = K_k \cup E_1$. □

4 Concluding Remarks

We end this paper with a few questions and conjectures. While Conjecture 1.1 is still open, it would also be interesting to consider the class of n-vertex k-chromatic claw-free connected graphs for $k > 4$, and to show that the analogous statement to Theorem 1.2 holds for these k. It would also be interesting to investigate the maximizing graphs for n-vertex k-chromatic ℓ-connected graphs for other values of ℓ.

There are also plenty of questions related to independent sets. We have given results for maximizing $i_t(G)$ for graphs G which are k-chromatic and ℓ-connected for $\ell = 0, 1$. The following is a natural question.

Question 4.1. *Let $\ell < k$ and let G be an ℓ-connected k-chromatic graph with n vertices. Fix $t \ge 3$. Is it true that*

$$i_t(G) \le i_t(K_\ell \vee (K_{k-\ell} \cup E_{n-k}))?$$

In our paper, we answer Question 4.1 in the affirmative when $\ell = 0$ (Theorem 1.6) and $\ell = 1$ (Theorem 1.9). What about other values of k and ℓ? Consider $k = 2$ and $\ell \ge 1$. Notice that ℓ-connected graphs have minimum degree at least ℓ. Maximizing in this larger family of fixed minimum degree graphs is already known.

Theorem 4.2 ([1]). *Let n, δ, and $t \ge 3$ be positive integers with $n \ge 2\delta$. If G is a bipartite graph on n vertices with minimum degree at least δ, then*

$$i_t(G) \le i_t(K_{\delta,n-\delta})$$

with equality if and only if $G = K_{\delta,n-\delta}$.

In other words, this shows that for $n \ge 2\ell$ we have $K_{\ell,n-\ell}$ is the unique 2-chromatic ℓ-connected graph that maximizes $i_t(G)$ for $t \ge 3$. This leads to a natural question.

Question 4.3. *Let $\ell \ge k$ and let G be a k-chromatic ℓ-connected graph with $n \ge 2\ell$ vertices. Fix $t \ge 3$. Is it true that*

$$i_t(G) \le i_t((K_{k-1} \cup E_{\ell-k+1}) \vee E_{n-\ell})?$$

We remark that the question of minimizing the number of independent sets of size t over graphs with n vertices and fixed connectivity are studied in [18].

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