A Direct Proof of the Integral Formulae for the Inverse Hyperbolic Functions

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Abstract

In the same vein of Arnold Insel's capsule [4], we present a direct geometric derivation of the integral formulae for the inverse hyperbolic functions. We then use these formulae to obtain the derivatives of the various hyperbolic trig functions.

1 Introduction

The mere mention of hyperbolic (trig) functions in the classroom setting typically invokes a collective student yawn, so consideration of the *inverse* hyperbolic functions is a recipe for mass napping! Moreover, it has been our experience that even professional mathematicians tend to have a purely analytical understanding of the hyperbolic functions. That is, most are well-aware that these functions are built upon the identities

$$\cosh t = \frac{1}{2}\left(e^t + e^{-t}\right) \quad \text{and} \quad \sinh t = \frac{1}{2}\left(e^t - e^{-t}\right), \tag{1}$$

and then facts about the hyperbolic functions are obtained by manipulation of these identities, using known facts about the exponential. Along these lines, the typical calculus textbook development introduces the hyperbolic functions via equation (1), and then the inverses afterward via a purely algebraic manipulation of (1). Finally, to wrap up the discussion, differentiation of the inverses yields the usual integral equations for the inverse hyperbolic functions (see, e.g., [1, pp. 365–371]).

However, there is a well-known, beautiful way to think about these functions that, unfortunately, rarely makes its way into calculus textbooks. In contrast, this approach has the benefit of being described very geometrically instead of analytically, mirroring the development of the ordinary trig functions on the unit circle $x^2 + y^2 = 1$. What's more, this way of thinking will reveal a direct geometric proof, independent of (1), that the inverse hyperbolic functions can be written as certain definite integrals, which in turn delivers all the derivatives of the various hyperbolic functions! In the capsule [4], Arnold Insel gave a parallel argument for the ordinary trig functions.

2 Parametrizing the "unit hyperbola" by area

There are many lucid presentations of the material in this section; see, for example, the wonderful exposition given by Isaac Greenspan in [2].

We define the "unit hyperbola" to be the right branch of the hyperbola $x^2 - y^2 = 1$, so that $x \geq 1$. Given (x, y) on the unit hyperbola in the first quadrant, let's consider the segment joining the origin to this point. This segment, together with the positive x-axis and the unit hyperbola encloses a region of a particular area, which we denote by $t/2$ (see Figure 1); the reason for the factor $1/2$ will be clear soon. Points on the unit hyperbola belonging to the fourth quadrant will be considered later.

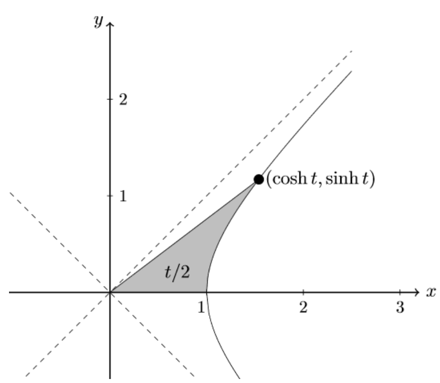

Figure 1: The "unit hyperbola" parametrized by area.

We challenge the interested reader to check that, if we instead started with (x, y) a point on the unit circle $x^2 + y^2 = 1$, and $\theta/2$ defined as the area of the circular sector subtended by the segment joining the origin to (x, y) and the positive x-axis, then we would have $x = \cos\theta$ and $y = \sin\theta$. In other words, for the usual trigonometric parametrization of the unit circle, the argument θ of the trig functions is equal to twice the area of the subtended circular sector. Of course, θ *also* happens to be the central angle (measured in radians) of this sector!

Emboldened by this observation in the case of ordinary trig functions, we *define* the hyperbolic sine and cosine functions by $x = \cosh t$ and $y = \sinh t$, where (to repeat) $t/2$ is the area of the "hyperbolic sector" in Figure 1. To be clear, this parametrization by area goes back to at least an 1895 article of Mellon W. Haskell [3]. This immediately begs the question: Is this alternative definition for the hyperbolic functions equivalent to (1)? Of course, the answer is "yes"; let us see why. By our definition of t and integrating with respect to the vertical direction, we have

$$\begin{aligned} t &= 2\int_0^{\sinh t} \sqrt{1+v^2}\,dv - \cosh t \sinh t \qquad (v = \tan u) \\ &= 2\int_0^{\arctan(\sinh t)} \sec^3 u\,du - \cosh t\sinh t \\ &= \Big(\ln|\sec u + \tan u| + \sec u \tan u\Big)\Big|_0^{\arctan(\sinh t)} - \cosh t \sinh t \\ &= \ln(\cosh t + \sinh t)\,; \end{aligned} \tag{2}$$

here, we have used the fact that $\cosh^2 t - \sinh^2 t = 1$ (since we are on the unit hyperbola), as well as $\sec(\arctan\alpha) = \sqrt{1+\alpha^2}$. We leave it to the interested reader to use (2) in tandem with $\cosh^2 t - \sinh^2 t = 1$ to show that (1) holds.

