BiLinear, Bicubic, and In Between Spline Interpolation

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1. # Goal of Interpolation

1D Interpolation

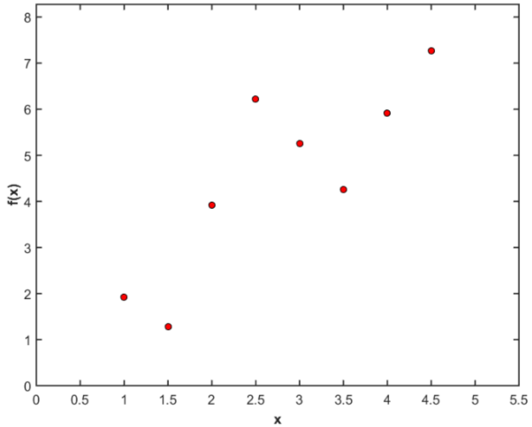
2. Linear
3. Cubic
4. Cubic Spline

2D Interpolation

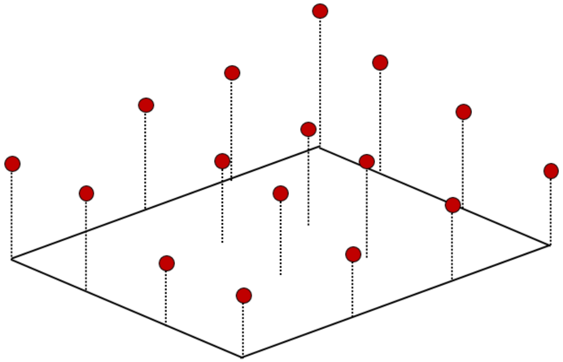
5. BiLinear
6. BiCubic
7. BiCubic Spline

8. # MRI Example

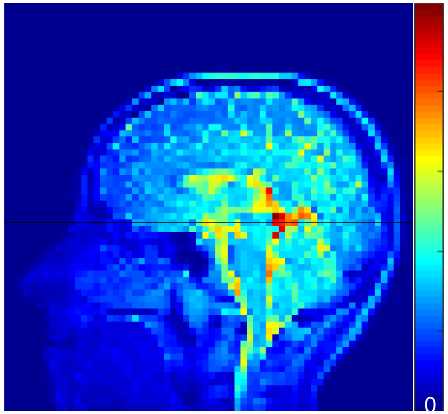
1D Interpolation



2D Interpolation



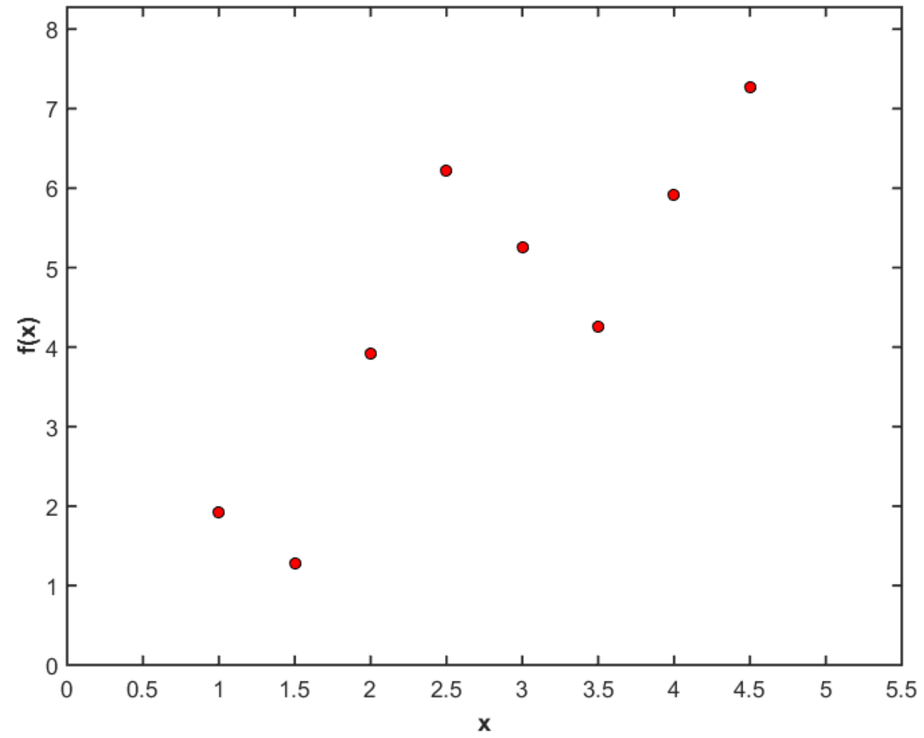
MRI Example



1. Goal of Interpolation

How do you determine how to get from one point to another?

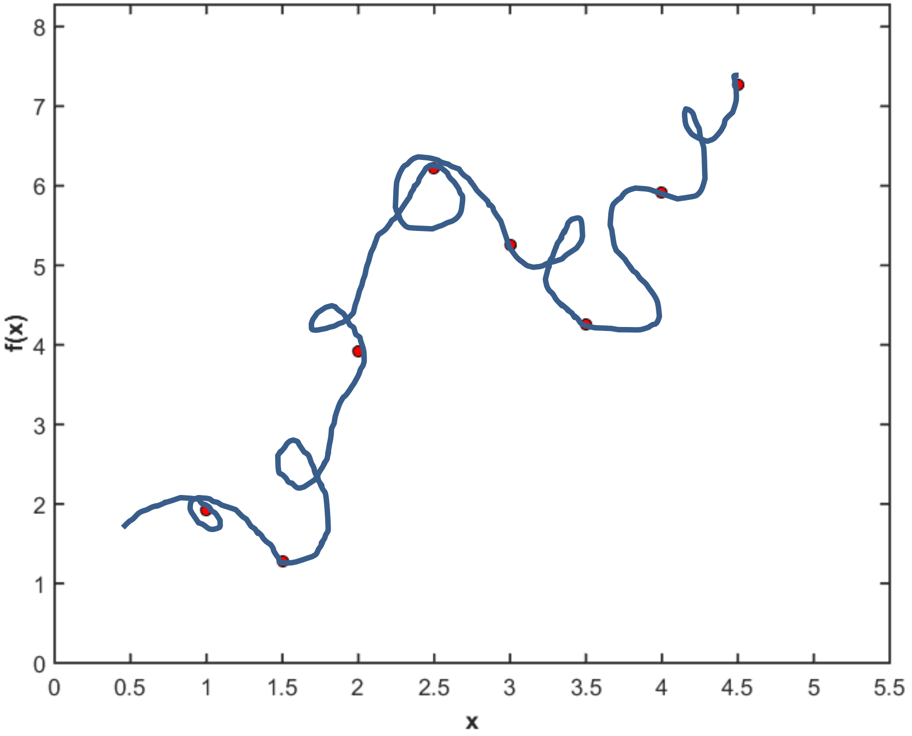
Can we estimate a path to traverse through the points, then interpolate intermediate values along our path?



1. Goal of Interpolation

How do you determine how to get from one point to another?

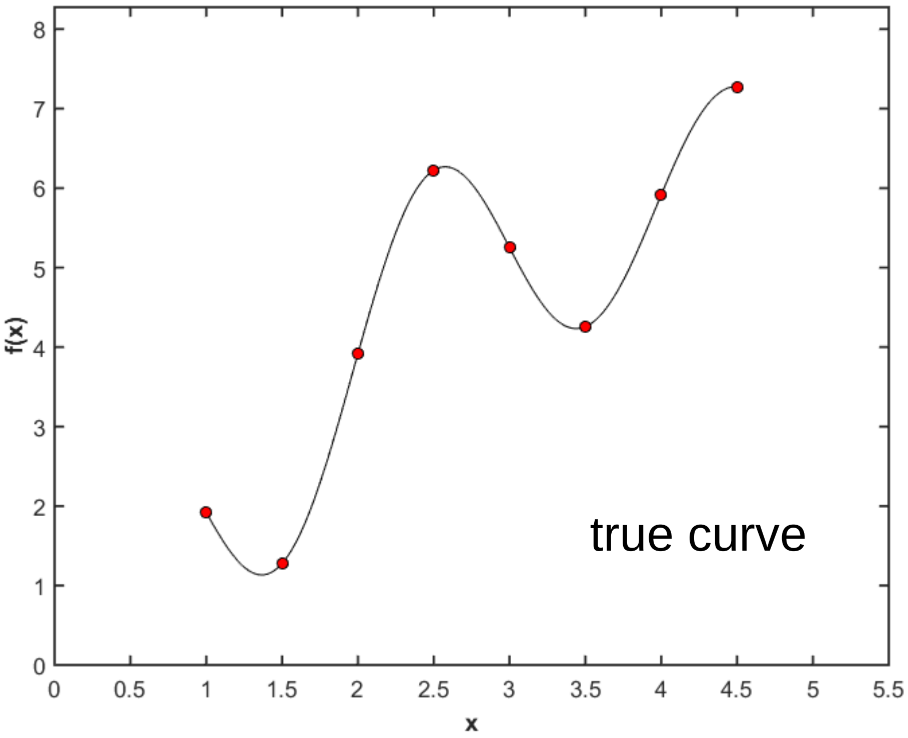
Not complicated like this!



1. Goal of Interpolation

How do you determine how to get from one point to another?

But some smooth progression through the points.



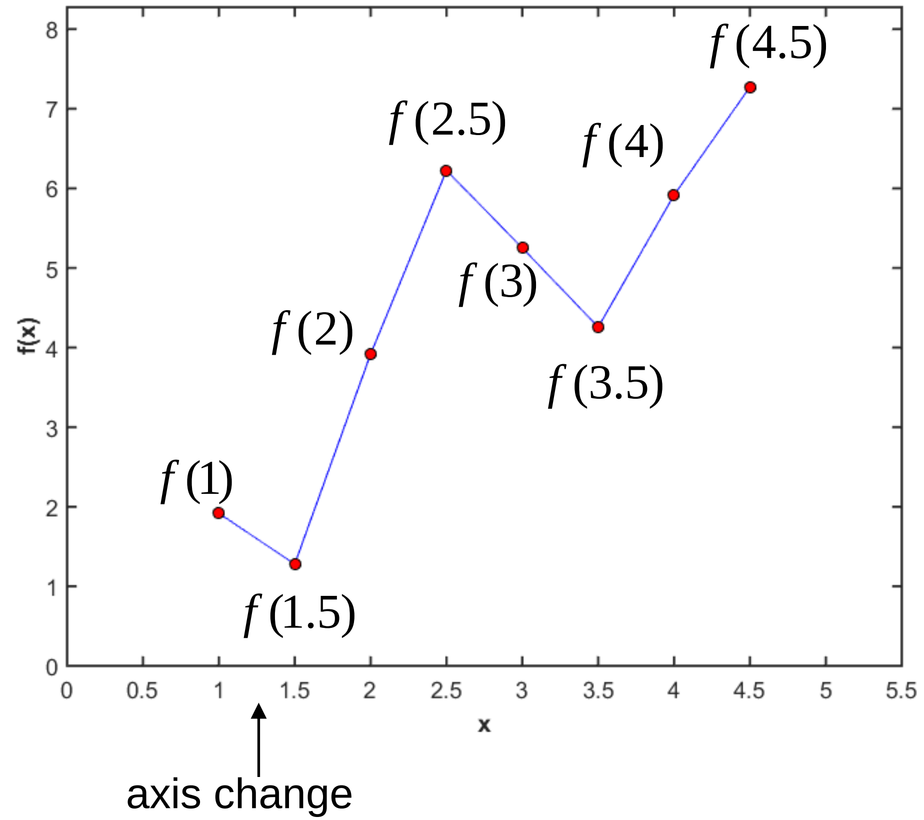
2. 1D Linear Interpolation

Easiest to draw straight lines between points and use values along the lines.

Interpolate:

Two points define a line.

Find the equation of the line between points.

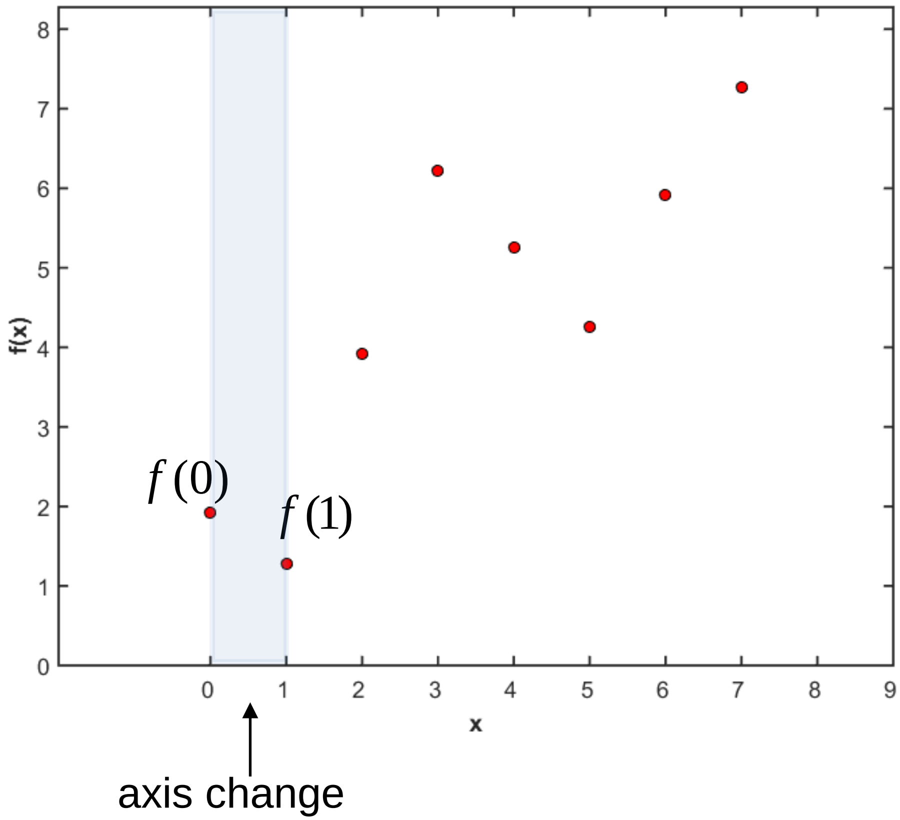


2. 1D Linear Interpolation

Easiest to draw straight lines between points and use values along the lines.

Normalization: $f(0), f(1)$

For regularly spaced points.

