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TOPOLOGICAL PARTITIONS

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A space X *partitions* a space Y if Y is the union of pairwise disjoint subsets, each of which is homeomorphic to X. We study the topological partition relation, particularly in the context of separable metric spaces, obtaining topological analogues to well-known problems in the theory of geometric partitions.

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0. Definitions and Notations

We will use very few notational devices which are at variance with accepted practice. Our basic definition is the following: a family $\mathcal{F}$ of topological embeddings of X into Y is a *partition of* Y *by* X (in symbols $\mathcal{F}: X \ll Y$) if the family $\{f[X] : f \in \mathcal{F}\}$ of images is a cover of Y by pairwise disjoint sets. We say X *partitions* Y (in symbols $X \ll Y$) if there is a partition $\mathcal{F}: X \ll Y$. X and Y are *partition equivalent* (in symbols $X \equiv Y$) if $X \ll Y$ and $Y \ll X$. The *partition spectrum* $\sigma(X, Y)$ of the pair $\langle X, Y\rangle$ is the set of cardinal numbers κ such that there is an $\mathcal{F}: X \ll Y$ with $|\mathcal{F}| = \kappa$ (vertical bars denote cardinality). X is *partition unique* (resp. *stable*) if whenever $Y \equiv X$, then $Y \simeq X$ (resp. $\sigma(X, X) = \{1\}$). Clearly partition stability implies partition uniqueness but the converse is false [Let $\mathbf{Q}$ denote the rational line. Then $\mathbf{Q} \simeq \mathbf{Q} \times \mathbf{Q}$ so $\omega \in \sigma(\mathbf{Q}, \mathbf{Q})$. However if $Y \ll \mathbf{Q}$ then Y is countable, second countable, and T_3 ($=$ regular T_1). Also if $\mathbf{Q} \ll Y$ then Y is self-dense (i.e. has no isolated points). Thus $Y \simeq \mathbf{Q}$ by a well-known folklore result (see [3])].

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The paper is divided into four sections. The first involves zero-dimensional spaces, for the most part, and is consequently "set-theoretic" in its approach. The next two sections involve higher-dimensional spaces and are more "geometric" in flavor. The fourth section delves more deeply into the notions of partition uniqueness and partition stability.

Our notation for set theory and topology follows custom: an ordinal α is the set of its predecessors (i.e. $2=\{0,1\}$, $\omega=\{0,1,2,\ldots\}$); the product of $\langle X_i : i\in I\rangle$ is denoted by $\Pi_{i\in I}X_i$ (and by X^I if $X_i = X$ for all $i\in I$) and is endowed with the Tichonov topology if each X_i is a space; the product of partitions $\langle \mathscr{F}_i : X_i \ll Y_i\rangle$ is again a partition

$$\Pi_{i\in I}\mathscr{F}_i : \Pi_{i\in I}X_i \ll \Pi_{i\in I}Y_i$$

(where for $f\in\Pi_{i\in I}\mathscr{F}_i$, $x\in\Pi_{i\in I}X_i$, $(f(x))_i = f_i(x_i)$); and the composition $\mathscr{G}\circ\mathscr{F}$ of partitions $\mathscr{F}: X \ll Y$ and $\mathscr{G}: Y \ll Z$ is also a partition (where $\mathscr{G}\circ\mathscr{F} = \{g\circ f : g\in\mathscr{G}, f\in\mathscr{F}\}$). The familiar spaces which we consider are $\mathbf{N}$ (= the nonnegative integers), $\mathbf{Q}$ (= the rational numbers), $\mathbf{P}$ (= the irrational numbers), $\mathbf{R}$ (= the real numbers), $\mathbf{C}$ (= the Cantor middle-thirds set), $\mathbf{I}$ (= the closed unit interval, $[0,1]$), l_2 (= the Hilbert space of square-summable real sequences), $\mathbf{H}$ (= the Hilbert cube in l_2), and $S(n)$ (= the unit n-sphere in $\mathbf{R}^{n+1}$, $n<\omega$). By well-known theorems (see [8], say) there are homeomorphisms: $\mathbf{P}\simeq\mathbf{N}^\omega$, $\mathbf{C}\simeq 2^\omega$, $l_2\simeq\mathbf{R}^\omega$, and $\mathbf{H}\simeq\mathbf{I}^\omega$.

Before proceeding, we would like to acknowledge our indebtedness to a number of individuals for their interest and helpful counsel during the preparation of this paper. Among these are F. Galvin, J. Roitman, and S. Stahl. Wherever possible we will indicate their specific contributions.

1. Partitions involving zero-dimensional spaces

The following theorem, due to F. Galvin, strengthens a result of ours, and is used by him to characterize those Hausdorff spaces which can be partitioned by $\mathbf{N}$.

1.1. Theorem. *Let X be a Hausdorff space with a family Φ of infinite closed subsets such that $|\Phi|\leq|F|$ for each $F\in\Phi$, and $\bigcap\Phi=\emptyset$. Then $\mathbf{N}\ll X$.*

Proof. We use the well-known combinatorial fact that if $\langle A_\alpha : \alpha<\kappa\rangle$ is an infinitive family of κ sets each of power at least κ then there is a family $\langle B_\alpha : \alpha<\kappa\rangle$ of pairwise disjoint sets with $B_\alpha\subset A_\alpha$ and $|B_\alpha|=\kappa$ for each $\alpha<\kappa$.

To prove the theorem, we first write Φ as a well-ordered family $\langle F_\alpha : \alpha<\kappa\rangle$. For $\alpha<\kappa$ let $B_\alpha\subset F_\alpha$ be such that $|B_\alpha|=\kappa$ and $B_\alpha\cap B_\beta=\emptyset$ for $\alpha<\beta<\kappa$. Since each B_α is infinite Hausdorff we can pick a copy $N_\alpha\subset B_\alpha$ of $\mathbf{N}$ for each $\alpha<\kappa$. Extend the family $\langle N_\alpha : \alpha<\kappa\rangle$ to a maximal family $\mathfrak{N}$ of pairwise disjoint copies of $\mathbf{N}$. Then $X\setminus\bigcup\mathfrak{N}$ is finite, say equal to $\{x_1,\ldots,x_n\}$, and $\bigcap\{\overline{N_\alpha} : \alpha<\kappa\}=\emptyset$. For $1\leq i\leq n$ let

$f(i) =$ least $\alpha < \kappa$ such that $x_i \notin \overline{N_\alpha}$. Then

$$\mathfrak{N}' = (\mathfrak{N} \setminus \{N_\alpha : \alpha < \kappa\}) \cup \{N_\alpha \cup f^{-1}(\alpha) : \alpha < \kappa\}$$

is a partition of X by $\mathbf{N}$. □

1.2. Theorem. *Let X be nonempty Hausdorff. Then $\mathbf{N} \ll X$ iff X is infinite but not the one-point compactification of an infinite discrete space.*

Proof. If $\mathbf{N} \ll X$ then X is infinite of course. Suppose X were $D \cup \{\infty\}$ for some infinite discrete D. Then, since every infinite subset of D has ∞ for a limit point, ∞ could not lie in a copy of $\mathbf{N}$.