3 Proof of the integral formulae

We are now ready to deliver on our promise to explain, geometrically, the various inverse hyperbolic integral formulae. First, by our very definition, since $x = \cosh t$ and $y = \sinh t$ we have that $\operatorname{arccosh} x = \operatorname{arcsinh} y = t$ is twice the area of the hyperbolic sector in Figure 1. On the other hand, the unit hyperbola written in polar coordinates is given by

$$r^2\cos^2\theta - r^2\sin^2\theta = 1 \implies r^2 = \frac{1}{\cos^2\theta - \sin^2\theta} = \frac{\sec^2\theta}{1-\tan^2\theta}.$$

Thus, we can express the area t in polar coordinates as below:

$$t = \int_0^{\arctan(y/x)} \frac{\sec^2\theta}{1-\tan^2\theta}\,d\theta \qquad \big(u = \tan\theta;\ du = \sec^2\theta\,d\theta\big)$$

$$= \int_0^{y/x} \frac{du}{1-u^2} \qquad \left(1-u^2 = \frac{1}{1+v^2};\ \frac{du}{1-u^2} = \frac{(v/u)\,dv}{1+v^2}\right) \tag{3}$$

$$= \int_0^{y} \frac{dv}{\sqrt{1+v^2}} \qquad \left(1+v^2 = w^2;\ \frac{dv}{w} = \frac{dw}{v}\right) \tag{4}$$

$$= \int_1^{x} \frac{dw}{\sqrt{w^2-1}}. \tag{5}$$

Since $z := \tanh t = y/x$, from (3)–(5) we obtain

$$\operatorname{arccosh} x = \int_1^x \frac{dw}{\sqrt{w^2-1}}, \quad \operatorname{arcsinh} y = \int_0^y \frac{dv}{\sqrt{1+v^2}},$$
$$\operatorname{arctanh} z = \int_0^z \frac{du}{1-u^2}. \tag{6}$$

Throughout, we have assumed that we are in the first quadrant; this means that our formulas hold for $y > 0$. For $y = 0$, we have $x = 1$ and so our formulas clearly hold. For points on the unit hyperbola in the fourth quadrant, we have $y < 0$. By letting the area of the sector be negative (to agree with integration), the area of the sector is $t/2$ with $t < 0$, and by symmetry we have $x = \cosh(t) = \cosh(-t)$ and $y = \sinh(t) = -\sinh(-t)$ (where $-t > 0$). Since with this interpretation for $t < 0$ we still have our two integral formulas $t = 2\int_0^{\sinh(t)} \sqrt{1+v^2}\,dv - \cosh t \sinh t$ and $t = \int_0^{\arctan(y/x)} \frac{\sec^2\theta}{1-\tan^2\theta}\,d\theta$ (as our angle $\arctan(y/x)$ is negative here), our formulae (2)–(6) hold for fourth quadrant points as well.

Note that the fundamental theorem of calculus applied to (6) gives the derivatives of the inverse hyperbolic functions, and then the inverse function theorem can be applied to obtain

$$\frac{d}{dt}(\sinh t) = \sqrt{1+\sinh^2 t} = \cosh t, \quad \frac{d}{dt}(\cosh t) = \sqrt{\cosh^2 t - 1} = \sinh t,$$
$$\frac{d}{dt}(\tanh t) = 1 - \tanh^2 t = \operatorname{sech}^2 t. \tag{7}$$

Any other derivative of one's favorite hyperbolic function is now easily obtained from these!

Remark. The careful reader will note that formula (2), $t = \ln(\cosh t + \sinh t)$, is equivalent to the identities

$$\operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right) \quad \text{and} \quad \operatorname{arcsinh} y = \ln\left(\sqrt{1+y^2} + y\right), \tag{8}$$

from which the first two formulas in (6) may alternatively be obtained upon differentiating.

References

[1] Bruce Edwards and Ron Larson, *Calculus: Early Transcendental Functions*, 6th ed., Brooks Cole, Boston, 2014.

[2] Isaac Greenspan, *Deriving the Hyperbolic Trig Functions*, presentation at the MMC Conference of Workshops, The University of Chicago Laboratory of Schools, Chicago (January, 2009; available at `http://talks.isaacgreenspan.com/DerivingTheHyperbolicTrigFunctions.pdf`).

[3] Mellon W. Haskell, *On the introduction of the notion of hyperbolic functions*, Bulletin of the American Mathematical Society, Vol. 1, No. 6 (March, 1895), pp. 155–159.

[4] Arnold J. Insel, *A Direct Proof of the Integral Formula for Arctangent*, The College Mathematics Journal, Vol. 20, No. 3 (May, 1989), pp. 235–237.