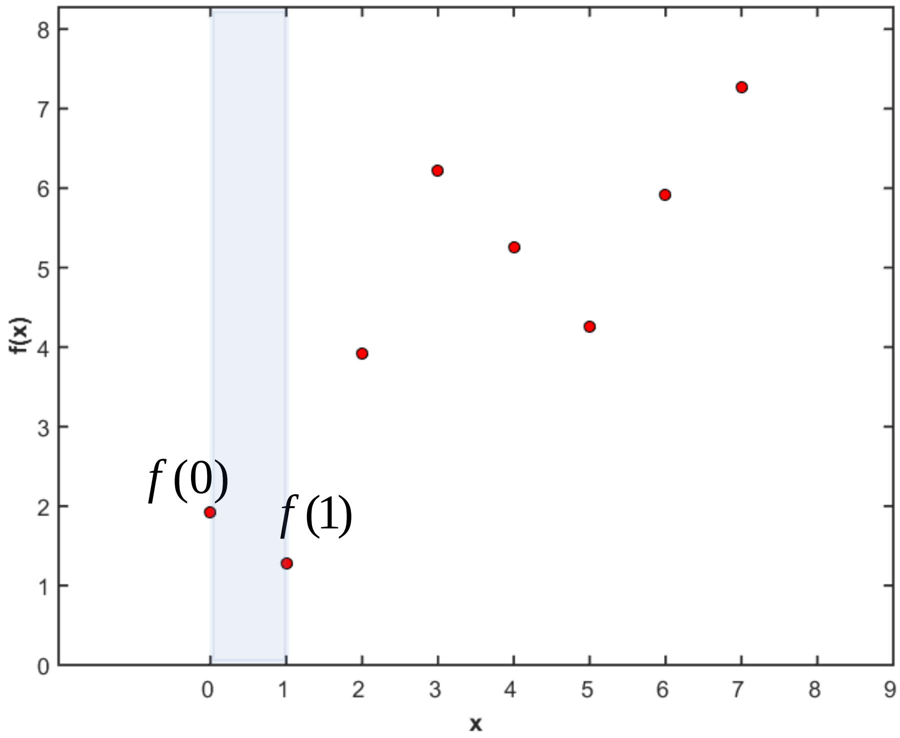


2. 1D Linear Interpolation

Easiest to draw straight lines between points and use values along the lines.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1$
$x = 0,1$



2. 1D Linear Interpolation

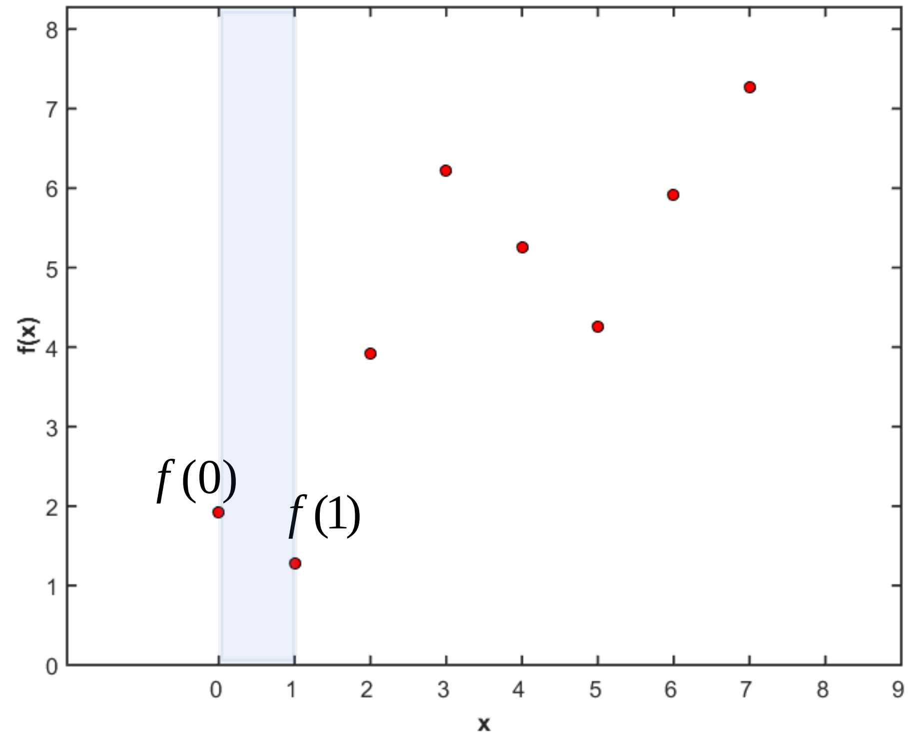
Easiest to draw straight lines between points and use values along the lines.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1$
$x = 0,1$

Solve: (a_0, a_1)

$$f(0) = a_0(1) + a_1(0)$$
$$f(1) = a_0(1) + a_1(1)$$



2. 1D Linear Interpolation

System of Equations: 2 equations, 2 unknowns

$$f(0) = a_0(1) + a_1(0)$$
$$f(1) = a_0(1) + a_1(1)$$

System of Equations

$$\underbrace{\begin{bmatrix} f(0) \\ f(1) \end{bmatrix}}_{y} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{a} \Bigg\} \text{two points}$$

System of Equations

$$y = Xa$$

Solution

$$\underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{a} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{X^{-1}} \underbrace{\begin{bmatrix} f(0) \\ f(1) \end{bmatrix}}_{y}$$

Solution

$$a = X^{-1}y$$

2. 1D Linear Interpolation

Easiest to draw straight lines between points and use values along the lines.

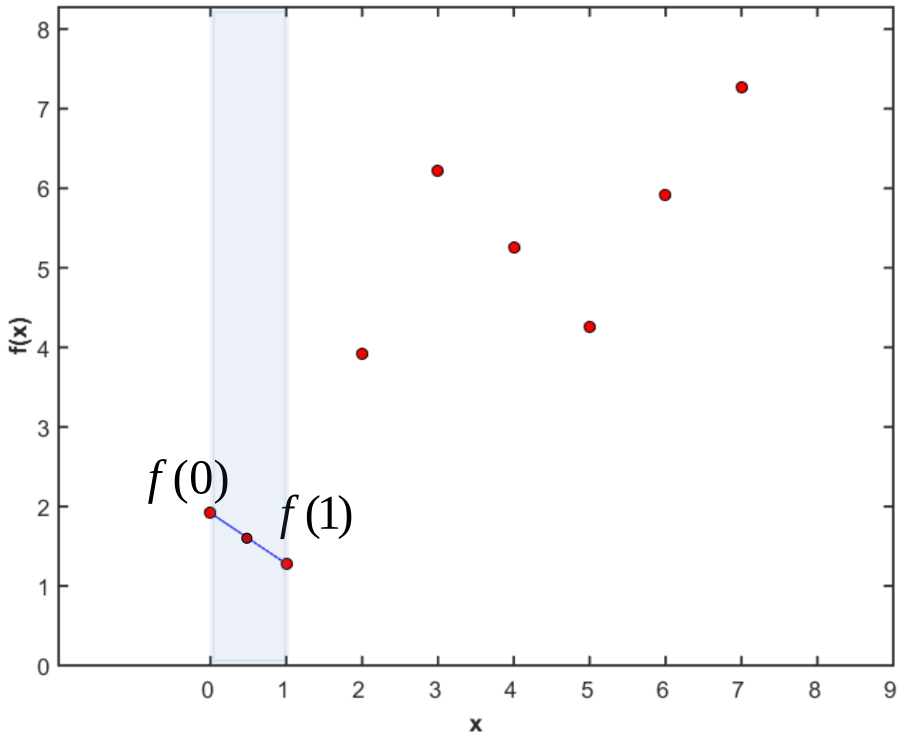
Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1$

$x = 0,1$

Solve: (a_0, a_1)

Interpolate: .5



$$f(.5) = a_0(.5)^0 + a_1(.5)^1$$

2. 1D Linear Interpolation

Repeat the process between all pairs of points to interpolate values between.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1$

$x = 0,1$

Solve: (a_0, a_1)

Interpolate: .5

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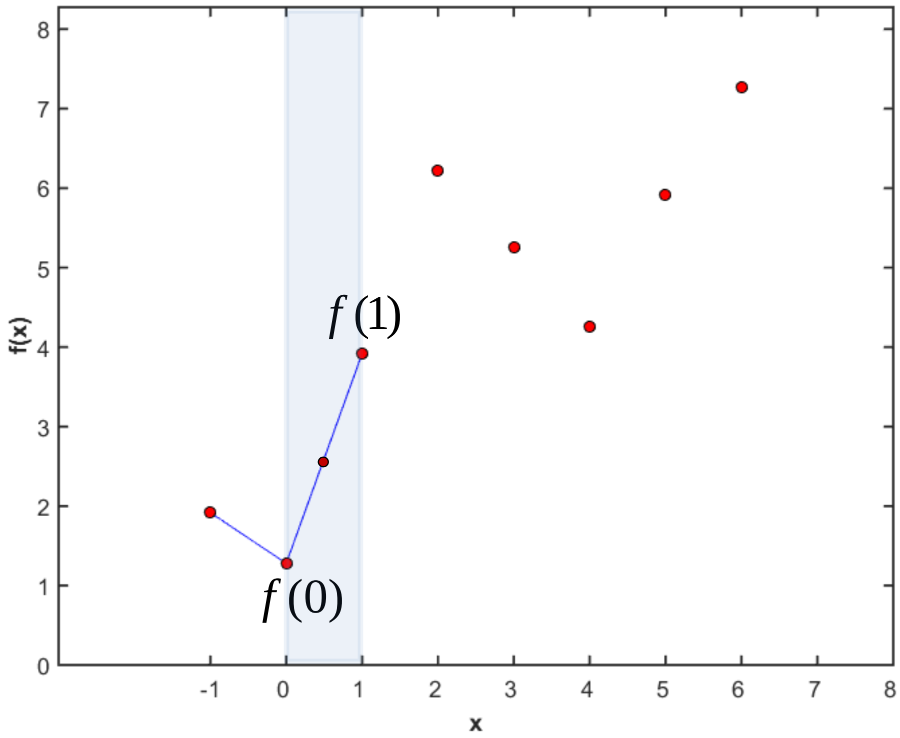

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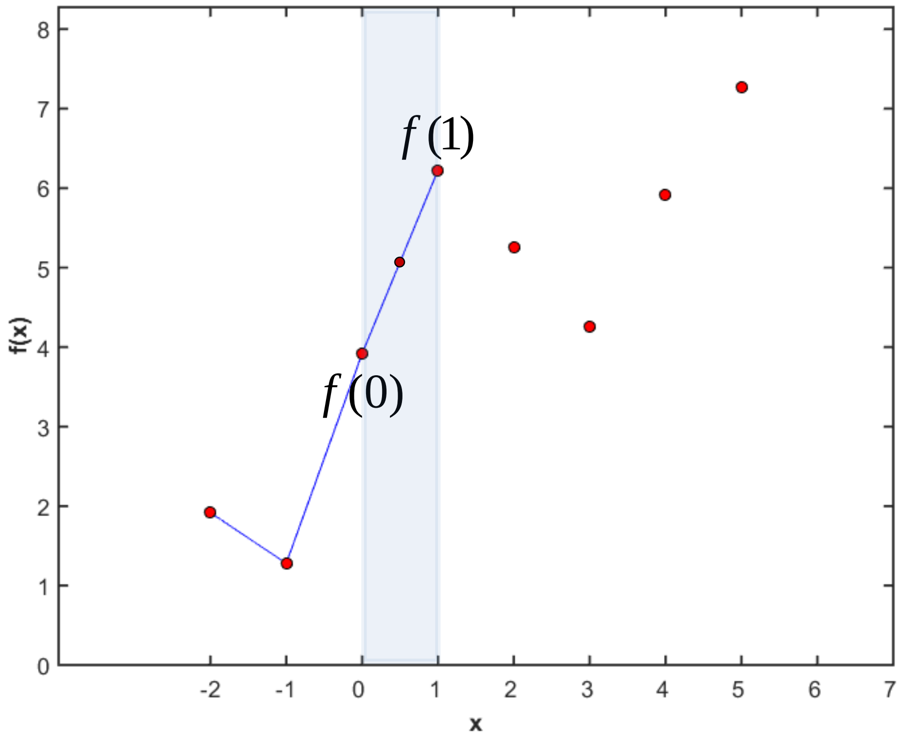
$$f(.5) = a_0(.5)^0 + a_1(.5)^1$$



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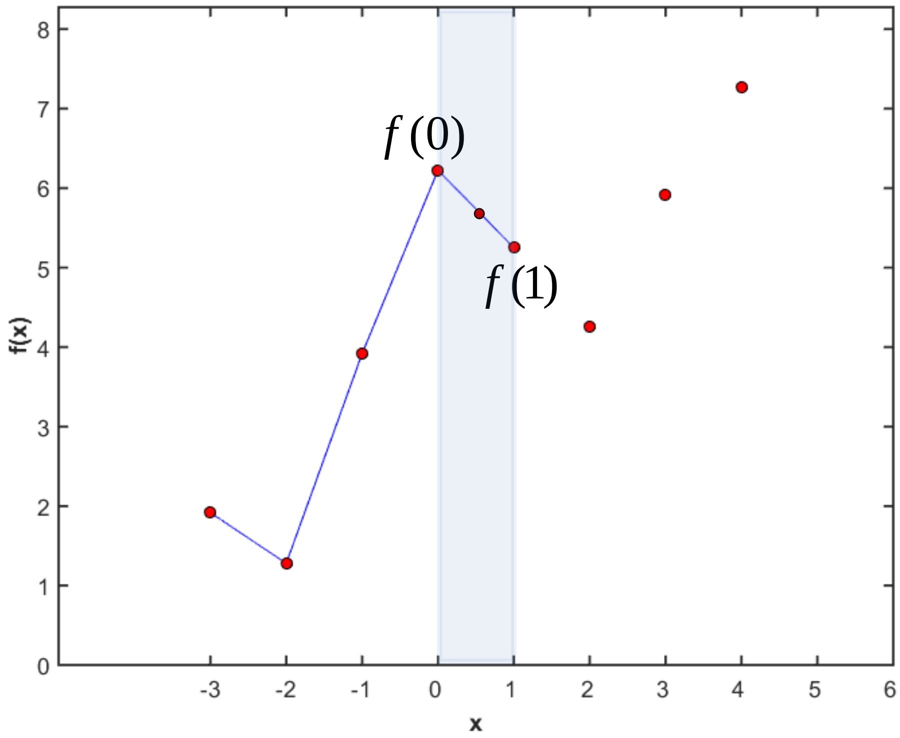
$$f(.5) = a_0(.5)^0 + a_1(.5)^1$$



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Repeat the process between all pairs of points to interpolate values between.