For the converse we consider two cases.

Case (i), X is compact: If X is infinite and not $D \cup \{\infty\}$ then X has at least two limit points, say x and y. Since X is T_3 there are open sets U of x, V of y with $\bar{U} \cap \bar{V} = \emptyset$. Since U, V are infinite, we invoke (1.1) to conclude that $\mathbf{N} \ll X$.

Case (ii), X is noncompact: Let $\mathfrak{U} = \langle U_\alpha : \alpha < \kappa \rangle$ be an open cover with no finite subcover, and assume κ is the minimal cardinal of such a cover. Let $F_\alpha = X \setminus U_\alpha$ and let $\Phi = \langle F_\alpha : \alpha < \kappa \rangle$. Then $\bigcap \Phi = \emptyset$. Moreover if $|F_\alpha| < \kappa$ for some $\alpha < \kappa$, then $\mathfrak{U}$ would have a subcover of power $< \kappa$ which in turn would have no finite subcover, a contradiction. Thus $|\Phi| \leq \min\{|F_\alpha| : \alpha < \kappa\}$. By (1.1), $\mathbf{N} \ll X$. □

1.3. Corollary. $\mathbf{C} \equiv \mathbf{P}$.

Proof. Since $2 \ll \mathbf{N}$, we have $\mathbf{C} \simeq 2^\omega \ll \mathbf{N}^\omega \simeq \mathbf{P}$. By (1.2), $\mathbf{N} \ll \mathbf{C}$, whence $\mathbf{P} \ll \mathbf{C}^\omega \simeq \mathbf{C}$. □

Remark. $\mathbf{C}$ and $\mathbf{P}$ give one of many instances of partition equivalent non-homeomorphic spaces; $[0, 1)$ and $[0, 1) \dot{\cup} [0, 1)$ (= the disjoint union) give another. In particular we see that compactness and connectedness are not preserved by $\equiv$.

We next consider spaces which are partitionable by the rational line.

1.4. Theorem. *Let X be T_3, first countable, and self-dense. If either*

(a) *X is hereditarily Lindelöf*; *or*

(b) *there are only countably many points of X with neighborhoods of power $< |X|$*,

then $\mathbf{Q} \ll X$.

Proof. For any T_3, first countable, self-dense space X and $x_0 \in X$, we construct a set $Q_0 \simeq \mathbf{Q}$ with $x_0 \in Q_0 \subset X$ as follows. Using first countability assign to each pair $\langle x, U \rangle$, $x \in U \subset X$, U open, a copy $s(x, U)$ of the ordinal space $\omega + 1$ with x as its limit point. Set $A_0 = s(x_0, X)$. Assuming A_n is a countable subset of X which has been defined, define $A_{n+1} = A_n \cup \bigcup \{s(a, U_a)$: a is isolated in A_n and U_a is a neighborhood of a which misses $A_n \setminus \{a\}\}$. Then, letting $Q_0 = \bigcup_{n<\omega} A_n$, we have a countable, first countable T_3 space with no isolated points; hence a copy of $\mathbf{Q}$.

Now assume (a) and let $\mathfrak{Q}$ be a maximal family of pairwise disjoint copies of $\mathbf{Q}$ in X. Using the easily deduced fact that hereditary Lindelöfness is equivalent to the property that no uncountable subset is scattered, we have that $X\backslash\bigcup\mathfrak{Q}$ is countable [otherwise there would be a self-dense $A\subset X\backslash\bigcup\mathfrak{Q}$ in which we could contradict the maximality of $\mathfrak{Q}$ by building a new copy of $\mathbf{Q}$]. Let $X\backslash\bigcup\mathfrak{Q}=\{x_n : n<\omega\}$. For each $n<\omega$ let $\mathfrak{Q}_n\subset\mathfrak{Q}$ be countable such that $x_n\in\overline{\bigcup\mathfrak{Q}_n}$. This is possible first since $\bigcup\mathfrak{Q}$ is dense in X [otherwise there is a self-dense open $U\subset X$ with $U\cap\bigcup\mathfrak{Q}=\emptyset$, contradicting the maximality of $\mathfrak{Q}$]; and second since X is first countable. Now let

$$\mathfrak{Q}'=(\mathfrak{Q}\backslash\textstyle\bigcup_{n<\omega}\mathfrak{Q}_n)\cup\{Q'\}$$

where $Q'=\bigcup_{n<\omega}(\bigcup\mathfrak{Q}_n\cup\{x_n\})$. Then $Q'\simeq\mathbf{Q}$, hence $\mathfrak{Q}'$ is a partition of X by $\mathbf{Q}$.

Next assume (b) holds. Let $\kappa=|X|$ and let $\langle x_\alpha : \alpha<\kappa\rangle$ be a well ordering of X. If $\kappa=\omega$, then $X\simeq\mathbf{Q}$, so assume κ is uncountable. For all $\alpha<\kappa$ we define subsets $A_\alpha\subset X$ by induction as follows: Let α be the smallest ordinal with A_α undefined, $x_\alpha\notin\bigcup_{\beta<\alpha}A_\beta$, and such that for all $\beta<\alpha$, $x_\beta\in A_\beta$, distinct A_β's are disjoint, and A_β is either $\{x_\beta\}$ or a copy of $\mathbf{Q}$. To get A_α, let S be the countable set of points $x\in X$ such that x has an open neighborhood of power $<\kappa$. We then let $A_\alpha=\{x_\alpha\}$ if $x_\alpha\in S\cap\overline{\bigcup_{\beta<\alpha}A_\beta}$. If $x_\alpha\notin\overline{\bigcup_{\beta<\alpha}A_\beta}$, let U be an open neighborhood of x_α missing $\bigcup_{\beta<\alpha}A_\beta$. U is self-dense so we let $A_\alpha\subset U$ be a copy of $\mathbf{Q}$ containing x_α. If $x_\alpha\notin S$, we have that $x_\alpha\in X\backslash(S\cup\bigcup_{\beta<\alpha}A_\beta)$, a T_3, first countable self-dense [since every neighborhood of x_α is uncountable] space. We can thus define A_α using the argument in the first paragraph. To complete the proof, let $T=\{x_\alpha : A_\alpha=\{x_\alpha\}\}$ and $\mathfrak{A}=\langle A_\alpha : \alpha<\kappa\rangle$. $T\subset S$ is countable; and since for each $x_\alpha\in T$ we have $x_\alpha\in\overline{\bigcup_{\beta<\alpha}A_\beta}$, we can find countable $\mathfrak{A}_\alpha\subset\mathfrak{A}$ with $x_\alpha\in\overline{\bigcup\mathfrak{A}_\alpha}$. Thus

$$\mathfrak{A}'=(\mathfrak{A}\backslash\textstyle\bigcup\{\mathfrak{A}_\alpha : x_\alpha\in T\})\cup\{Q'\},$$

where $Q'=\bigcup\{\bigcup\mathfrak{A}_\alpha\cup\{x_\alpha\} : x_\alpha\in T\}$, is a partition of X by $\mathbf{Q}$. □

