CONSTRUCTING PIECEWISE LINEAR 2-KNOT COMPLEMENTS

JONATHAN DENT, JOHN ENGBERS, AND GERARD VENEMA

Introduction

The groups of high dimensional knots have been characterized by Kervaire [7], but there is still no general description of all 2-knot groups. Kervaire identified a large class of groups that are natural candidates to serve as 2-knot groups and proved that each of these groups is the group of the complement of a smooth 2-sphere in a homotopy 4-sphere. Freedman's solution to the 4-dimensional Poincaré conjecture implies that the groups Kervaire identified are the groups of locally flat topological 2-knots, but it is not known whether all of them are groups of piecewise linear (PL) 2-knots in S^4. The best result is due to Levine [9], who observed that the Andrews-Curtis conjecture can be used to show that all the groups identified by Kervaire are groups of PL 2-knots in S^4.

In this note we outline a new proof of Levine's theorem. The proof given here is entirely 4-dimensional; the Kirby calculus of links is used to give an explicit picture of the 2-knot and its complement. By contrast, the usual proof of Levine's theorem involves constructing a 5-dimensional ball pair whose boundary is the knot. Our proof leads to a piecewise linear knot with one nonlocally flat point. It is clear from the construction that the link of the exceptional vertex is a ribbon link. The proof in this paper is based on a recent construction of Lickorish [10].

1. Properties of knot groups

Before we can state the Kervaire theorem we need several definitions.

Definition. An *n-knot* is a topological embedding $h : S^n \to S^{n+2}$.

It is usually assumed that the embedding h is either smooth or PL, although knots in other categories can be profitably studied as well.

Definition. The *group* of the knot $h : S^n \to S^{n+2}$ is $\pi_1(S^{n+2} \setminus h(S^n))$.

This research was partially supported by an REU supplement to NSF grant number DMS-0206647.

Definition. A group π has *weight* 1 if there exists one element $z \in \pi$ such that π is generated by conjugates of z and z^{-1}. Such an element z is called a *meridian* of the group.

It is relatively easy to see that the group π of any smooth or PL knot has the following properties:

(1) π is finitely presented.
(2) The abelianized group $\pi/[\pi, \pi]$ is infinite cyclic.
(3) $H_2(\pi) = 0$.
(4) π has weight 1.

Kervaire [7] proved that these four properties completely characterize the groups of high dimensional knots.

Theorem 1.1 (Kervaire)**.** *A group π is the group of a smooth n-knot, $n \geq 3$, if and only if π satisfies* (1) – (4)*.*

Since a 2-knot can be suspended to a 3-knot, every 2-knot group is a 3-knot group and therefore satisfies (1) – (4). But not every 3-knot group is a 2-knot group—see [3], for example. Hence stronger conditions are needed to characterize 2-knot groups. The following condition is a natural one to try in place of condition (3).

($3'$) π has deficiency 1.

Definition. The *deficiency* of a finite presentation for a group is

$$(\#\text{ generators}) - (\#\text{ relations}).$$

The *deficiency* of a finitely presented group is the maximal deficiency of its presentations.

Kervaire [7] proved that any group satisfying (1), ($3'$), and (4) is the fundamental group of the complement of a smooth 2-sphere in a homotopy 4-sphere. Combining that result with Freedman's solution [5] to the 4-dimensional topological Poincaré conjecture yields the following theorem.

Theorem 1.2 (Kervaire-Freedman)**.** *If π satisfies* (1), ($3'$)*, and* (4)*, then π is the group of a locally flat topological 2-knot in S^4.*

Remark. It is relatively easy to see [6, page 17] that

$$(1) + (3') + (4) \Rightarrow (2),$$

so it is not necessary to assume (2) in the Kervaire-Freedman theorem.

The Kervaire-Freedman theorem exhibits a large class of groups that are 2-knot groups, but it does not characterize 2-knot groups. While the conditions listed in the theorem are sufficient to guarantee that a group is a 2-knot group, they are not necessary. In particular, condition ($3'$) is not a necessary condition since the group of the 2-knot described by Fox in [4, Example 12] does not have deficiency 1. In addition, Freedman's theorem gives only a topological homeomorphism between the homotopy 4-sphere and S^4, so the theorem does not answer the question of whether or not groups satisfying (1), ($3'$), and (4) are the groups of smooth or PL 2-knots. It is the latter aspect of the theorem that will be investigated in this paper; we will add an additional condition which allows us to construct a PL 2-knot realizing the group.