Normalization: $f(0), f(1)$

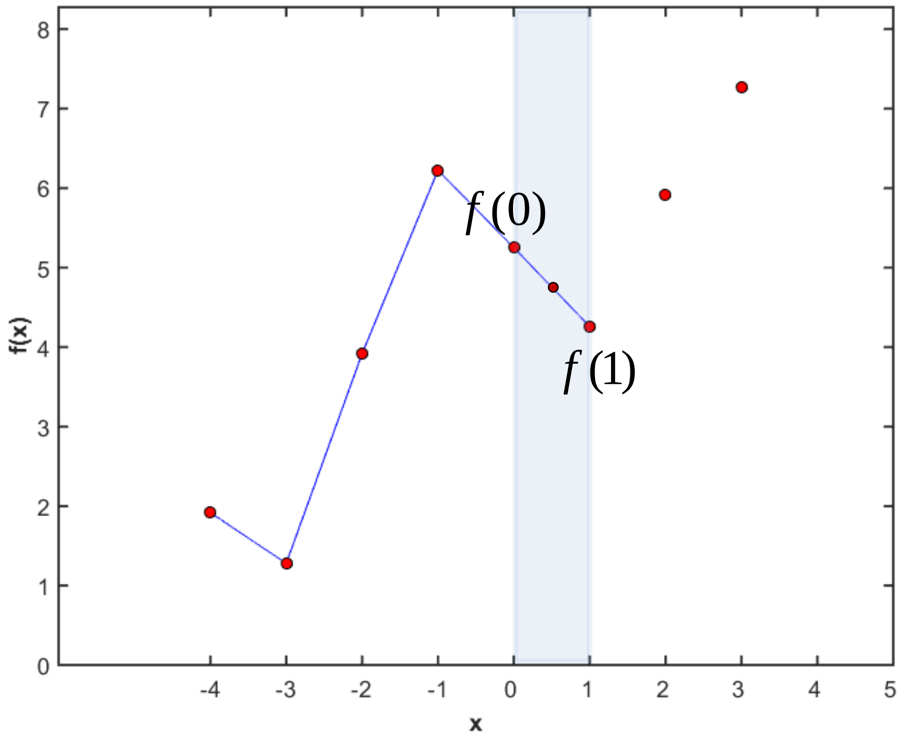
Model: $f(x) = a_0 x^0 + a_1 x^1$

$x = 0,1$

Solve: (a_0, a_1)

Interpolate: .5

$$f(.5) = a_0 (.5)^0 + a_1 (.5)^1$$



2. 1D Linear Interpolation

Repeat the process between all pairs of points to interpolate values between.

Normalization: $f(0), f(1)$

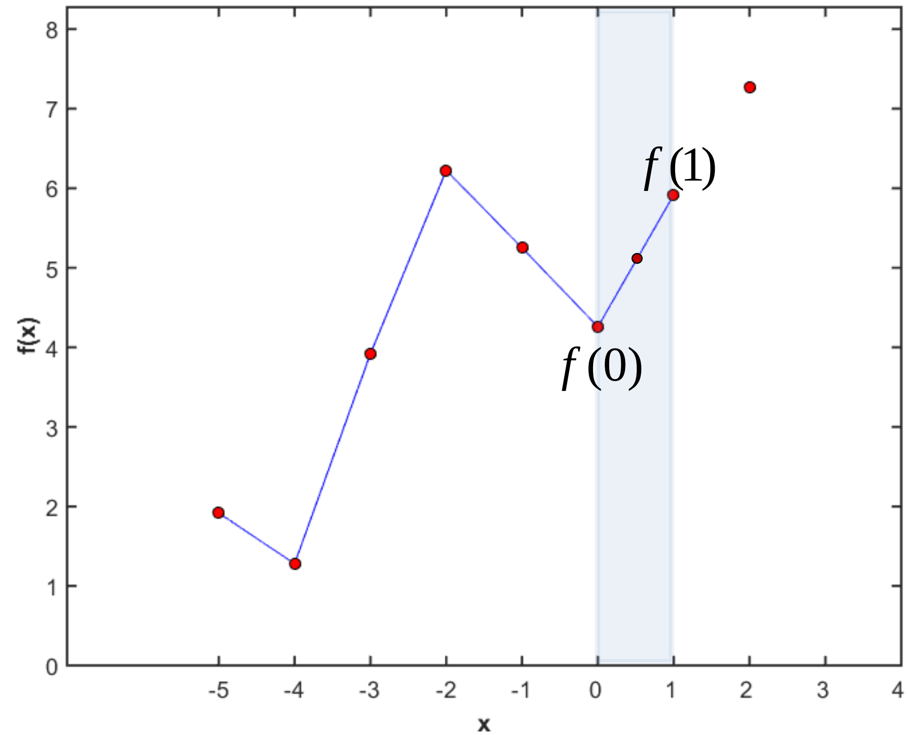
Model: $f(x) = a_0 x^0 + a_1 x^1$

$x = 0,1$

Solve: (a_0, a_1)

Interpolate: .5

$$f(.5) = a_0(.5)^0 + a_1(.5)^1$$



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Repeat the process between all pairs of points to interpolate values between.

Normalization: $f(0), f(1)$

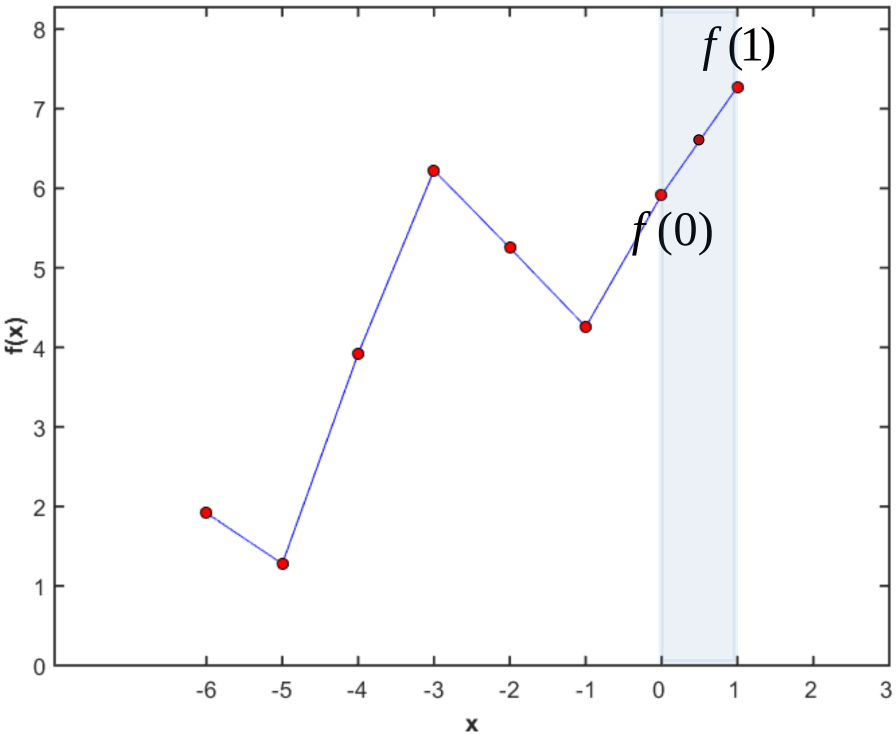
Model: $f(x) = a_0 x^0 + a_1 x^1$

$x = 0,1$

Solve: (a_0, a_1)

Interpolate: .5

$$f(.5) = a_0 (.5)^0 + a_1 (.5)^1$$



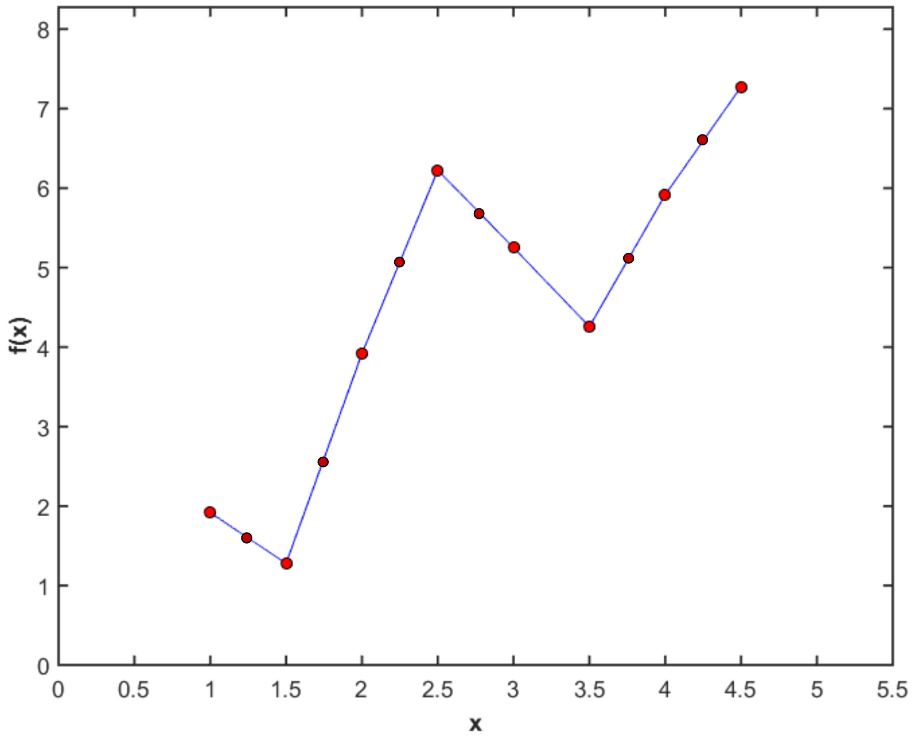
2. 1D Linear Interpolation

Repeat the process between all pairs of points to interpolate all values between.

If we need more interpolated Values, then use more than just .5.

Interpolate at $0<x<1$

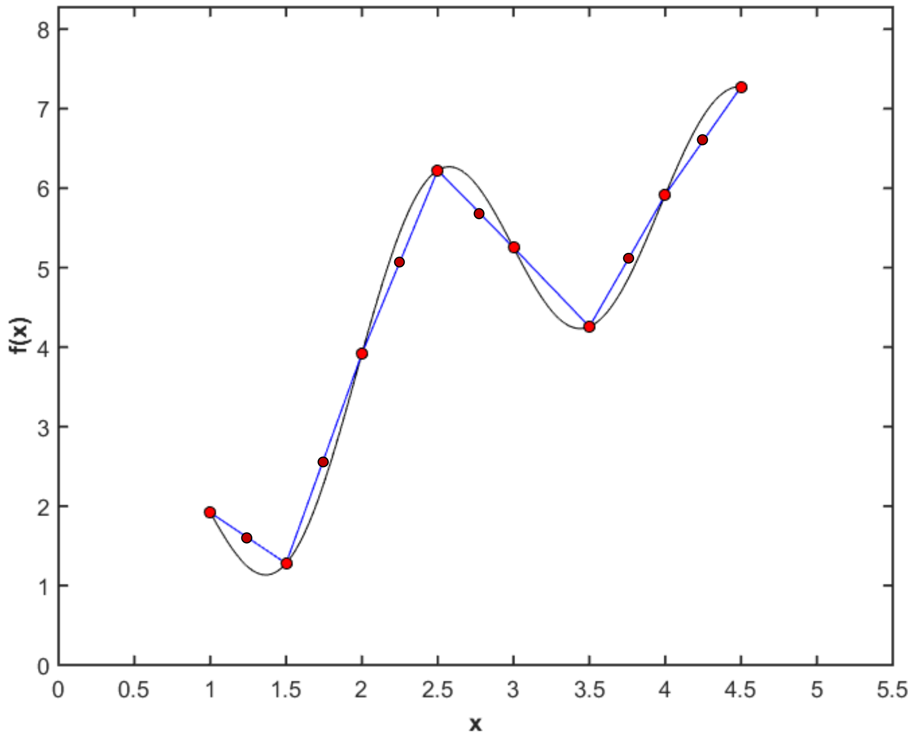
$$f(x) = a_0 x^0 + a_1 x^1$$



2. 1D Linear Interpolation

Repeat the process between all pairs of points to interpolate all values between.

But regardless of how many points we interpolate, the intrinsic curvature through the points is not captured!



3. 1D Cubic Interpolation

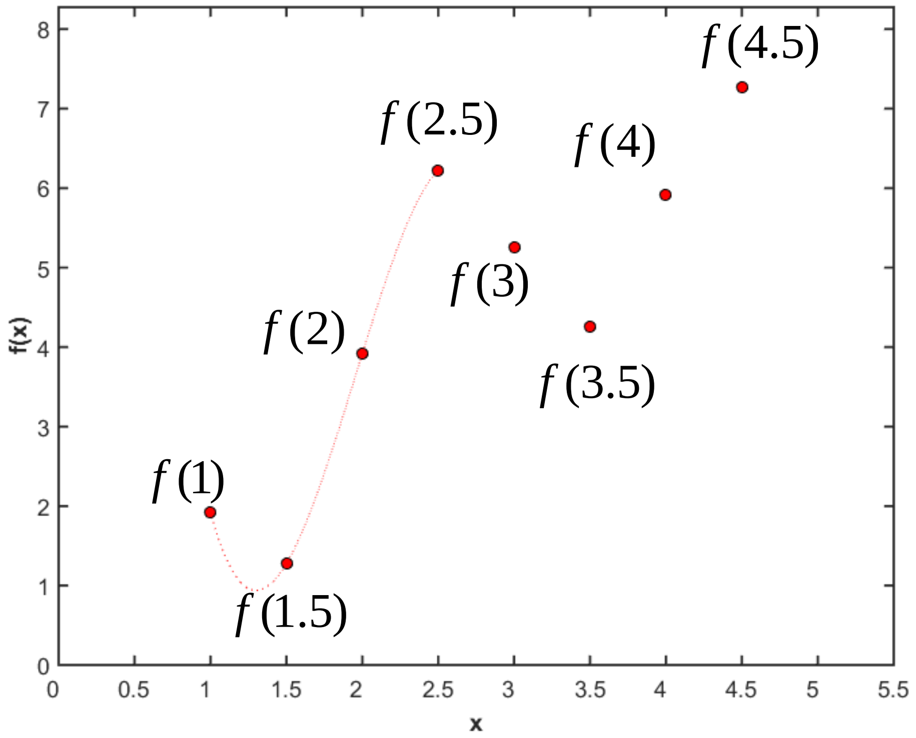
Using adjacent points, we can estimate a cubic (third order polynomial) between points.

Interpolate:

Four points define a cubic equation.

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

Find the coefficients of the cubic eqn. between points.

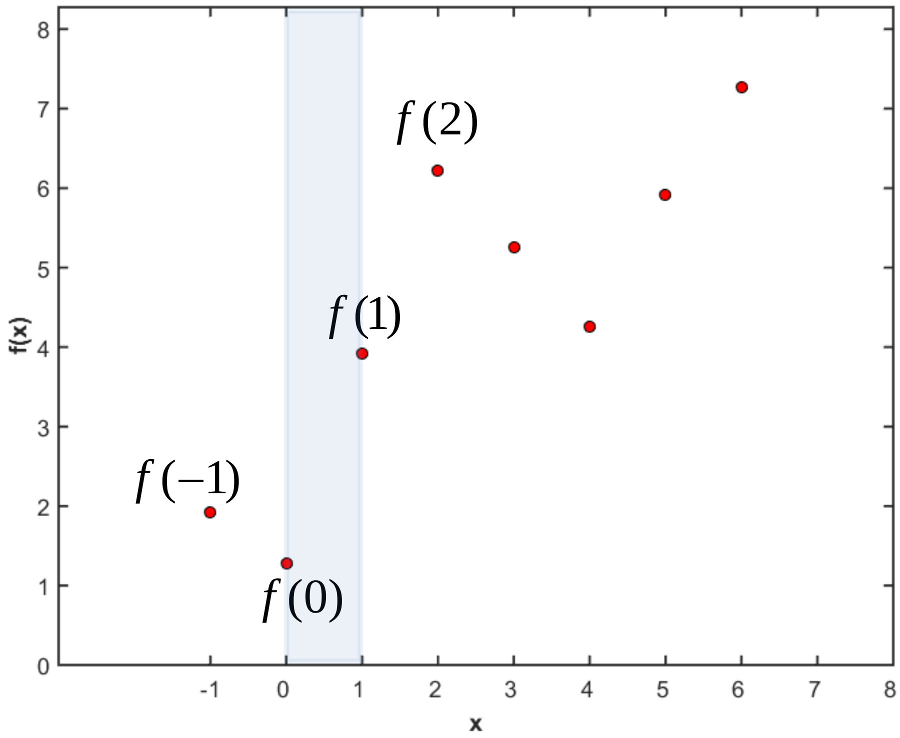


3. 1D Cubic Interpolation

Using adjacent points, we can estimate a cubic (third order polynomial) between points.