1.5. Corollary. *Let X be separable metric. Then $\mathbf{Q}\ll X$ iff X is self-dense.*

Proof. If X is self-dense separable metric then X is also first countable, T_3, and hereditarily Lindelöf. □

1.6. Question. Does $\mathbf{Q}$ partition every self-dense metric space?

1.7. Corollary. $\mathbf{Q}^\omega\equiv\mathbf{C}$.

Proof. By (1.4) $\mathbf{Q}\ll\mathbf{C}$, so $\mathbf{Q}^\omega\ll\mathbf{C}$. On the other hand $2\ll\mathbf{Q}$ so $\mathbf{C}\ll\mathbf{Q}^\omega$. □

Remark. $\mathbf{Q}^\omega$, $\mathbf{C}$, and $\mathbf{P}$ are three distinct partition equivalent spaces. $\mathbf{Q}^\omega$ os topologically distinct from the others since it contains $\mathbf{Q}$, a noncomplete metric space, as a closed subset. Hence $\mathbf{Q}^\omega$ is not complete-metrizable. We now know that topological completeness is not preserved by partition equivalence.

1.8. Proposition. *If X is nonempty, countable, first countable, and T_3, then $X \ll \mathbf{Q}$.*

Proof. $X \ll X \times \mathbf{Q} \simeq \mathbf{Q}$. □

Lest the reader wonder whether we need extra clauses (such as (a), (b)) in (1.4) we offer the following example due to J. Roitman.

1.9. Example. A first countable, T_3, self-dense (but nonmetrizable) space X which is not $\mathbf{Q}$-partitionable.

Construction. Suppose X has an uncountable discrete subset D with countable complement. Then clearly $\mathbf{Q} \not\ll X$. We construct such a space having the desired properties. A space so constructed can of course never be metric since it would then also be separable metric with an uncountable discrete subset.

Let A be the lexicographically ordered $\mathbf{Q} \cdot \omega$ (i.e. $A = \bigcup_{n<\omega} Q_n$, $Q_n \cong \mathbf{Q}$, $\langle r, m\rangle < \langle s, n\rangle$ iff $m < n$ or $m = n$ and $r < s$), and let $\langle S_\alpha : \alpha < c\rangle$ ($c = \exp(\omega)$) be an uncountable collection of almost disjoint subsets of ω (i.e. pairwise intersections are finite). For each $\alpha < c$ let p_α be a point not in A and different from p_β for $\beta < \alpha$. Let $X = A \cup \{p_\alpha : \alpha < c\}$ where A has the order topology and subbasic neighborhoods of p_α are of the form

$$\{p_\alpha\} \cup \{x : r <_\alpha x \text{ for some } r\}$$

where $u <_\alpha v$ iff $u, v \in \bigcup_{n \in S_\alpha} Q_n$ and $u < v$. □

Remark. With the removal of first countability and/or regularity other examples of spaces can be found which are not $\mathbf{Q}$-partitionable. Indeed the Stone–Čech remainder $\beta\mathbf{N}\backslash\mathbf{N}$ is T_3, self-dense, satisfies clause (b) of (1.4), but contains no copies of $\mathbf{Q}$. On a different tack, the real line with basic sets of the form (open interval)\(countable set) is T_2, connected, and hereditarily Lindelöf but likewise contains no copies of $\mathbf{Q}$.

We now look at partitions involving $\mathbf{C}$ and $\mathbf{P}$. Since $\mathbf{C} \equiv \mathbf{P}$, qualitative results about one will also hold for the other. Quantitative results will vary, however, when it comes to partition spectra (e.g. we will see that $\sigma(\mathbf{C}, \mathbf{P}) \neq \sigma(\mathbf{P}, \mathbf{C})$). Indispensible to the proofs of many of our assertions is the following characterization of $\mathbf{C}$ and $\mathbf{P}$ (see [3], [8]).

1.10. Lemma. (i) *If X is a nonempty, zero-dimensional, self-dense, compact, metric space, then $X \simeq \mathbf{C}$.*

(ii) *If X is a nonempty, zero-dimensional, self-dense, separable, completely metrizable space in which no nonempty open set is compact, then $X \simeq \mathbf{P}$.*

(iii) (*Mazurkiewicz*) *If X is a totally disconnected dense G_δ-subset of* $\mathbf{R}$, then $X \simeq \mathbf{P}$. □

1.11. Theorem. *Let X be a nonempty, zero-dimensional, separable metric space. Then $X \ll \mathbf{P}$. Thus $X \ll \mathbf{P}$ iff $\emptyset \neq X \subset \mathbf{P}$.*

Proof. It will suffice to show that $X \ll \mathbf{P}^3 \subset \mathbf{R}^3$. Let $\langle p_\alpha : \alpha < c \rangle$ well-order $\mathbf{P}^3$. If Π (resp. Λ) is a plane (resp. line) in $\mathbf{R}^3$ we say Π (resp. Λ) is *oblique* if all of its coördinate projections are surjective. Clearly if Π is an oblique plane and $p \in \Pi$, then there are c oblique lines Λ with $p \in \Lambda \subset \Pi$. We now claim that if Λ is any oblique line then $\Lambda \cap \mathbf{P}^3 \simeq \mathbf{P}$. Indeed if Λ is oblique then the restrictions of the three coördinate projections to Λ are one-one, so we can map $\Lambda \backslash P^3$ one-one into a countable set; whence $\Lambda \cap \mathbf{P}^3$ is a totally disconnected dense G_δ-subset of $\Lambda \simeq \mathbf{R}$. By (1.10 (iii)), $\Lambda \cap \mathbf{P}^3 \simeq \mathbf{P}$.