2. Andrews-Curtis moves and the theorem of Levine

Levine [9] observed that a conjecture of Andrews and Curtis [1] can be used to prove that groups satisfying (1), ($3'$), and (4) are the groups of smooth or PL 2-knots.

Definition. Let $\mathcal{P} = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ be a presentation of a group. The following are called *Andrews-Curtis moves* on $\mathcal{P}$:

- Replace r_i by $r_i a_j a_j^{-1}$ or $r_i a_j^{-1} a_j$.
- Replace r_i by a cyclic permutation of r_i.
- Replace r_i by r_i^{-1}.
- Add a new generator a_{n+1} and a new relation $a_{n+1} w^{-1}$, where w is an arbitrary word in $a_1, \ldots, a_n$.

Definition. Two finite presentations are said to be *AC equivalent* if it is possible to get from one to the other by a finite sequence of Andrews-Curtis moves or their inverses.

Suppose
$$\mathcal{P} = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_{n-1} \rangle$$
is a deficiency 1 presentation of a group π. If π has weight 1 and z is a meridional element for π, then
$$\mathcal{P}' = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_{n-1}, z \rangle$$
is presentation of the trivial group. Note that $\mathcal{P}'$ has deficiency 0.

Definition. The presentation $\mathcal{P}'$ is called the *induced* presentation of the trivial group. The presentation $\langle a \mid a \rangle$ is called the *trivial presentation* of the trivial group.

The Andrews-Curtis Conjecture. The Andrews-Curtis conjecture states that if a finitely presented group has weight 1 and deficiency 1, then the induced presentation of the trivial group is AC equivalent to the trivial presentation.[1] This conjecture is deep and has thus far resisted all attempts at proof. Levine's theorem [9] asserts that if the Andrews-Curtis conjecture holds for a particular presentation, then the group is the group of a PL 2-knot.

Theorem 2.1 (Levine)**.** *If π satisfies* (1), (3′)*, and* (4) *and if π has a presentation such that the induced presentation of the trivial group is AC equivalent to the trivial presentation, then π is the group of a locally flat PL 2-knot.*

3. Construction of PL 2-knots

In this section we will outline a proof of Levine's Theorem. The first step is to get a presentation of the group that reflects the additional structure given by the Andrews-Curtis moves.

Definition. Let π be a group of weight 1 with meridional element z. A *meridional presentation* of π is a presentation of the form

$$\mathcal{P} = \langle a_1, \ldots, a_n, z \mid r_1, \ldots, r_n \rangle$$

in which each r_i is a product of conjugates of z and z^{-1}.

More specifically, each r_i has the form

$$r_i = w_{i1} z^{\epsilon_1} w_{i1}^{-1} w_{i2} z^{\epsilon_2} w_{i2}^{-1} \ldots w_{im_i} z^{\epsilon_{m_i}} w_{im_i}^{-1}$$

in which each $\epsilon_k = \pm 1$ and w_{ij} is a word in $a_1, \ldots, a_n, z$.

Lemma 3.1. *If π satisfies* (1), (3′)*, and* (4) *and if π has a presentation such that the induced presentation of the trivial group is AC equivalent to the trivial presentation, then π has a meridional presentation.*

Sketch of proof. Assume π has a presentation $\mathcal{P}$ such that the induced presentation $\mathcal{P}'$ of the trivial group is AC equivalent to the trivial presentation. Do the same Andrews-Curtis moves to $\mathcal{P}$ as to $\mathcal{P}'$, but without canceling the z's. The same moves that transform $\mathcal{P}'$ to the trivial presentation will transform $\mathcal{P}$ to a meridional presentation. □

Here is a statement of the theorem we will prove.

Theorem 3.2. *If π satisfies* (1), (3′)*, and* (4) *and if π has a presentation such that the induced presentation of the trivial group is AC equivalent to the trivial presentation, then π is the group of a PL 2-knot. Furthermore, the PL 2-knot is locally flat except at one point and the complementary disk is a ribbon disk.*

[1]More generally, the Andrews-Curtis conjecture asserts that any two presentations of a group that have the same deficiency are AC equivalent.

We will sketch the proof in a simple case. It will be evident how to modify the proof to cover the general case.