Normalization: $f(0), f(1)$

For regularly spaced points.



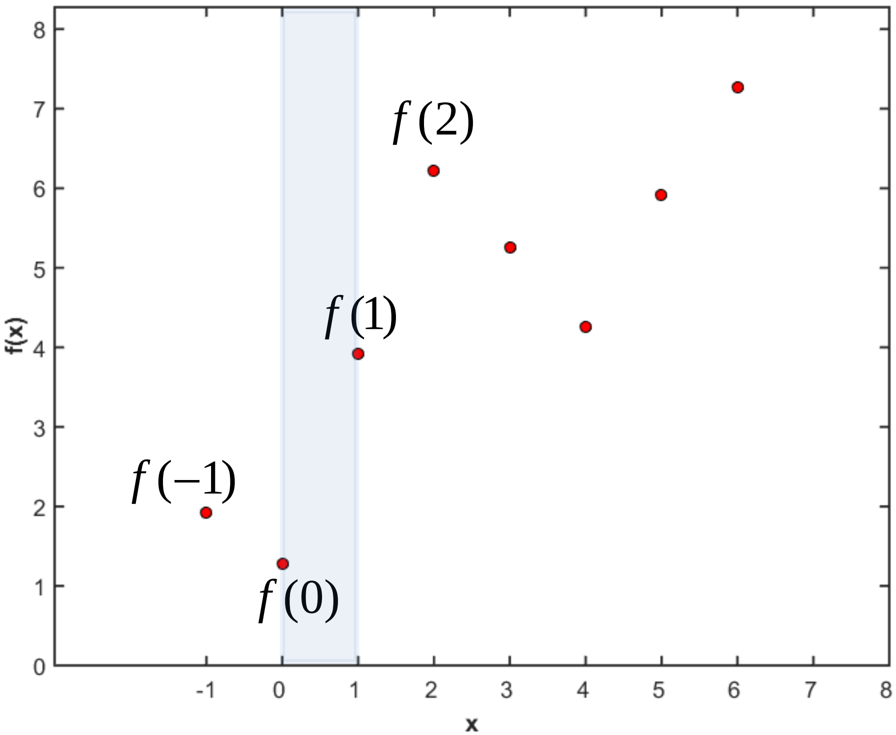
3. 1D Cubic Interpolation

Using adjacent points, we can estimate a cubic (third order polynomial) between points.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$$x = -1, 0, 1, 2$$



3. 1D Cubic Interpolation

Using adjacent points, we can estimate a cubic (third order polynomial) between points. 4 equations, 4 unknowns

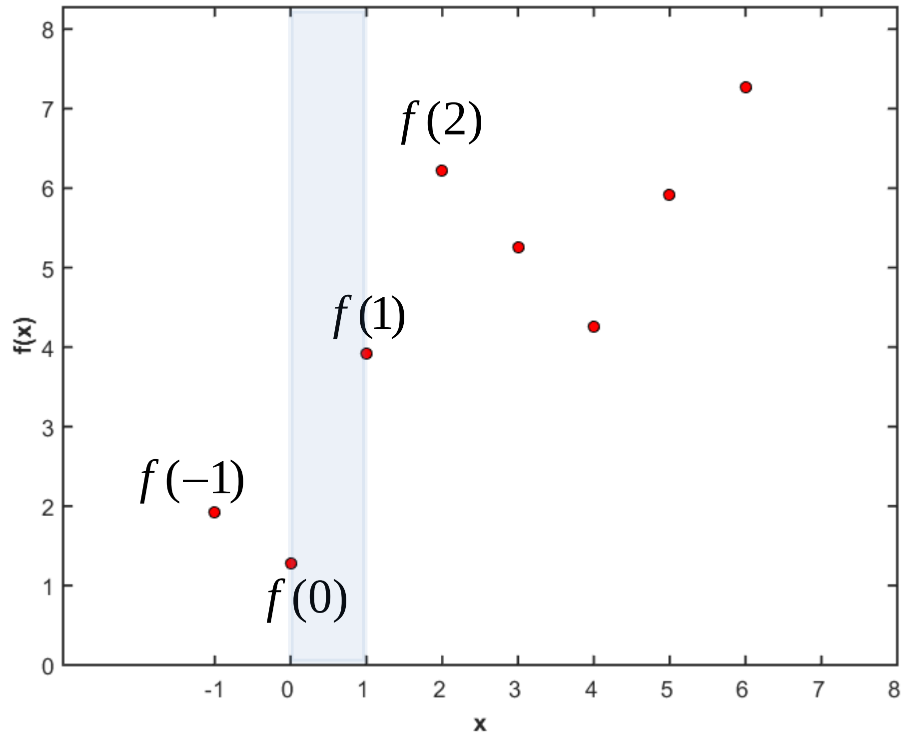
Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

$$f(-1) = a_0(-1)^0 + a_1(-1)^1 + a_2(-1)^2 + a_3(-1)^3$$
$$f(0) = a_0(0)^0 + a_1(0)^1 + a_2(0)^2 + a_3(0)^3$$
$$f(1) = a_0(1)^0 + a_1(1)^1 + a_2(1)^2 + a_3(1)^3$$
$$f(2) = a_0(2)^0 + a_1(2)^1 + a_2(2)^2 + a_3(2)^3$$



$$f(-1) = a_0(-1)^0 + a_1(-1)^1 + a_2(-1)^2 + a_3(-1)^3$$
$$f(0) = a_0(0)^0 + a_1(0)^1 + a_2(0)^2 + a_3(0)^3$$
$$f(1) = a_0(1)^0 + a_1(1)^1 + a_2(1)^2 + a_3(1)^3$$
$$f(2) = a_0(2)^0 + a_1(2)^1 + a_2(2)^2 + a_3(2)^3$$

3. 1D Cubic Interpolation

System of Equations: 4 equations, 4 unknowns

System of Equations

$$\begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

previous point

current points

next point

Solution

$$y = Xa$$

Solution

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 0 & 6 & 0 & 0 \\ -2 & -3 & 6 & -1 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

recycle matrix

Solution

$$a = X^{-1}y$$

3. 1D Cubic Interpolation

System of Equations: 4 equations, 4 unknowns

Solution

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 6 & 0 & 0 \\ -2 & -3 & 6 & -1 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

Interpolate at $0<x<1$

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

Solution

$$a = X^{-1}y$$

Interpolate at $0<x<1$

$$f(x) = [x^0, x^1, x^2, x^3]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

3. 1D Cubic Interpolation

System of Equations: 4 equations, 4 unknowns

Interpolate at $0<x<1$

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

Interpolate at $0<x<1$

$$f(x) = [x^0, x^1, x^2, x^3]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Interpolate at $.5$

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$

Interpolate at $.5$

$$f(.5) = [.5^0, .5^1, .5^2, .5^3]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

3. 1D Cubic Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

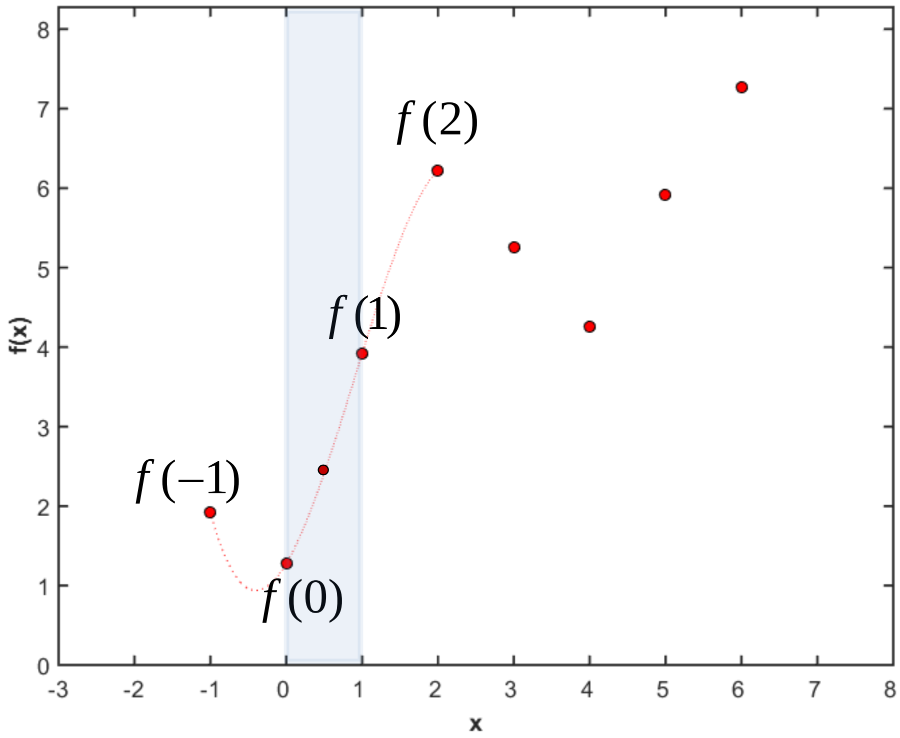
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

Interpolate:

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



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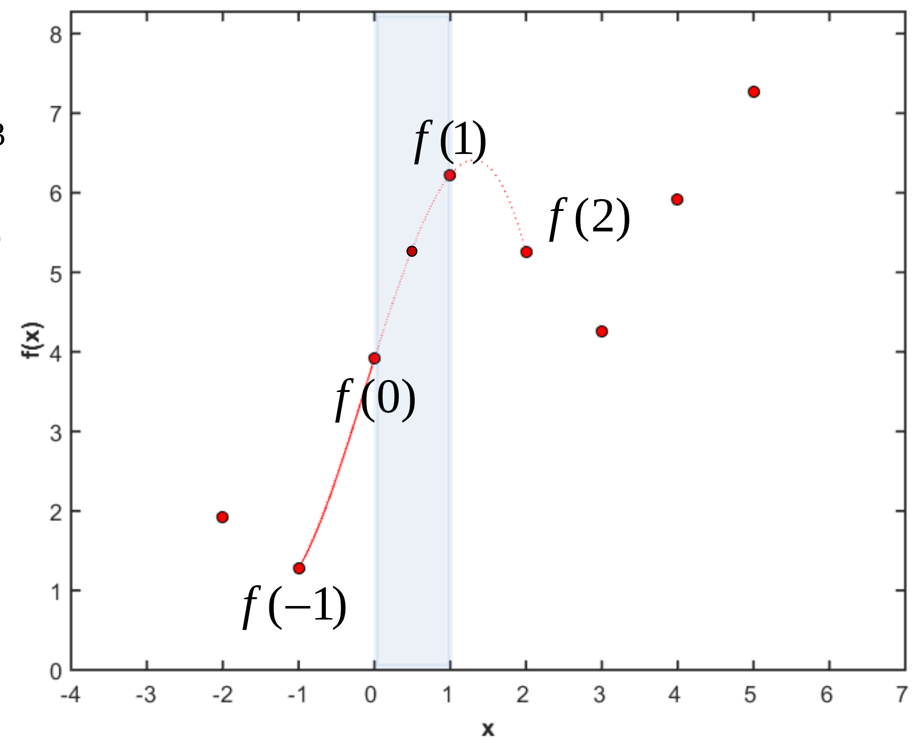
$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

Interpolate:

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



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This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

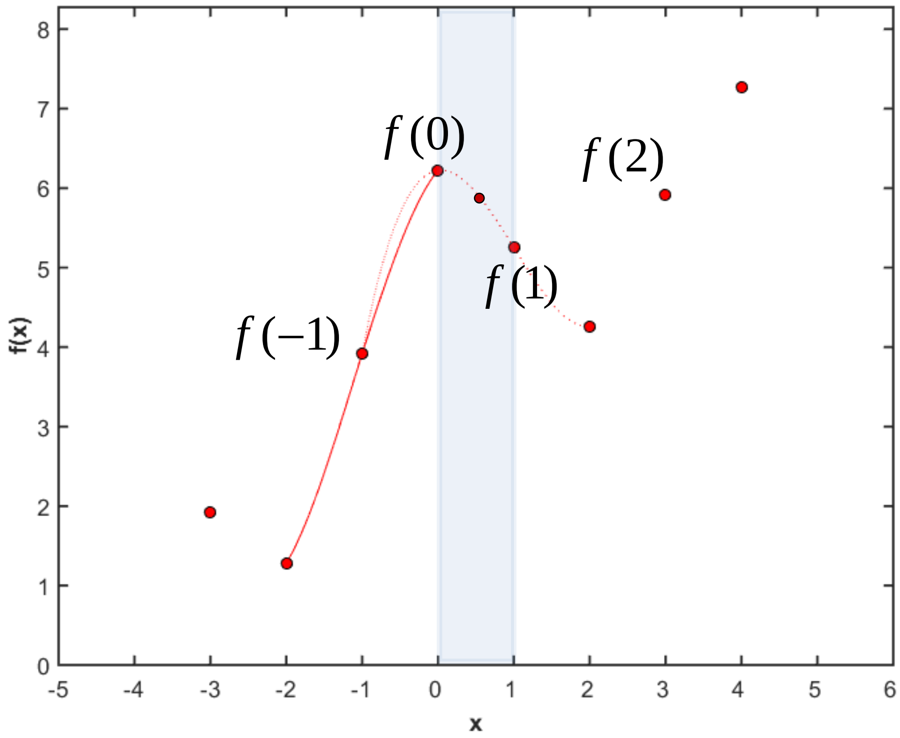
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

Interpolate:

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



3. 1D Cubic Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

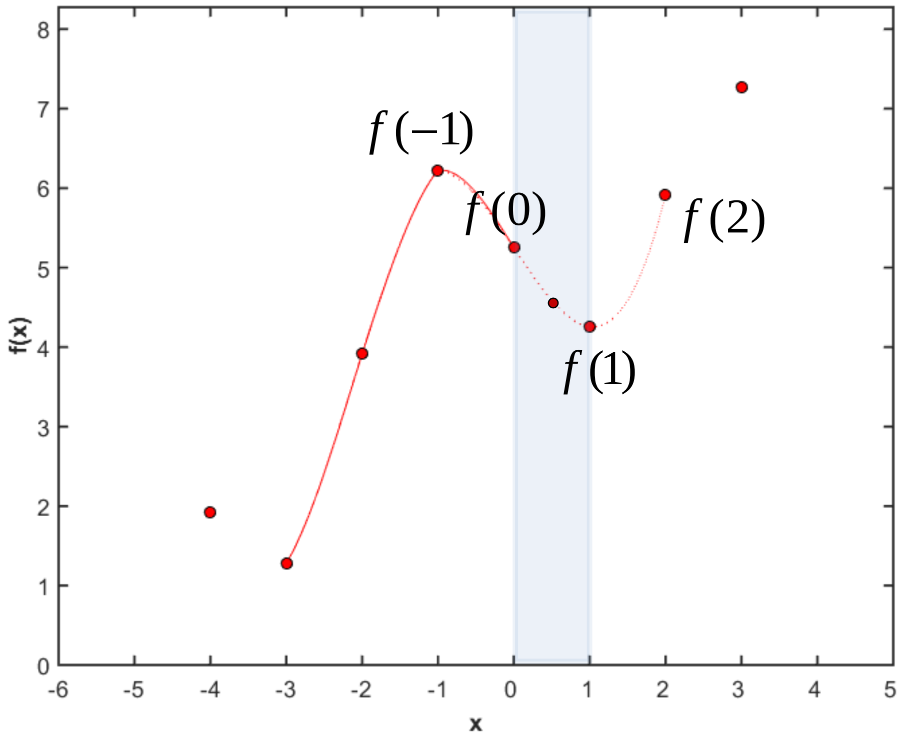
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

Interpolate:

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



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This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

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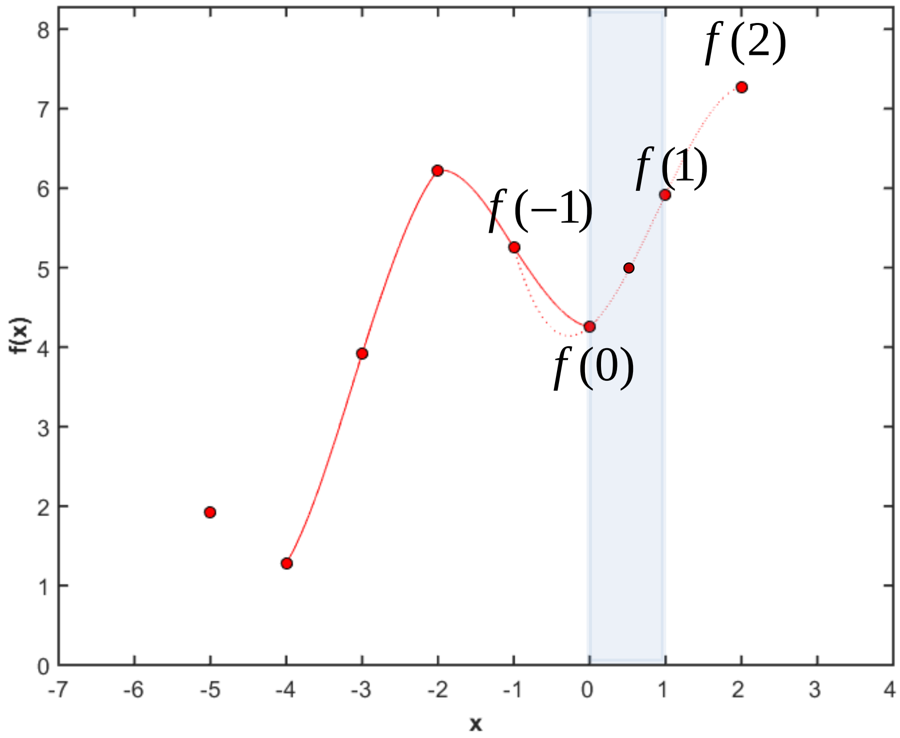
$x = -1, 0, 1, 2$

Solve: (a_0, a_1, a_2, a_3)

Interpolate:

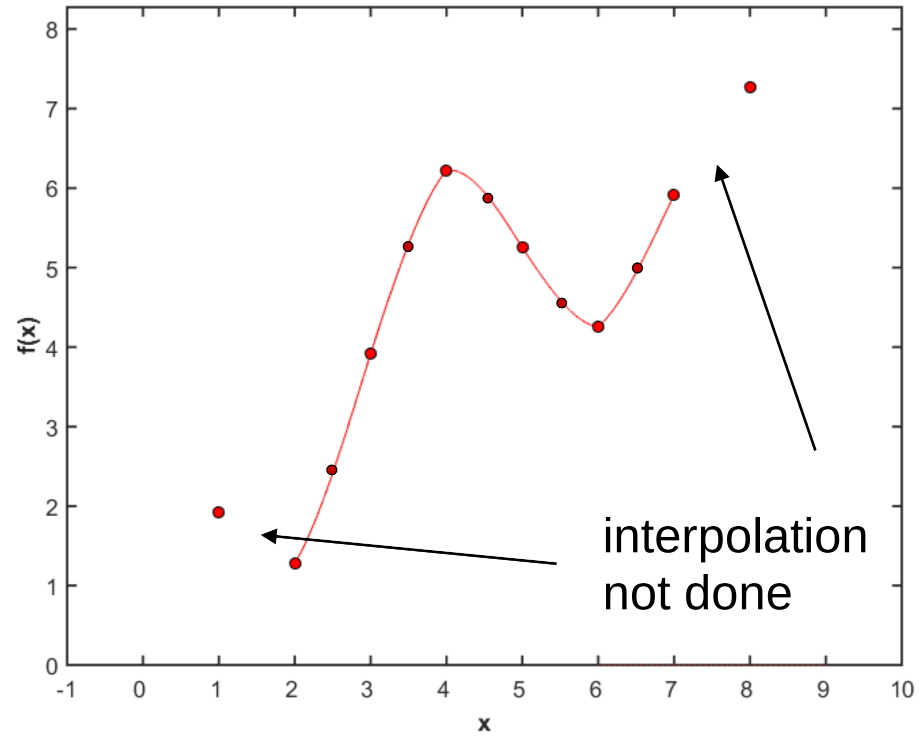
$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



3. 1D Cubic Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Return to unnormalized axis.

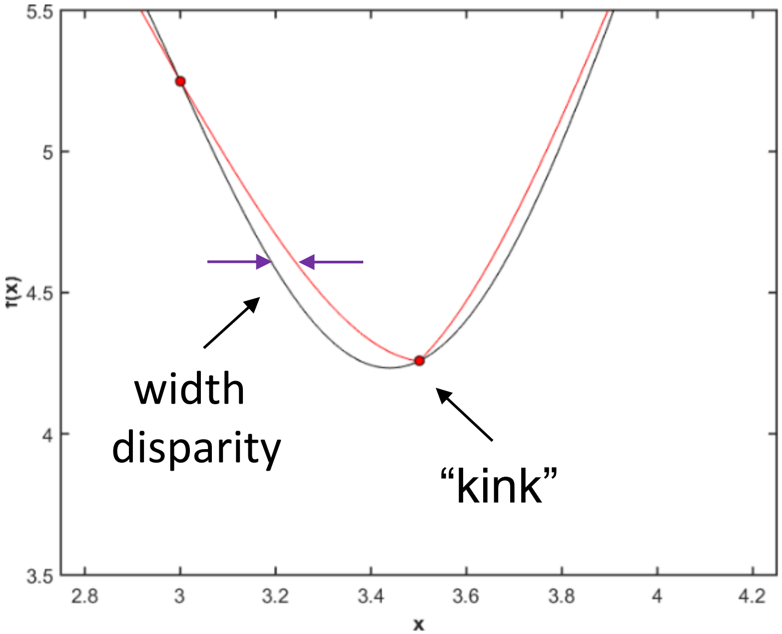


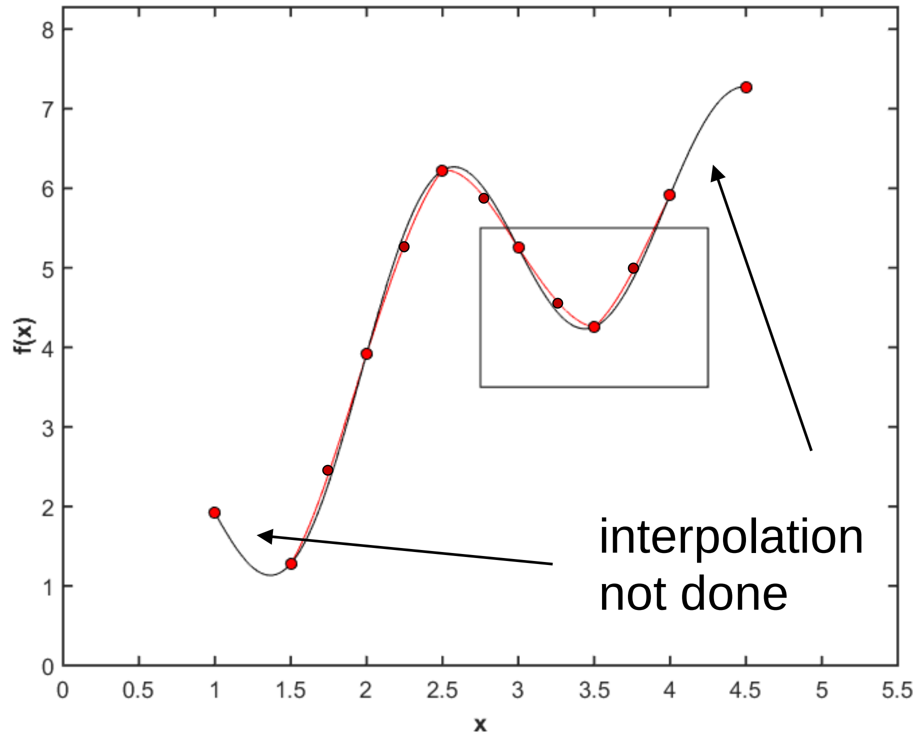
Interpolation not done
-ends of cubic
-linear ends
-quadratic ends

3. 1D Cubic Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Cubic still “kinky” at points.

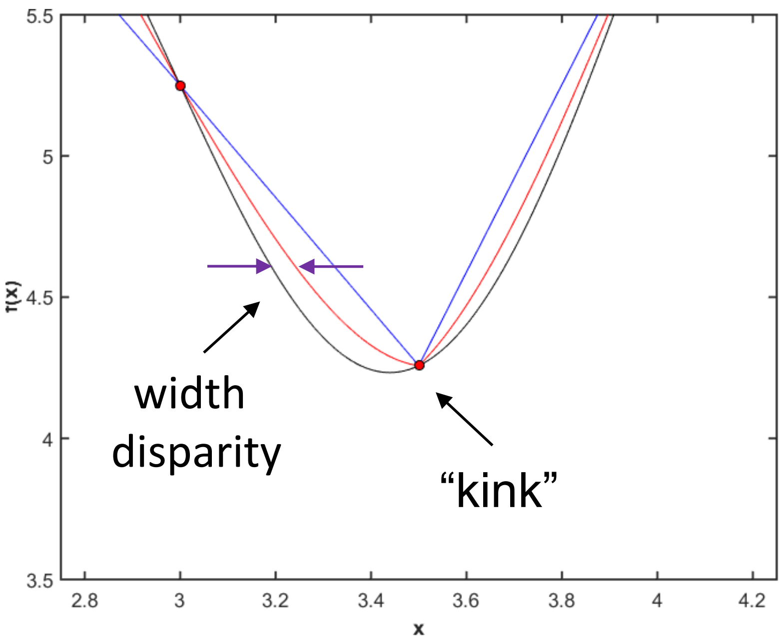


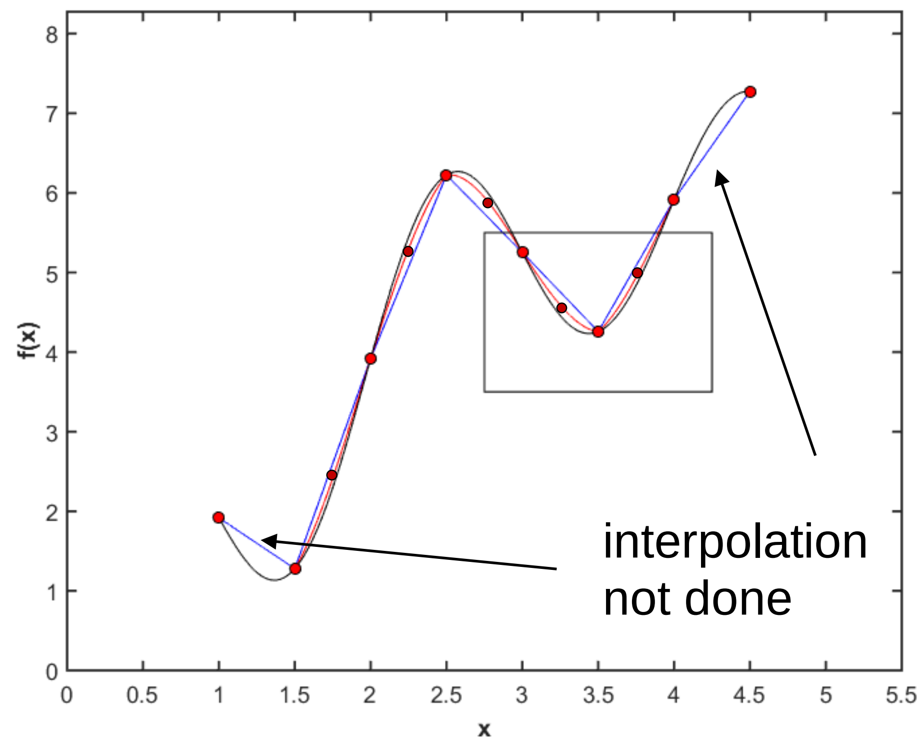


3. 1D Cubic Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

But much better than linear!





4. 1D Cubic Spline Interpolation

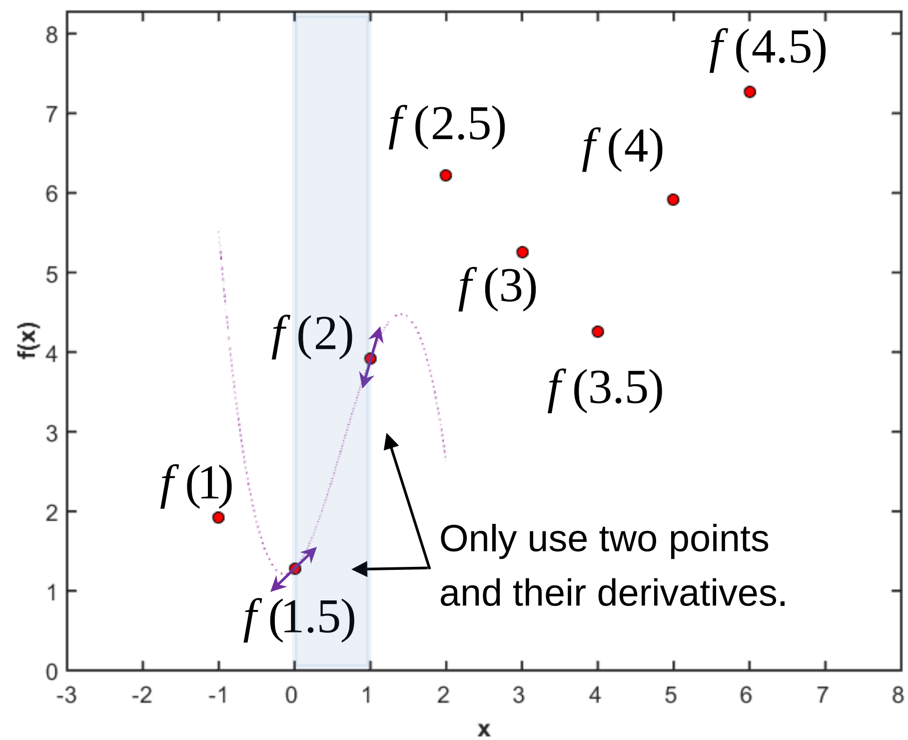
With two points and two derivatives we can fit a cubic polynomial between points.