So we partition $\mathbf{P}^3$ by X using induction as in the proof of (1.4). Assume p_α is the first uncovered point, $p_\beta \in X_\beta \simeq X$ for all $\beta < \alpha$, distinct X_β's being disjoint, and embedded in oblique lines in $\mathbf{P}^3$. Since $|\alpha| < c$ we can pick an oblique plane Π_α containing p_α but failing to contain any X_β, $\beta < \alpha$. Thus $|\Pi_\alpha \cap X_\beta| \leq 1$ for $\beta < \alpha$, so $|\Pi_\alpha \cap \bigcup_{\beta<\alpha} X_\beta| < c$. Since there are c oblique lines in Π_α containing p_α, there is one such, say Λ_α, which misses $\bigcup_{\beta<\alpha} X_\beta$. Since $\Lambda_\alpha \cap \mathbf{P}^3 \simeq \mathbf{P}$, a homogeneous space, we can embed a copy X_α of X in $\Lambda_\alpha \cap \mathbf{P}^3$ with $p_\alpha \in X_\alpha$. This completes the induction. □

Remark. By (1.3, 1.11), then, a space is partitionable by every nonempty zero-dimensional separable metric space iff it is partitionable by $\mathbf{C}$. A well-known fact about self-dense complete metric spaces is that every one of their points lies in an embedded Cantor set. This motivates the next theorem, one which uses extra set-theoretic axioms. We assume the reader to be moderately familiar with the Continuum Hypothesis (CH), which says that $c = \omega_1$; and with Martin's Axion (MA), which says (in one of its forms) that in a compact T_2 space in which there are no uncountable families of pairwise disjoint open sets, the intersection of $<c$ dense open sets is dense.

1.12. Theorem. (i) (CH) *Let X be a complete self-dense metric space of power c. Then $\mathbf{C} \ll X$.*

(ii) (MA) *Let X be a complete self-dense separable metric space. Then $\mathbf{C} \ll X$.*

Proof. re(i). Let $\langle p_\alpha : \alpha < c \rangle$ well-order X. We partition X into Cantor sets by induction. Assume p_α is the first uncovered point, $p_\beta \in C_\beta \simeq \mathbf{C}$, distinct C_β's are disjoint, and each C_β is nowhere dense in X, $\beta < \alpha$. Let $Y_\alpha = X \backslash \bigcup_{\beta<\alpha} C_\beta$. By CH and the Baire Category Theorem (BCT), we have that Y_α is a self-dense G_δ-subset of X; hence a self-dense completely metrizable space in its own right. We thus let $C_\alpha \simeq \mathbf{C}$ with $p_\alpha \in C_\alpha \subset Y_\alpha$. Clearly we can arrange for C_α to be nowhere dense in Y_α (hence in X) and we are done.

re(ii). Repeat the above proof to the point where we define Y_α. Our problem is that α may not be countable, so we resort to the following generalization of a theorem of Martin–Solovay (see [1]): (MA) Let Z be any second countable space (countable

π-weight will do). Then the union of $<c$ nowhere dense sets is of the first category in Z.

Thus by MA we know that $\bigcup_{\beta<\alpha} C_\beta$ is contained in a countable union $\bigcup_{n<\omega} A_n$ of closed nowhere dense sets. Let $S_\alpha = X \setminus \bigcup_{n<\omega} A_n$. Then S_α is a self-dense G_δ. Let U be an open neighborhood of p_α. Then $U \cap S_\alpha \neq \emptyset$ by BCT, whence $p_\alpha \in \overline{S_\alpha}$. Since $\{p_\alpha\}$ is a G_δ, $S_\alpha \cup \{p_\alpha\}$ is a self-dense completely metrizable space, so we can construct C_α as before. □

1.13. Corollary. (MA) *Let X be a nonempty zero-dimensional separable metric space and let Y be any self-dense complete separable metric space. Then X partitions Y.* □

Using Martin's Axiom we showed that $\mathbf{C}$ partitions many spaces. In restricted cases, we can carry out the partitionings more constructively.

1.14. Theorem. *Both the real line and the closed unit interval are* $\mathbf{C}$*-partitionable.*

Proof. The same proof works, whether we wish to partition $\mathbf{R}$ or any other interval (with or without endpoints). Let A_0 be any Cantor set in $\mathbf{R}$. Then $\mathbf{R} \setminus A_0$ is a countable union of pairwise disjoint open intervals, say $\bigcup_{n<\omega} I_n$, so let $C_n \subset I_n$ be a Cantor set and let $A_1 = \bigcup_{n<\omega} C_n$. Proceed by induction. At every stage, $\mathbf{R} \setminus \bigcup_{m<n} A_m$ is a countable union of pairwise disjoint open intervals, so A_n can be defined as a countable union of pairwise disjoint Cantor sets. Let $A = \bigcup_{n<\omega} A_n$. Since each A_n is closed nowhere dense, $\mathbf{R} \setminus A$ is a dense G_δ-set. Clearly no interval lies in $\mathbf{R} \setminus A$, so by (1.10(iii)), $\mathbf{R} \setminus A \simeq \mathbf{P}$. Since $\mathbf{C} \ll A$ and $\mathbf{C} \ll \mathbf{P}$ we have the result. □

We complete this section of the paper with a close analysis of the partition spectra $\sigma(X, Y)$ with X and Y chosen among $\mathbf{C}$, $\mathbf{P}$, $\mathbf{R}$. Of the nine possible sets only seven are nonempty (clearly $\sigma(\mathbf{R}, \mathbf{C}) = \sigma(\mathbf{R}, \mathbf{P}) = \emptyset$), and $\sigma(\mathbf{R}, \mathbf{R}) = \{1\}$ since $\mathbf{R}$ is connected. What we know of the remaining six cases is contained in the following.

1.15. Theorem. (i) $\sigma(\mathbf{C}, \mathbf{C})$ *and* $\sigma(\mathbf{P}, \mathbf{P})$ *contain c, ω, and all nonzero finite cardinals.*

(ii) $\sigma(\mathbf{P}, \mathbf{C})$ and $\sigma(\mathbf{P}, \mathbf{R})$ *contain c and ω, but no finite cardinals.*

(iii) $\sigma(\mathbf{C}, \mathbf{P})$ *and* $\sigma(\mathbf{C}, \mathbf{R})$ *contain c, but no countable cardinals.*

Proof. (We are thankful to F. Galvin for providing the essential ideas to prove (ii).)

re(i). $c \in \sigma(\mathbf{C}, \mathbf{C})$ (resp. $c \in \sigma(\mathbf{P}, \mathbf{P})$) since $\mathbf{C} \simeq \mathbf{C} \times \mathbf{C}$ (resp. $\mathbf{P} \simeq \mathbf{P} \times \mathbf{P}$). $\omega \in \sigma(\mathbf{C}, \mathbf{C})$ (resp. $\omega \in \sigma(\mathbf{P}, \mathbf{P})$) since $\mathbf{C} \simeq \mathbf{C} \times (\omega + 1)$ (resp. $\mathbf{P} \simeq \mathbf{P} \times \mathbf{N}$); and for positive $n < \omega$, $n \in \sigma(\mathbf{C}, \mathbf{C})$ (resp. $n \in \sigma(\mathbf{P}, \mathbf{P})$) since $\mathbf{C} \simeq \mathbf{C} \times n$ (resp. $\mathbf{P} \simeq \mathbf{P} \times n$).