Proof. By the lemma, we may assume that π has a meridional presentation. Suppose, for example, that π has presentation

$$\mathcal{P} = \langle x, y, z \mid x = y^{-1}xzx^{-1}y, y = xy^{-1}zyx^{-1}\rangle.^2$$

In order to simplify the notation we use α to denote the relation $x^{-1}y^{-1}xzx^{-1}y$ and β to denote the relation $y^{-1}xy^{-1}zyx^{-1}$.

We begin by constructing the knot complement. Figure 1 shows a Kirby diagram of a compact 4-manifold X. We are using the standard Kirby calculus notation: a 1-handle attached to B^4 is the same as an unknotted 2-handle subtracted from B^4, so a 1-handle is indicated by an unknotted circle with a large dot on it. It is clear that $\pi_1(X) \cong \pi$.

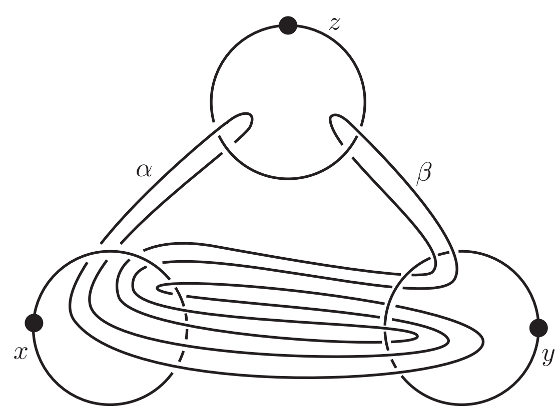

FIGURE 1. Kirby diagram of the complement X

Observe that X can be constructed as a subset of S^4. To see this, note that both $x \cup y \cup z$ and $\alpha \cup \beta$ are unlinks. We think of these two links as lying on $\partial B^4 \subset S^4$. Since both links are trivial, we can attach disjoint disks to $x \cup y \cup z$ in B^4 and attach disjoint disks to $\alpha \cup \beta$ in $S^4 \smallsetminus \operatorname{Int} B^4$. Then X is realized in S^4 by starting with B^4, subtracting neighborhoods of the inside disks, and adding neighborhoods of the outside disks.

We now consider $M = \overline{S^4 \smallsetminus X}$. The proof of the theorem will be completed by showing that M is a regular neighborhood of a PL 2-sphere. We will accomplish this by first constructing a Kirby diagram for M and then canceling handles to show that M has a handle decomposition consisting of one 0-handle and one 2-handle.

[2]This is a meridional presentation of the figure-eight knot group.

Note that M is obtained from the 4-ball $S^4 \smallsetminus \mathrm{Int}\, B^4$ by adding neighborhoods of the disks attached to $x \cup y \cup z$ and subtracting neighborhoods of the disks attached to $\alpha \cup \beta$. This means that the diagram of M is obtained from the diagram of X by removing the dots from x, y, z and placing new dots on α, β—see Figure 2.

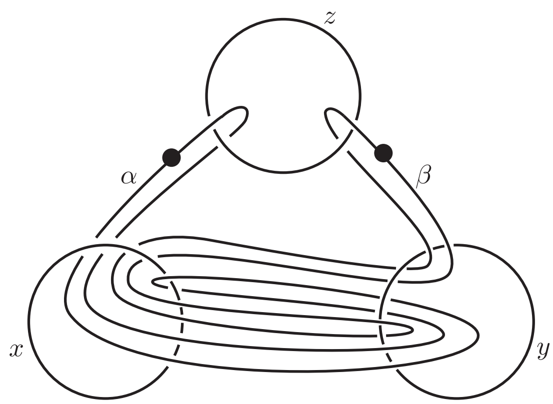

FIGURE 2. Kirby diagram of $M = \overline{S^4 \smallsetminus X}$

We now wish to simplify the handle decomposition of M. The decomposition indicated in Figure 2 consists of one 0-handle, two 1-handles, and three 2-handles. We will cancel two $(1,2)$-handle pairs, leaving just a 0-handle with a single 2-handle attached. The first step is to straighten out α and β—see Figure 3.