Interpolate:

Two points plus two derivatives. No "kinks?"

Smooth transition through.

Find the equation of the cubic eqn. between points.

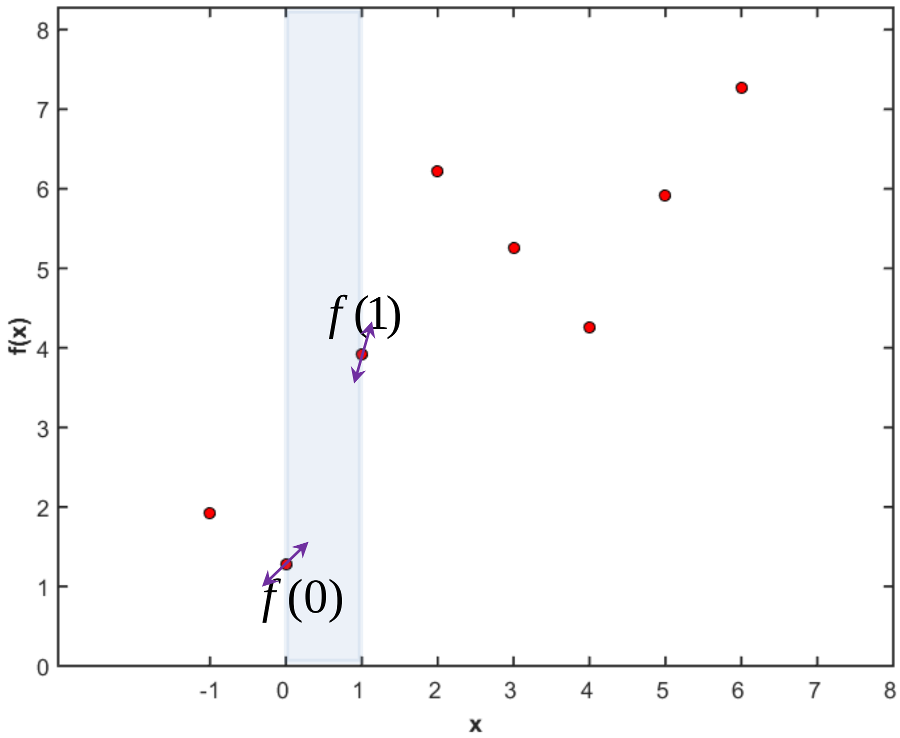


4. 1D Cubic Spline Interpolation

With two points and two derivatives we can fit a cubic polynomial between points.

Normalization: $f(0), f(1)$

For regularly spaced points.



4. 1D Cubic Spline Interpolation

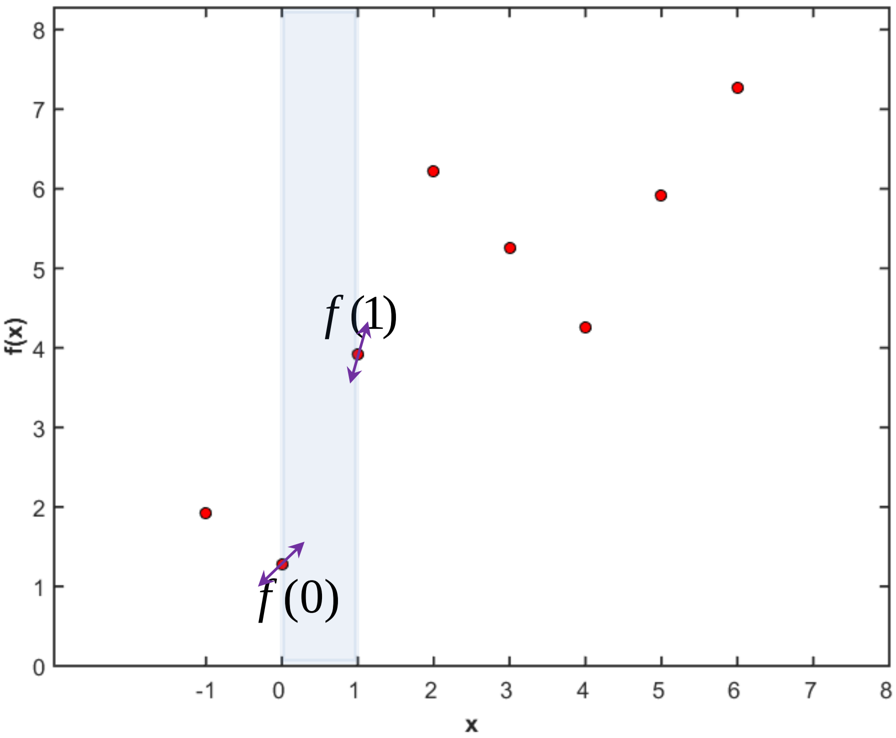
With two points and two derivatives we can fit a cubic polynomial between points.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$

$$f'(x) = 1a_1x^0 + 2a_2x^1 + 3a_3x^2$$

$$x = -1, 0, 1, 2$$



4. 1D Cubic Spline Interpolation

With two points and two derivatives we can fit a cubic polynomial between points.

4 equations, 4 unknowns

Normalization: $f(0), f(1)$

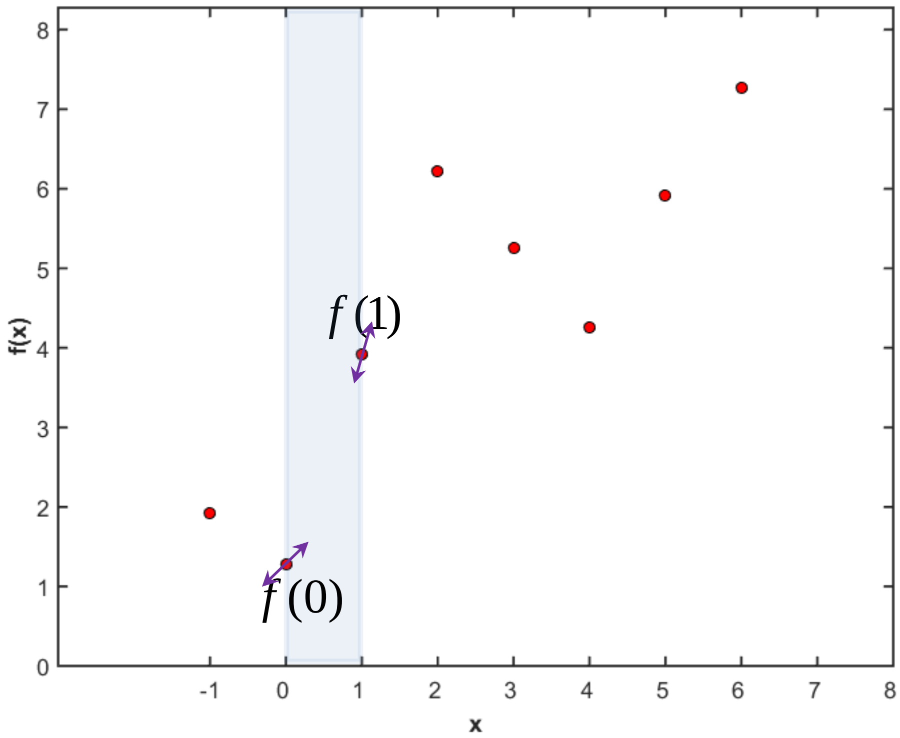
Model: $f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$

$f'(x) = 1a_1x^0 + 2a_2x^1 + 3a_3x^2$

Solve: (a_0, a_1, a_2, a_3)

$x = -1, 0, 1, 2$

$$f(0) = a_0(0)^0 + a_1(0)^1 + a_2(0)^2 + a_3(0)^3$$
$$f(1) = a_0(1)^0 + a_1(1)^1 + a_2(1)^2 + a_3(1)^3$$
$$f'(0) = a_1(0)^0 + 2a_2(0)^1 + 3a_3(0)^2$$
$$f'(1) = a_1(1)^0 + 2a_2(1)^1 + 3a_3(1)^2$$



4. 1D Cubic Spline Interpolation

With two points and two derivatives we can fit a cubic polynomial between points.

4 equations, 4 unknowns

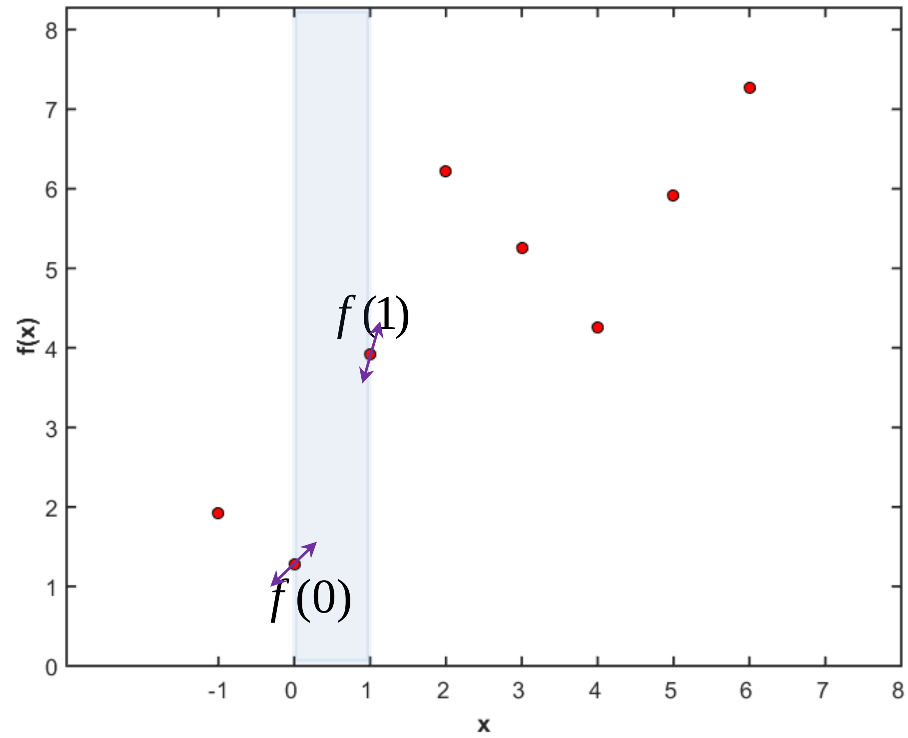
Need time series discrete derivatives at $x=0,1$.

Derivative at $x=0$:

$$f'(0) = \frac{f(1) - f(-1)}{2}$$

Derivative at $x=1$:

$$f'(1) = \frac{f(2) - f(0)}{2}$$



4. 1D Cubic Spline Interpolation

$$f(0) = a_0(0)^0 + a_1(0)^1 + a_2(0)^2 + a_3(0)^3$$
$$f(1) = a_0(1)^0 + a_1(1)^1 + a_2(1)^2 + a_3(1)^3$$
$$f'(0) = a_1(0)^0 + 2a_2(0)^1 + 3a_3(0)^2$$
$$f'(1) = 1a_1(1)^0 + 2a_2(1)^1 + 3a_3(1)^2$$

System of Equations: 4 equations,

4 unknowns

note: don't use *f*(-1) and *f*(2)!
If did, 6 eqn. 4 unknown,
interpolation not through points.

System of Equations $y = Xa$

$$\begin{bmatrix} f(0) \\ f(1) \\ f'(0) \\ f'(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Derivatives $y = Df$

$$\begin{bmatrix} f(0) \\ f(1) \\ f'(0) \\ f'(1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

Solution $a = X^{-1}y = X^{-1}Df$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

recycle matrix

4. 1D Cubic Spline Interpolation

System of Equations: 4 equations, 4 unknowns

Solution for cubic spline

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

Interpolate at $0<x<1$

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

Interpolate at .5

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$

Solution for cubic

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 6 & 0 & 0 \\ -2 & -3 & 6 & -1 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

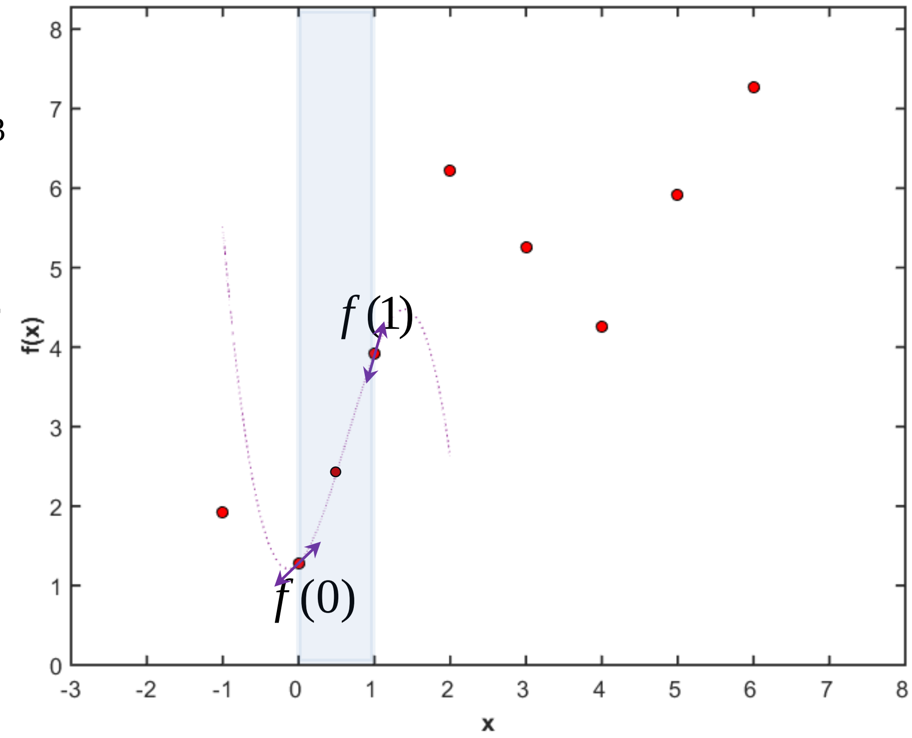
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$$f'(x) = 1a_1 x^0 + 2a_2 x^1 + 3a_3 x^2$$