re(ii). We saw in (1.3) that $c \in \sigma(\mathbf{P}, \mathbf{C})$; and in (1.14) that $c \in \sigma(\mathbf{C}, \mathbf{R})$. Thus $c \in \sigma(\mathbf{P}, \mathbf{R})$. Let us prove that $\sigma(\mathbf{P}, \mathbf{C})$ contains no finite cardinals. A similar argument will show the same result for $\sigma(\mathbf{P}, \mathbf{R})$. Let $\{P_1, \ldots, P_n\}$ be a partition of $\mathbf{C}$ by n copies of $\mathbf{P}$. We know $n > 1$ and each P_i is a G_δ-set in $\mathbf{C}$. Let $B \subset \mathbf{C}$ be a basic clopen set hitting a minimum number of P_j's. Then, supposing $P_i \cap B \neq \emptyset$, P_i is dense in B [otherwise there is a nonempty clopen $B' \subset B$ with $B' \cap P_i = \emptyset$, so B' intersects fewer P_i's than does B]. Now $B \simeq \mathbf{C}$ is a Baire space (i.e. satisfies the BCT) and $|i: P_i \cap B \neq \emptyset| > 1$, so B can be written as a disjoint union of two dense G_δ-subsets, an impossibility.

Next we show $\omega \in \sigma(\mathbf{P}, \mathbf{C})$. We can then couple this result with the proof of (1.14) to infer that $\omega \in \sigma(\mathbf{P}, \mathbf{R})$. For convenience we will identify $\mathbf{C} \simeq 2^\omega$ with the power set of ω via the correspondence $f \mapsto \{n : f(n) = 1\}$. Let $\mathscr{B}$ be the standard clopen basis for 2^ω (a typical member of which is of the form $\{x \subset \omega : s \subset x, x \cap t = \emptyset\}$ for $s, t \subset \omega$ finite). Then (1.10(ii)) translates to the following:

(*) $X \subset 2^\omega$ is homeomorphic to $\mathbf{P}$ iff X is a nonempty G_δ-set and whenever $B \in \mathscr{B}$ and $X \cap B \neq \emptyset$, then $\bar{X} \cap B \neq X \cap B$.

From (*) it follows that whenever $X \subset 2^\omega$ is homeomorphic to $\mathbf{P}$ and $x \in \bar{X}$, then $X \cup \{x\} \simeq \mathbf{P}$. So choose a partition $\langle N_i : i < \omega \rangle$ of ω into infinite sets and set $N_\omega = \emptyset$. For $i \leq \omega$ define

$$A_i = \{x \subset \omega : x \text{ infinite}, x \cap N_i = \emptyset, \text{ and for all } j < i,\ x \cap N_j \neq \emptyset\}.$$

Then the A_i's are pairwise disjoint, $\overline{A_i} = \{x \subset \omega : x \cap N_i = \emptyset\}$, $\bigcup_{i \leq \omega} A_i = \{x \subset \omega : x \text{ infinite}\}$; and, by (*), $A_i \simeq P$. Now for each finite $x \subset \omega$ there are infinitely many i's with $x \in \bar{A}_i$, hence infinitely many i's with $A_i \cup \{x\} \simeq P$. So attach the finite sets to distinct A_i's obtaining a countable partition of $\mathbf{C}$ by $\mathbf{P}$.

re(iii). That $c \in \sigma(\mathbf{C}, \mathbf{P}) \cap \sigma(\mathbf{C}, \mathbf{R})$ is immediate. That no countable cardinal lies in $\sigma(\mathbf{C}, \mathbf{R})$ follows from BCT and the fact that no Cantor set in $\mathbf{R}$ can have nonempty interior. Since $\mathbf{P}$ is Baire it suffices to show that no Cantor set in $\mathbf{P}$ can have non-empty interior. We again appeal to (1.10(ii)) to assert that no nonempty open subset of $\mathbf{P}$ is compact. If C is any Cantor set in $\mathbf{P}$, suppose B is a nonempty basic clopen subset of $\mathbf{P}$ with $B \subset C$. Then B is clopen in C, whence B is compact. □

Remark. If we assume CH, then (1.15) implies trivially that
(i) $\sigma(\mathbf{C}, \mathbf{C}) = \sigma(\mathbf{P}, \mathbf{P}) = \{1, 2, \ldots, \omega\} \cup \{c\}$;
(ii) $\sigma(\mathbf{P}, \mathbf{C}) = \sigma(\mathbf{P}, \mathbf{R}) = \{\omega, c\}$; and
(iii) $\sigma(\mathbf{C}, \mathbf{P}) = \sigma(\mathbf{C}, \mathbf{R}) = \{c\}$.

We do not know whether these equations hold in the usual set theory (which we take to be Zermelo-Fraenkel with Choice, ZFC); but a privately communicated result of S. Shelah implies that they hold as a consequence of Martin's Axiom. More precisely, Shelah has proved the following Theorem: Let X be a complete separable metric space. If $\kappa > \omega$ and X is a union of κ pairwise disjoint G_δ-sets then R is the union of κ pairwise disjoint nowhere dense G_δ-sets.

2. Partitions involving higher-dimensional spaces

2.1. Lemma. *Let X be a locally compact, locally connected, connected Hausdorff space. Then there is no partition of X into countably many compact proper subsets.*

Proof. This follows immediately from the Chaining Lemma and the theorem of Sierpiński (see [3, vol. II, p. 173]) that no continuum is the union of a countably infinite number of pairwise disjoint closed subsets. □

2.2. Theorem. (i) $\mathbf{I}^m \ll \mathbf{R}^n$ *iff* $m < n$.
(ii) $\mathbf{I}^m \ll S(n)$ *iff* $m < n$.

Proof. re(i). Clearly if $\mathbf{I}^m \ll \mathbf{R}^n$, then $m \leqslant n$ by standard dimension theory. If $m = n$ then, by Brouwer Invariance of Domain (BID), a partition of $\mathbf{R}^n$ by $\mathbf{I}^n$ must be countable. By (2.1), this is impossible.