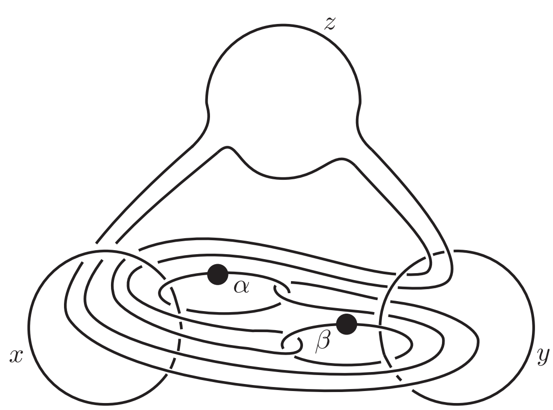

FIGURE 3. Make α and β look like the standard unlink

The next step is to slide the 2-handle z over over the 2-handle x and then to cancel $\alpha \cup x$—see Figure 4.

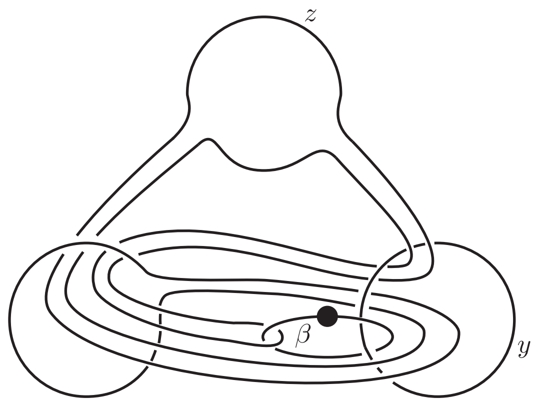

FIGURE 4. Slide z over x; then cancel $\alpha \cup x$

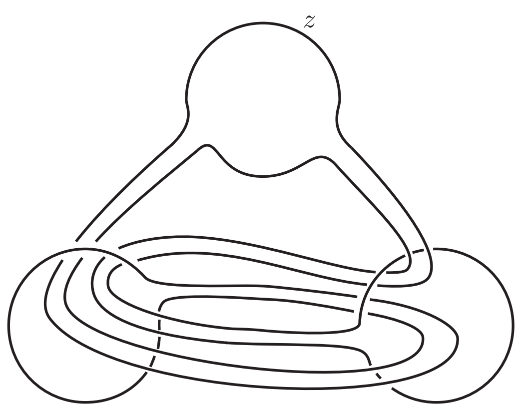

FIGURE 5. Slide z over y; then cancel $\beta \cup y$

The final step is to slide the 2-handle z over over the other 2-handle y and then to cancel $\beta \cup y$—see Figure 5.

We now see that M consists of a 0-handle with one 2-handle attached along z. Thus M is a regular neighborhood of a PL 2-sphere. The 2-sphere consists of the the core of the 2-handle plus the cone on z, so it has one nonlocally flat point. It is clear from the construction that the final 2-handle z is a ribbon knot and that the disk is the the ribbon disk. □

References

[1] J. J. Andrews and M. L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc., **16** (1965), 192–195.

[2] J. Dent and J. Engbers, *4-manifolds, Kirby calculus, and the construction of knotted 2-spheres in 4-space*, to appear.

[3] M. Š. Farber, *Duality in an infinite cyclic covering, and even-dimensional knots*, Izv. Akad. Nauk SSSR Ser. Mat., **41** (1977), 794–828.

[4] R. H. Fox, *A quick trip through knot theory*, in “Topology of 3-manifolds and Related Topics” (M. K. Fort, Jr., editor), Prentice Hall, Engelwood Cliffs, New Jersey (1962), 120–167.

[5] M. H. Freedman, *The topology of four-dimensional manifolds*, Journal Differential Geometry, **17** (1982), 357–453.

[6] J. Hillman, *2-knots and their groups*, Australian Mathematical Society Lecture Series, vol. 5, Cambridge University Press, Cambridge, 1989.

[7] M. Kervaire, *On higher dimensional knots*, in “Differential and Combinatorial Topology” (S. S. Cairns, editor), Princeton University Press, Princeton (1965), 225–271.

[8] R. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978), 35–56.

[9] J. Levine, *Some results on higher dimensional knot groups*, Lecture Notes in Mathematics, Springer-Verlag, **685** (1978), 243–269.

[10] W. B. R. Lickorish, *Knotted contractible 4-manifolds in S^4*, Pacific Journal of Mathematics, **208** (2003), 283–290.

Department of Mathematics and Statistics, Calvin College, Grand Rapids, MI 49546 USA

E-mail address: `venema@calvin.edu`