Solve: (a_0, a_1, a_2, a_3) $\quad x = -1, 0, 1, 2$

Interpolate: .5

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

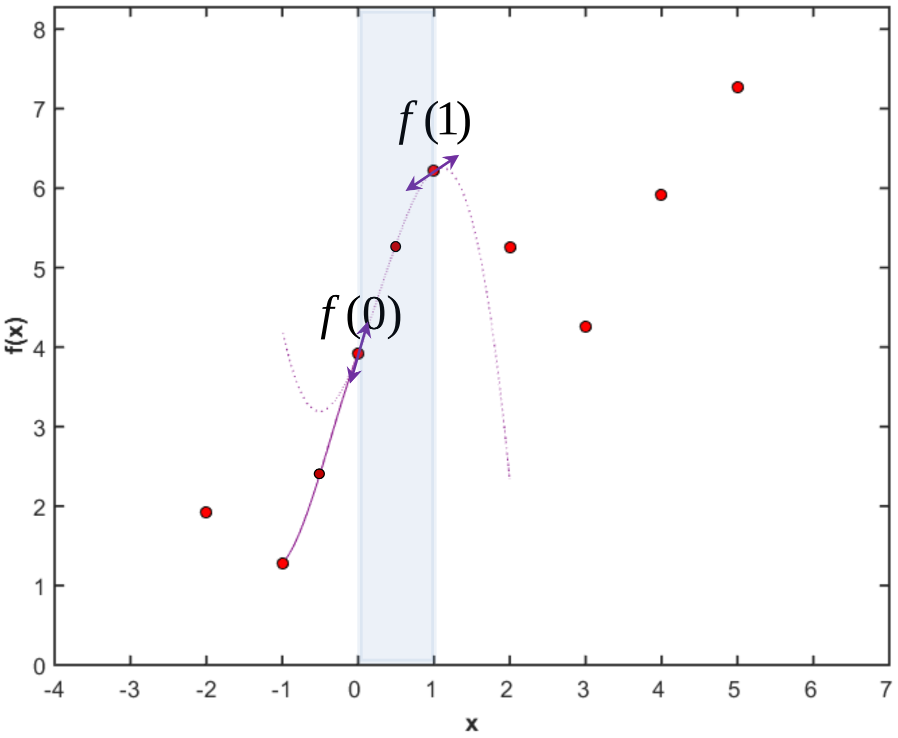
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$$f'(x) = 1a_1 x^0 + 2a_2 x^1 + 3a_3 x^2$$

Solve: (a_0, a_1, a_2, a_3) $\quad x = -1, 0, 1, 2$

Interpolate: .5

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

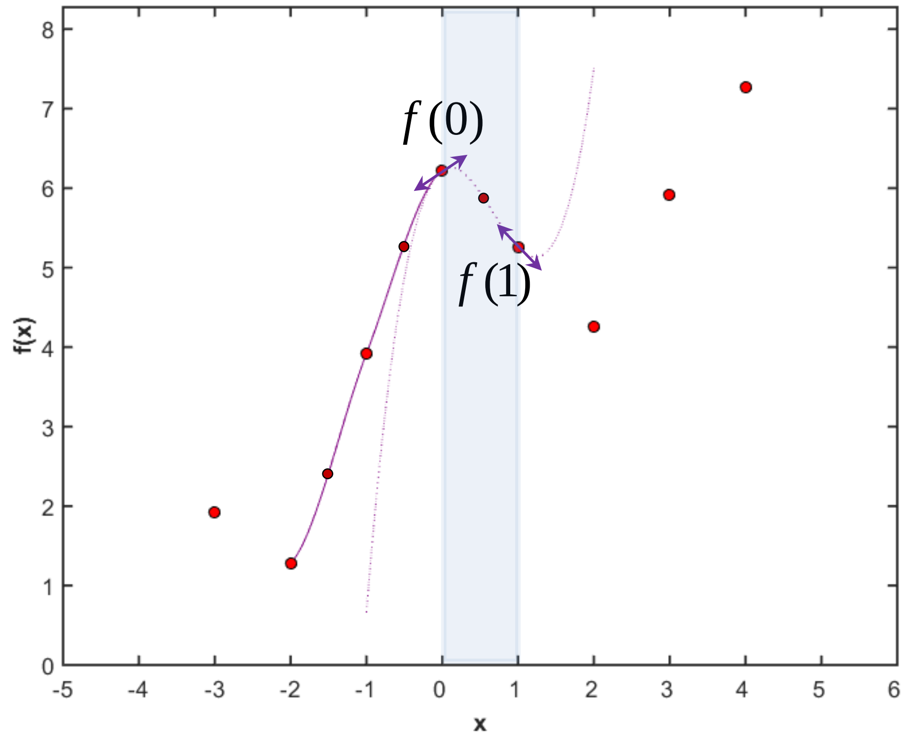
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$f'(x) = 1 a_1 x^0 + 2 a_2 x^1 + 3 a_3 x^2$

Solve: (a_0, a_1, a_2, a_3) $\quad x = -1, 0, 1, 2$

Interpolate: .5

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

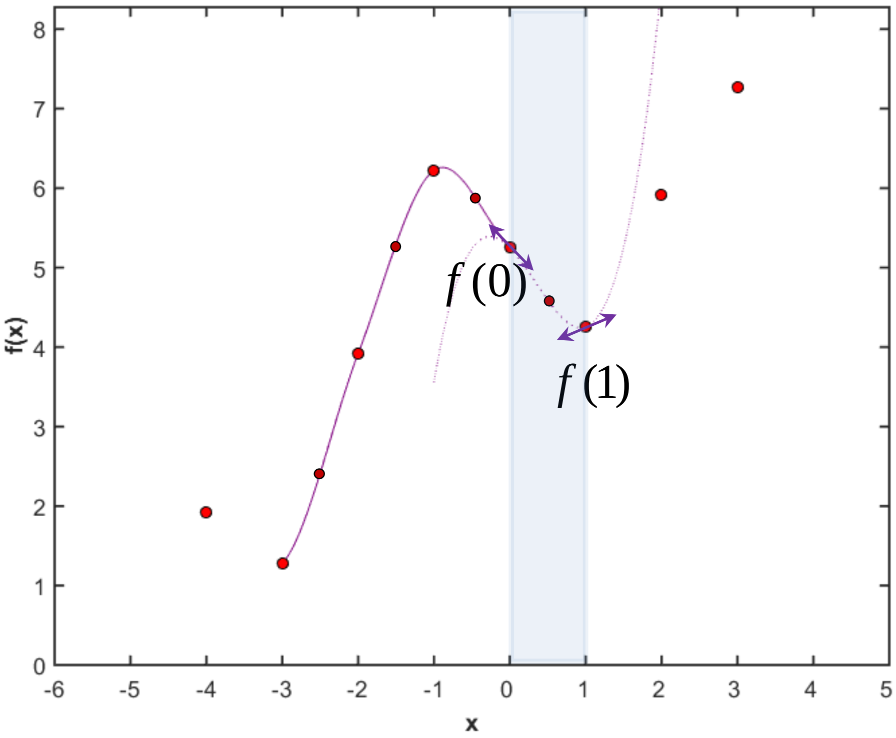
Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

$$f'(x) = 1a_1 x^0 + 2a_2 x^1 + 3a_3 x^2$$

Solve: (a_0, a_1, a_2, a_3) $\quad x = -1, 0, 1, 2$

Interpolate: .5

$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Normalization: $f(0), f(1)$

Model: $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

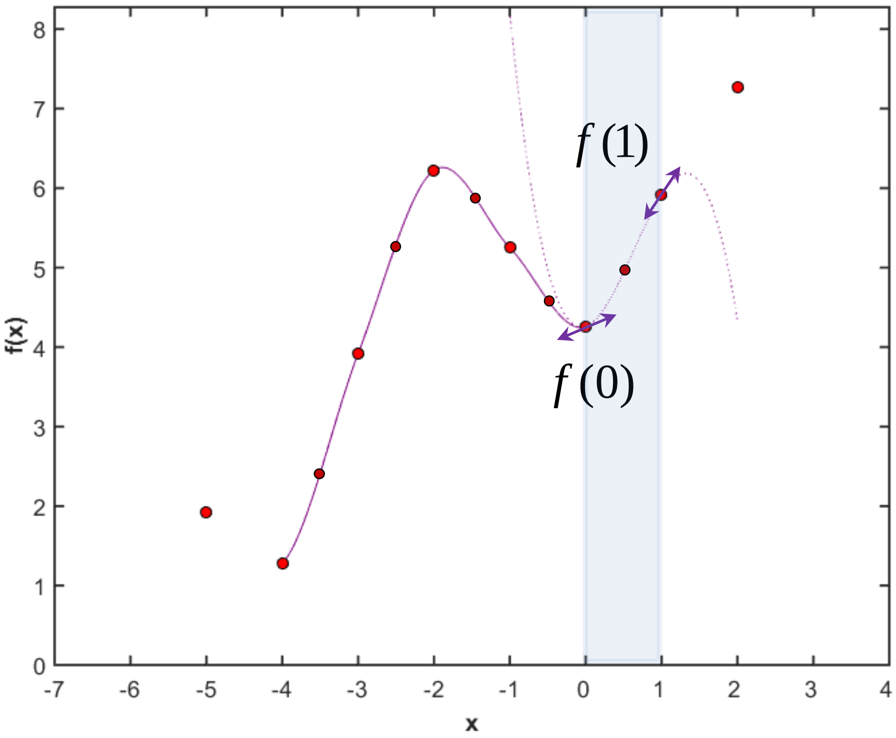
$$f'(x) = 1a_1 x^0 + 2a_2 x^1 + 3a_3 x^2$$

$$x = -1, 0, 1, 2$$

Solve: (a_0, a_1, a_2, a_3)

Interpolate: .5

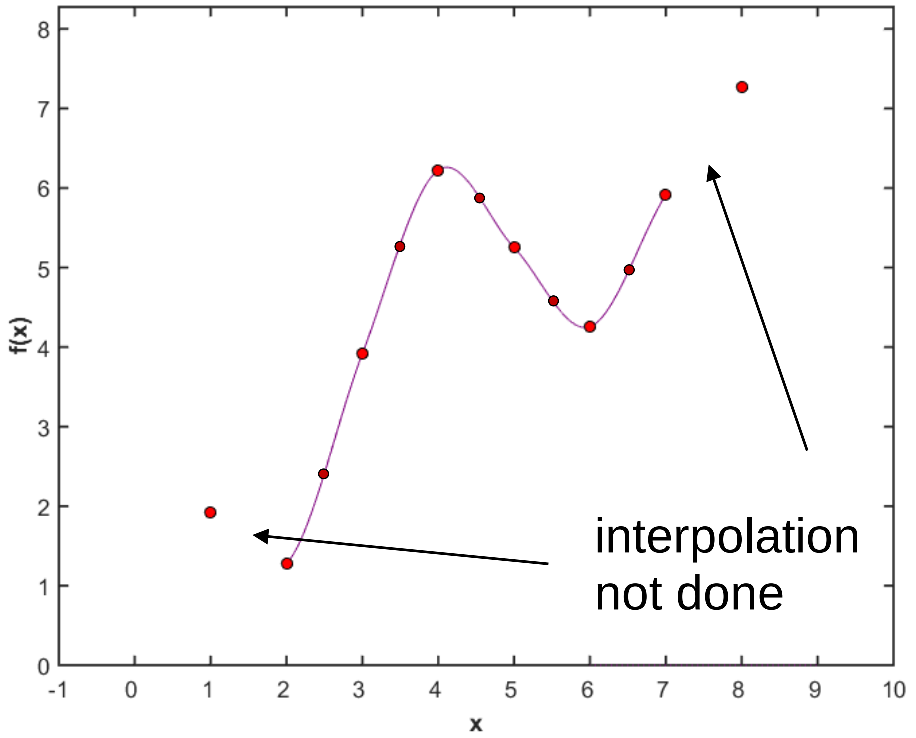
$$f(.5) = a_0 .5^0 + a_1 .5^1 + a_2 .5^2 + a_3 .5^3$$



4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

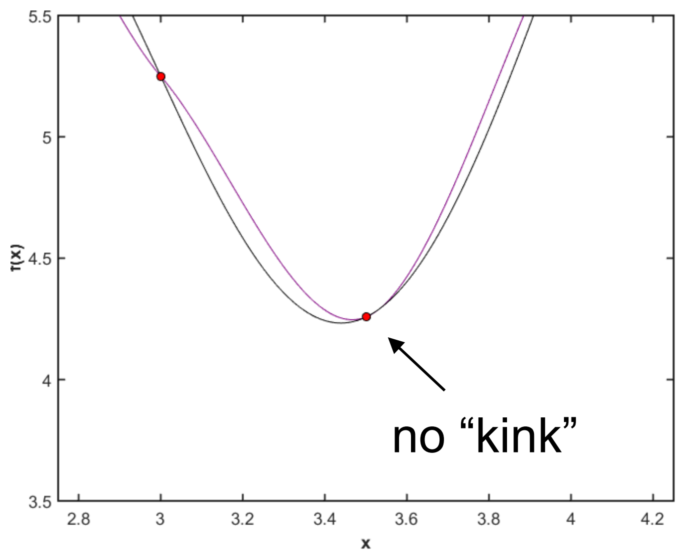
Return to unnormalized axis.

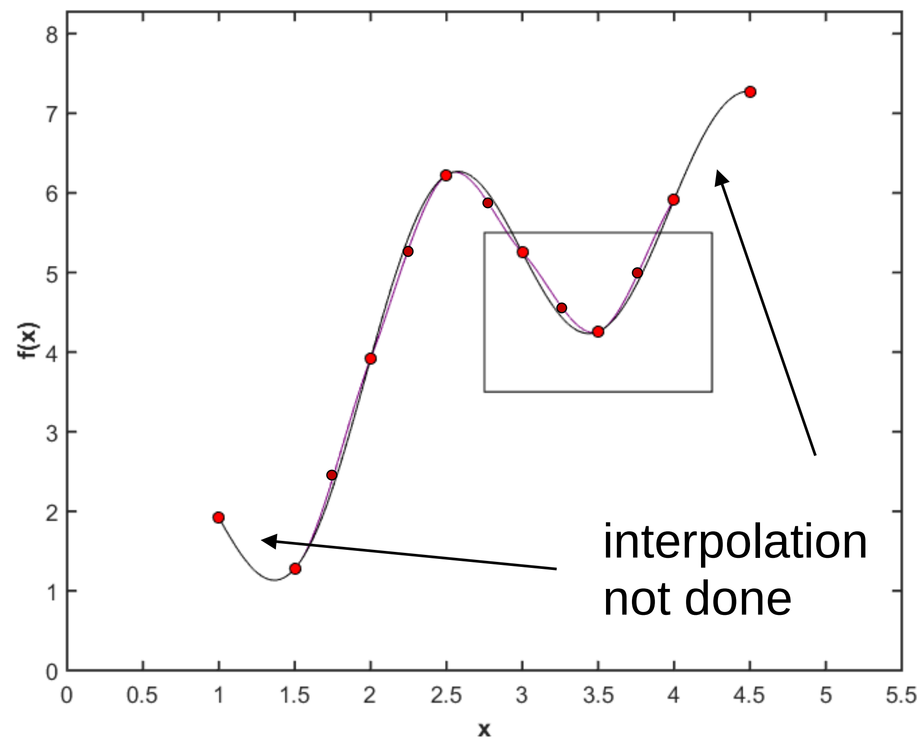


4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

Cubic Spline no “kinks.”