Conversely, we show $\mathbf{I}^n \ll \mathbf{R}^{n+1}$. Clearly $[0, 1) \ll \mathbf{R}$, so $[0, 1)^{n+1} \ll \mathbf{R}^{n+1}$. Also $[0, 1)^2 \simeq \mathbf{I} \times [0, 1)$, so by induction $[0, 1)^{n+1} \simeq \mathbf{I}^n \times [0, 1)$. Thus $\mathbf{I}^n \ll \mathbf{R}^{n+1}$.

re(ii). First suppose $m \geqslant n$. If $m > n$, then BID prevents $\mathbf{I}^m \ll S(n)$. If $m = n$ we again resort to (2.1). For the converse suppose $m < n$. If $n = m + 1$ let D be an m-disk in $S(m+1)$. Then $S(m+1)\backslash D \simeq \mathbf{R}^{m+1}$; so, since $\mathbf{I}^m \ll \mathbf{R}^{m+1}$, we have $\mathbf{I}^m \ll S(m+1)$. If $m + 1 < n$ we proceed inductively, partitioning $S(n)$ into its equator ($\simeq S(n-1)$) plus upper and lower hemispheres (each $\simeq \mathbf{R}^n$). □

2.3. Theorem. *If* $m \geqslant 1$ *and* $S(m) \ll \mathbf{R}^n$, *then* $m + 1 < n$.

Proof. It suffices to show that $S(m) \not\ll \mathbf{R}^{m+1}$. Let $\mathcal{S}$ be a partition of $\mathbf{R}^{m+1}$ by m-spheres. By the Jordan Curve Theorem (JCT), each $S \in \mathcal{S}$ separates $\mathbf{R}^{m+1}\backslash S$ into a bounded component B_S and an unbounded one, $R^{m+1}\backslash(S \cup B_S)$. If $S_1, S_2 \in \mathcal{S}$, define $S_1 < S_2$ iff $S_1 \subset B_{S_2}$. Pick $S_0 \in \mathcal{S}$ and let $\mathfrak{M} \subset \mathcal{S}$ be a maximal chain below S_0. Then $\bigcap_{S \in \mathfrak{M}} B_S = \bigcap_{S \in \mathfrak{M}} (B_S \cup S)$ contains a point x which cannot lie on any member of $\mathfrak{M}$. But x lies on some S' in $\mathcal{S}$ and $S' < S$ for all $S \in \mathfrak{M}$, contradicting the maximality of $\mathfrak{M}$. □

2.4. Theorem. *Let X be a compact space which partitions* $\mathbf{R}^n$, $n > 1$. *Then* $\dim(X) < n$. *If X is also connected, then X contains no* $(n-1)$*-sphere.*

Proof sketch. If $\dim(X) = n$ then, by a classic theorem of dimension theory (see [7]), any embedded copy of X in $\mathbf{R}^n$ must have nonempty interior. Thus any partition of $\mathbf{R}^n$ by X must be countable, contradicting (2.1). If $n > 1$ and X is connected and contains an $(n-1)$-sphere, we use the fact that X must be embedded nowhere densely in $\mathbf{R}^n$ and mimic the proof of (2.3). □

The following result expands on a technique used in [2] to show (nonconstructively) that $S(1) \ll \mathbf{R}^3$. The spirit of this technique also pervades earlier proofs in the present paper (e.g. (1.11)).

2.5. Theorem. *Let* $\emptyset \neq X \subset S(n)$. *Then* $X \ll \mathbf{R}^{2n+1}$.

Proof. Let $\langle p_\alpha : \alpha < c \rangle$ be a well-ordering of $\mathbf{R}^{2n+1}$. We partition $\mathbf{R}^{2n+1}$ by induction.

Let $\alpha < c$ and assume that for each $\beta < \alpha : p_\beta \in X_\beta \simeq X$, distinct X_β's are disjoint, $p_\alpha \notin \bigcup_{\beta<\alpha} X_\beta$, and each X_β is contained in an n-sphere $S_\beta \subset \mathbf{R}^{2n+1}$. Each S_β is contained in a unique $(n+1)$-plane E_β, so we need the

Lemma. *Let $s < t < \omega$, let $p \in \mathbf{R}^t$, let $\mathscr{E}$ be a family of $<c$ planes containing p and of dimension $\leqslant s$. Then there is a $(t-1)$-plane H containing p which fails to contain any member of $\mathscr{E}$.*

Proof of lemma. Let S be a standard $(t-1)$-sphere centered at p. If E is an r-plane containing p then there is a unique $(t-r)$-plane containing p which is perpendicular to E (call this plane $E^\perp$). Then $E^\perp \cap S = S(E)$ is a "great $(t-r-1)$-circle" (i.e. $q \in S(E)$ iff $q' \in S(E)$ where q' is the S-antipode of q). If E,F are planes about p then $E \subset F$ iff $S(F) \subset S(E)$. If H is a hyperplane about p then $S(H)$ is a pair of antipodal points of S. Now a sphere cannot be covered by $<c$ great circles of smaller dimension, so there is a point and its antipode on S which are not in $\bigcup\{S(E) : E \in \mathscr{E}\}$. This gives the hyperplane we want. □

So now we find a $2n$-plane H_α which fails to contain any E_β, $\beta < \alpha$. Thus for each $\beta < \alpha$, $H_\alpha \cap E_\beta$ is a (possibly empty) plane of dimension $\leqslant n$; so $H_\alpha \cap S_\beta$ is at worst a sphere of dimension $\leqslant n-1$. Now we work in H_α instead of $\mathbf{R}^{2n+1}$, proceeding backward in an obvious induction obtaining, after n steps, an $(n+1)$-plane $F_\alpha \subset H_\alpha$ about p_α which hits each X_β in at most two points (i.e. a zero-sphere). Since $|F_\alpha \cap \bigcup_{\beta<\alpha} X_\beta| < c$ we can place an n-sphere $S_\alpha \subset F_\alpha$ containing p_α and missing $\bigcup_{\beta<\alpha} X_\beta$. By homogeneity of spheres, we can then place a copy X_α of X in S_α containing p_α. □□

2.6. Corollary. *Let X be a nonempty separable metric space of dimension n. Then $X \ll \mathbf{R}^{4n+3}$. In particular any nonempty finite-dimensional separable metric space partitions Hilbert space.*

Proof. By dimension theory, X embeds as a subset of $S(2n+1)$; whence $X \ll \mathbf{R}^{4n+3}$ by (2.5). □

2.7. Theorem $\mathbf{H} \equiv l_2$.

Proof. $\mathbf{H} \simeq \mathbf{I}^\omega$ and $l_2 \simeq \mathbf{R}^\omega$. Since $\mathbf{I} \ll \mathbf{R}^2$ (by (2.3)) we have $\mathbf{I}^\omega \ll \mathbf{R}^\omega$. On the other hand the reader can easily check that $\mathbf{R} \ll \mathbf{I}^2$; whence $\mathbf{R}^\omega \ll \mathbf{I}^\omega$.

2.8. Corollary. *Let X be a nonempty finite-dimensional separable metric space. Then* $\mathbf{H} \equiv \mathbf{H} \times X$.

Proof. By (2.6, 2.7), $X \ll \mathbf{H}$. Thus $\mathbf{H} \times X \ll \mathbf{H} \times \mathbf{H} \simeq \mathbf{H}$. □

2.9. Question. Is there a nonempty separable metric space which does not partition the Hilbert cube?

3. Partitions involving connected low-dimensional spaces

In this section we address the question of whether any of the results in Section 2 can be improved if we keep the dimension down (say $\leqslant 3$). First of all we do not know whether the bound in (2.5) is sharp. For example if $\dim(X) = 0$ then (2.6) tells us that $X \ll \mathbf{R}^3$. But we already know from (1.11, 1.14) that $X \ll \mathbf{R}$. Some simple questions whose answers we have been unable to determine follow.

3.1. Questions. (i) Is there a 1-dimensional space which does not partition $\mathbf{R}^3$?
(ii) Does every nonempty subset of $\mathbf{R}$ partition $\mathbf{R}^2$?
(iii) Which subsets of $\mathbf{R}$ partition $\mathbf{R}$?
(iv) Does $S(2)$ partition $\mathbf{R}^4$?

Remark. In (2.5) we used the Axiom of Choice to prove that $S(n)$ partitions $\mathbf{R}^{2n+1}$. This weak form of (2.5) can be proved without AC but at consideraable notational expense in the cases $n \geqslant 2$. If $n = 1$, however, there is a simple pictorial proof: Let $A = \mathbf{I}^3 \subset \mathbf{R}^3$, and let $S \subset A$ be the simple closed curve formed by taking an arc on $\mathrm{Bd}(A)$ and joining the endpoints with an open segment in $\mathrm{Int}(A)$. Then

$$A' = A \backslash (S \cup \mathrm{Bd}(A)) \simeq (\text{open 3-disk} \backslash \text{diameter})$$

is $S(1)$-partitionable. By adding simple closed curves to thicken up the closed arc on the boundary of A', we obtain from $A' \cup S$ an open 3-disk with an open 2-disk on its boundary. This is clearly homeomorphic to $[0, 1)^3$ which then partitions R^3. The reader may wish to apply a similar method to partition I^3 with $S(1)$. We met with no success when we tried to do this; however the nonconstructive ploy (à la (2.5)) does work.

For the remainder of this section we will be concerned with partitioning 2-dimensional things with 1-dimensional things.

3.2. Theorem. *The unit interval is the only nondegenerate Peano continuum (= connected, locally connected compact metric space) which partitions the plane.*

Proof. Of course $\mathbf{I} \ll \mathbf{R}^2$, so suppose X is another nondegenerate Peano continuum which partitions $\mathbf{R}^2$. Then by (2.3) X cannot be $S(1)$. We now resort to the following results of R.L. Moore.

Lemma. (i) ([5]) *Let Y be a nondegenerate Peano continuum. Then Y is either an arc, a simple closed curve, or a space containing an embedded triod (= three arcs joined at a common endpoint).*
(ii) ([4]) *There is no uncountable planar collection of pairwise disjoint triods.*

This lemma, coupled with (2.1), gives an immediate contradiction. □

In light of the fact that $S(1)$ does not partition $S(2)$, together with the relatively straightforward classification theorem for compact connected 2-manifolds (see [6]), it seems reasonable to ask which of these manifolds can be partitioned by the circle. The answer, discovered with the much-appreciated help of S. Stahl is the following.

3.3. Theorem. *The torus and the Klein bottle are the only compact connected 2-manifolds which are $S(1)$-partitionable.*

Proof. Clearly the torus and the Klein bottle are $S(1)$-partitionable. So let M be a compact connected orientable 2-manifold and let $\mathcal{S}$ be a covering of M by pairwise disjoint simple closed curves. Restricting ourselves to the orientable case will suffice since every nonorientable 2-manifold has a two-sheeted orientable covering space. Thus if M were nonorientable with $\pi : \tilde{M} \to M$ the above-mentioned covering projection, then for each $S \in \mathcal{S}$, $\pi^{-1}(S)$ would be a union of at most two simple closed curves. This would mean that $S(1) \ll \tilde{M}$, hence $\tilde{M}$ would be a torus (given the theorem in the orientable case) and M would therefore be a Klein bottle.

Let $S \in \mathcal{S}$. We can assume that S is tame so that there is a regular neighborhood U of S in M with $U \simeq S \times [0, 1]$. Let M_S be the result of removing $\text{Int}(U)$ from M and identifying the two boundary circles to points p, q. Then M_S is an orientable compact 2-manifold with at most two components. Moreover $S(1) \ll M_S \backslash \{p, q\}$; and since S does not bound a disk in M [otherwise $S(1) \ll R^2$], the genus of each component of M_S is less than that of M. Let M_1, M_2 be the components of M_S, say $p \in M_1$, $q \in M_2$ [even if $M_1 \neq M_2$, each component intersects $\{p, q\}$]. Then $\mathcal{S} \backslash \{S\}$ naturally splits into $\mathcal{S}_1$, $\mathcal{S}_2$, partitions of $M_1 \backslash \{p\}$, $M_2 \backslash \{q\}$ respectively. We now repeat the process outlined above for M_1, M_2, iterating until, after a finite number of steps, we obtain a disjoint union of 2-spheres $\Sigma_0, \ldots, \Sigma_n$ and finite sets $F_i \subset \Sigma_i$, $i \leq n$, such that $S(1) \ll \Sigma_i \backslash F_i$ for each $i \leq n$. Moreover it is easy to check that if the genus of M is at least 2 then for some $i \leq n$, $|F_i| \geq 3$. We are done, therefore, once we prove the following.

Lemma. *Let F be a finite subset of $S(2)$ with at least three points. Then $S(1) \not\ll S(2) \backslash F$.*

Proof of lemma. We proceed by induction on $|F|$. Suppose first that $F = \{p_1, p_2, p_3\}$ and that $\mathcal{S}$ is a partition of $S(2) \backslash F$ into simple closed curves. For $i = 1, 2, 3$ define $<_i$ on $\mathcal{S}$ as follows: Let $S \in \mathcal{S}$, define B_S^i to be the open connected component of $S(2) \backslash S$ not containing p_i. Then define $S' <_i S$ if $S' \subset B_S^i$ (as in the proof of (2.3)). For simplicity let $i = 3$, and let $<$ be $<_3$. If $B_S \cap \{p_1, p_2\}$ is empty we get a contradiction in the same way we show $S(1) \not\ll \mathbf{R}^2$. Thus we partition $\mathcal{S}$ into $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, where for $i = 1, 2$, $S \in \mathcal{S}_i$ iff $B_S \cap \{p_1, p_2\} = \{p_i\}$; and $S \in \mathcal{S}_3$ iff $p_1, p_2 \in B_S$.

Claim (i). $\mathcal{S}_3$ is nonempty. Otherwise we let

$$A_i = \{p_i\} \cup \bigcup \{\overline{B_S} : S \in \mathcal{S}_i\}, \quad i = 1, 2.$$

Then $\langle A_1, A_2 \rangle$ forms a partition of $S(2) \backslash \{p_3\} \simeq \mathbf{R}^2$. Now A_i is either open (i.e. when $\mathcal{S}_i$ is nonempty and has no maximal element) or closed (i.e. when either $\mathcal{S}_i = \emptyset$ or $\mathcal{S}_i$ has a maximal element). Since $\mathbf{R}^2$ is connected we can assume A_1 is closed, A_2 is open. Then A_1 is either a point or a closed disk; whence A_2 cannot be simply connected. But on the contrary, A_2 is a chain union of open disks and is therefore contractible.

Claim (ii). $\mathscr{S}_3$ has no minimal element. Otherwise, if S is minimal in $\mathscr{S}_3$, then $B_S = \bigcap\{B_{S'}: S' \in \mathscr{S}_3\}$; hence $\mathscr{S}_1 \cup \mathscr{S}_2$ partitions $B_S \backslash \{p_1, p_2\}$. But this would imply that $R^2\backslash$(two points) is partitionable by simple closed curves, none of which enclose both points. This possibility was excluded in Claim (i).

Now the assertion that $\mathscr{S}_3$ is nonempty and has no minimal element in the $<_3$-ordering says that $\mathscr{S}_3$ has no maximal element in the $<_1$- or $<_2$-orderings. Thus $\{p_3\} \cup \bigcup \mathscr{S}_3$ is a union of open disks; whence $\bigcup \mathscr{S}_3$ is nonempty open in $S(2)\backslash F$. Similarly we show that $\bigcup \mathscr{S}_1$, $\bigcup \mathscr{S}_2$ are nonempty open sets, disconnecting $S(2)\backslash F$, a contradiction.

So assume for the inductive step that $F = \{p_1, \ldots, p_n\}$, $n > 3$. If $\mathscr{S}$ is a partition of $S(2)\backslash F$ into simple closed curves and $S \in \mathscr{S}$ encloses (with respect to $< = <_n$) $1 < k < n-1$ points of F, then we can partition $\mathbf{R}^2\backslash(k$ points), hence $S(2)\backslash(k+1$ points) into simple closed curves. But our inductive hypothesis bars this. Thus $\mathscr{S} = \bigcup_{1 \leq i \leq n} \mathscr{S}_i$ where for $1 \leq i < n$, $S \in \mathscr{S}_i$ iff $B_S \cap F\backslash\{p_n\} = \{p_i\}$; and $S \in \mathscr{S}_n$ iff $F\backslash\{p_n\} \subset B_S$. We then show directly as before that each $\bigcup \mathscr{S}_i$ is nonempty open in $S(2)\backslash F$; this being a disconnection of a connected space. This proves the lemma and thereby the theorem. □

Putting (3.2) and (3.3) together yields the following rather general statement.

3.4. Corollary. *Let X be a nondegenerate Peano continuum, with M a compact connected 2-manifold, $X \neq M$. Then $X \ll M$ iff either*

(a) $X \simeq \mathbf{I}$; *or*

(b) $X \simeq S(1)$ *and M is either a torus or a Klein bottle.*

Proof. Suppose $X \ll M$. Then X can contain no embedded triods (see (2.1), (3.2)). Thus either $X \simeq \mathbf{I}$ or $X \simeq S(1)$. If the latter holds then M must be a torus or a Klein bottle by (3.3).

Conversely, suppose $X \simeq \mathbf{I}$. Then $X \ll M$ since M can be written as a disjoint union of handles (each being $\simeq S(1) \times \mathbf{I}$), crosscaps (each being an annulus with antipodal points on the outer circle identified), and a 2-sphere with finitely many closed arcs removed. On the other hand if $X \simeq S(1)$ and M is a torus or Klein bottle then clearly $X \ll M$. □

4. Partition equivalence

The relation of partition equivalence, aside from leaving cardinality and dimension (in the separable metric case) intact, does not preserve very many topological properties. Examples we have presented show that compactness, connectedness, and completeness are among these properties. If we look at the nonmetrizable case, we find that $N^{\omega_1} \equiv 2^{\omega_1}$, hence even normality is not preserved [that N^{ω_1} is not normal is a famous theorem of A.H. Stone].

In this section we will be interested mainly in the partition-equivalence types of various spaces. The partition unique spaces are thus interesting because their $\equiv$-types are the same as their $\simeq$-types. We saw earlier that **Q** is partition unique but not partition stable. **N** also has this property. We will show that all manifolds which are either open (i.e. locally Euclidean) or compact are indeed partition stable. To this end we define a space X to be *strongly* (resp. *weakly*) *Brouwer* if whenever $h: X \to X$ is an embedding then $h[X]$ is open (resp. has nonempty interior) in X.

4.1. Theorem. *Let X be a connected space. Then X is partition stable if either*

(a) *X is strongly Brouwer; or*

(b) *X is weakly Brouwer, locally connected, compact T_2, and satisfies the countable chain condition.*

Proof. Suppose (a). Then X is partition stable simply by connectedness. If (b) holds we resort to (2.1) for the conclusion. □

4.2. Corollary. *If M is a connected manifold which is either open or compact, then M is partition stable.*

Proof. If M is open then M is strongly Brouwer since BID holds for all locally Euclidean spaces. If M is compact let ∂M denote the boundary of M, with $h: M \to M$ an embedding. Then $h[M \backslash \partial M] \subset M \backslash \partial M$; whence M is weak Brouwer. M also satisfies all the other hypotheses of (4.1(b)), so the conclusion follows. □

Remark. There are connected manifolds which are not partition unique; namely the half-open unit interval is partition equivalent to a disjoint union of two copies of itself.

4.3. Theorem. *Any compact Hausdorff space of power $<c$ is partition stable.*

Proof. Repeated application of a self-partitioning of X yields a tree of closed subsets of X, ordered by inclusion. By compactness, each branch of the tree (of which there are at least c) realizes a distinct point of X. □

Significant among those spaces which are not partition unique are the Cantor set and the Hilbert cube; and we can say very little by way of a reasonable determination of either of their $\equiv$-types. As far as partial answers go we have Corollary (2.8) as well as the following corollary of (1.11, 1.12).

4.4. Corollary. (MA) *Let X be a nonempty complete, separable, zero-dimensional metric space. Then $X \equiv$* **C**. □

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