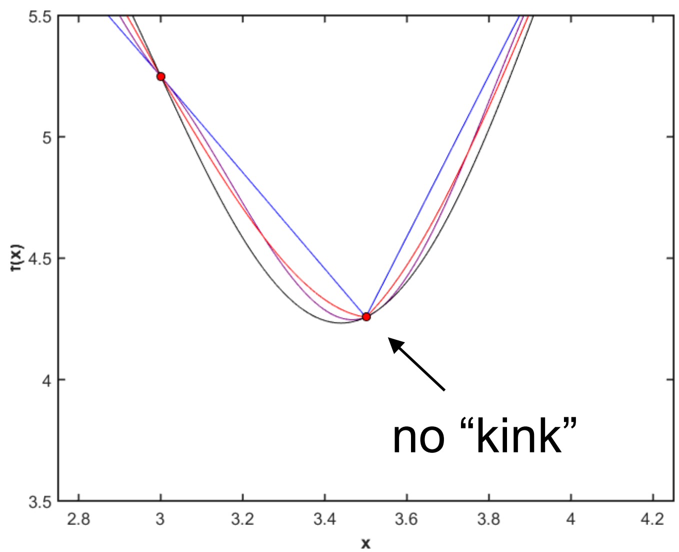


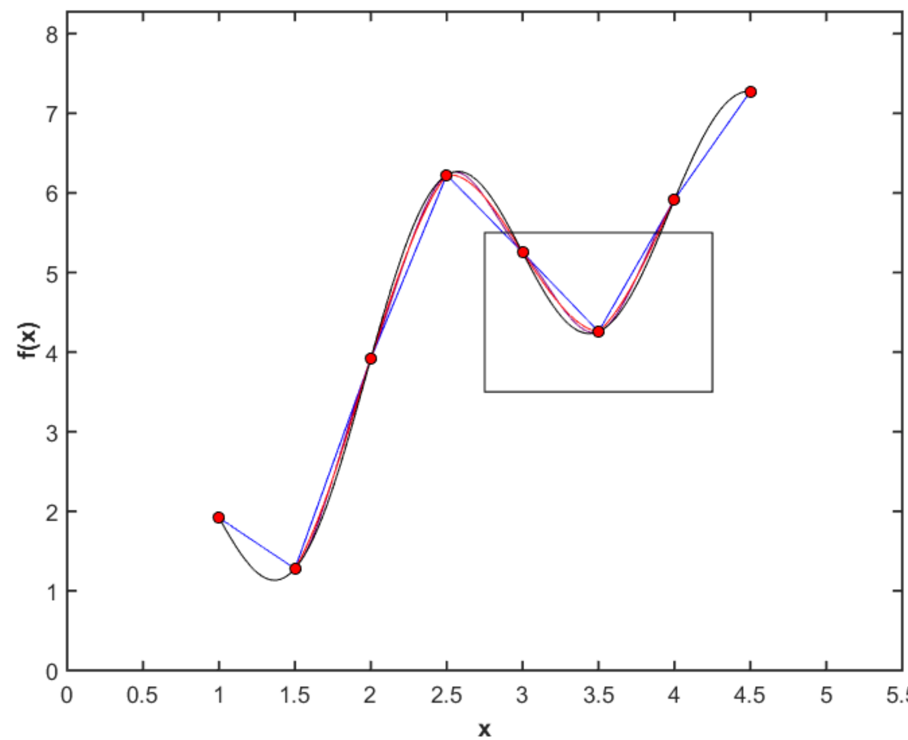


4. 1D Cubic Spline Interpolation

This involves 4 unknowns and 4 points to estimate them, then points along the polynomial.

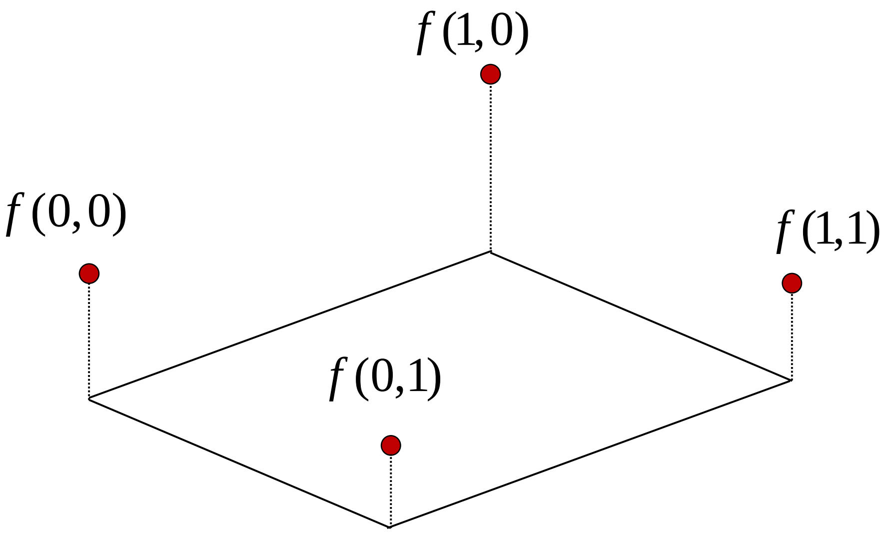
Cubic Spline better at points!





5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

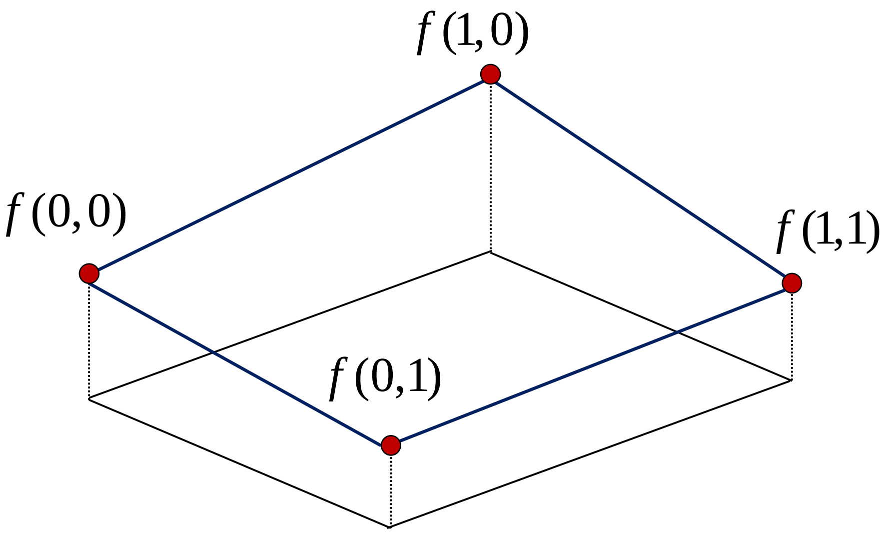


array/image coordinate system

5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

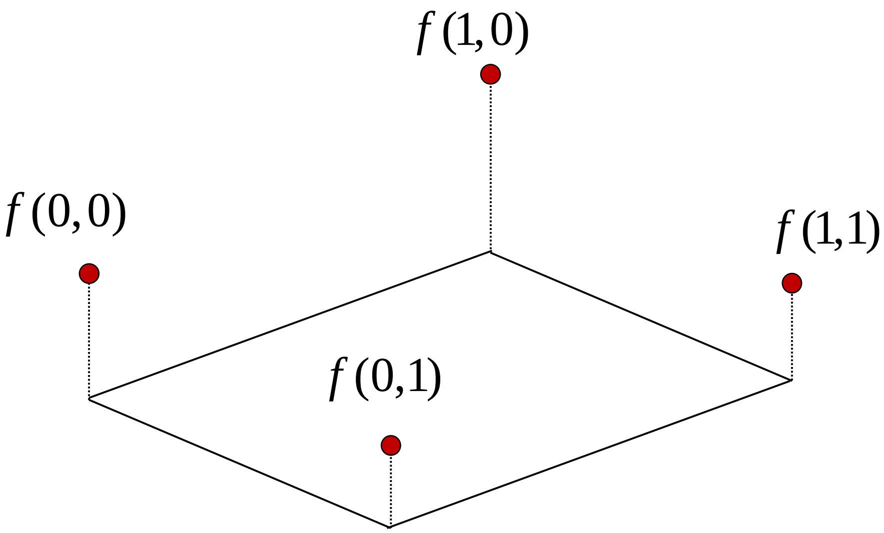
Interpolate:
Find the equation of the plane between points.



5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

Normalization: $f(0,0),\ f(1,0)$
$f(0,1),\ f(1,1)$



5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

Normalization: $f(0,0),\ f(1,0)$

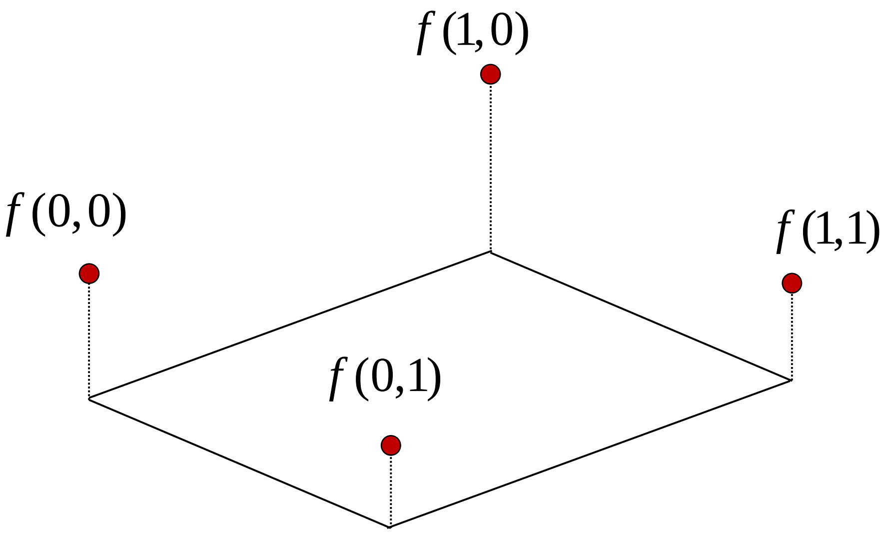
$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{1}\sum_{i=0}^{1}a_{ij}x^i y^j$$

$$x=-1,0,1,2$$

$$f(x,y)=a_{00}+a_{10}x+a_{01}y+a_{11}xy$$

$$x,y=-1,0,1,2$$



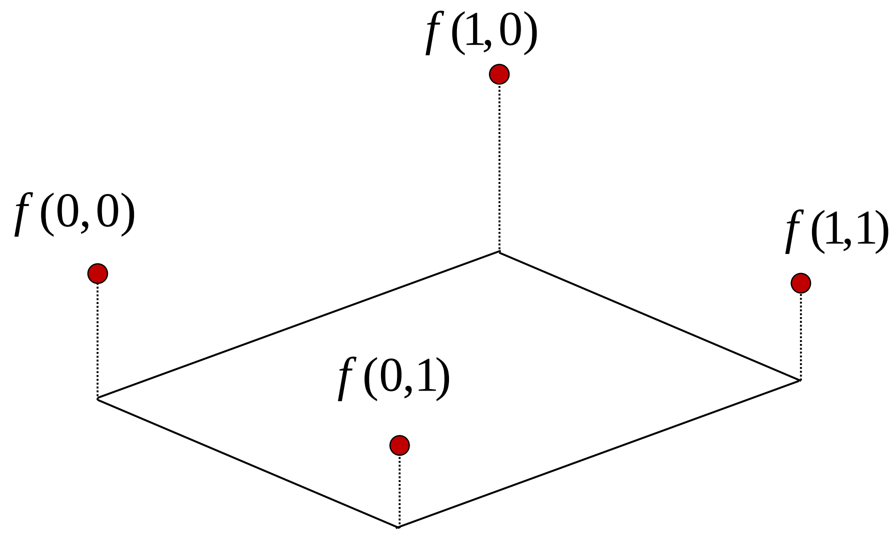
5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

Normalization: $f(0,0),\ f(1,0)$
$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{1}\sum_{i=0}^{1}a_{ij}x^i y^j$$

$$x=-1,0,1,2$$



Solve: a_{ij}

$$f(0,0)=a_{00}$$
$$f(1,0)=a_{00}+a_{10}$$
$$f(1,0)=a_{00}+a_{01}$$
$$f(1,1)=a_{00}+a_{10}+a_{01}+a_{11}$$

5. 2D BiLinear Interpolation

System of Equations: 4 equations, 4 unknow

$$f(0,0) = a_{00}$$
$$f(1,0) = a_{00} + a_{10}$$
$$f(1,0) = a_{00} + a_{01}$$
$$f(1,1) = a_{00} + a_{10} + a_{01} + a_{11}$$

System of Equations $y = Xa$

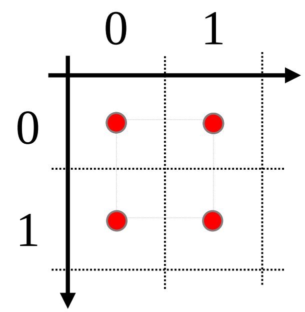
$$\begin{bmatrix} f(0,0) \\ f(1,0) \\ f(0,1) \\ f(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}$$

Solution $a = X^{-1} y$

$$\begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) \\ f(1,0) \\ f(0,1) \\ f(1,1) \end{bmatrix}$$

recycle matrix

Image



Interpolate

$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy$$

$0<x<1,\ 0<y<1$

5. 2D BiLinear Interpolation

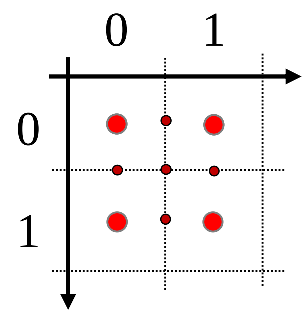
System of Equations: 4 equations, 4 unknow

$$f(0,0)=a_{00}$$
$$f(1,0)=a_{00}+a_{10}$$
$$f(1,0)=a_{00}+a_{01}$$
$$f(1,1)=a_{00}+a_{10}+a_{01}+a_{11}$$

Solution $a=X^{-1}y$

$$\begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) \\ f(1,0) \\ f(0,1) \\ f(1,1) \end{bmatrix}$$

$$f(x,y)=a_{00}+a_{10}x+a_{01}y+a_{11}xy$$



Interpolate one pixel $y_{\text{int}}=X_{\text{int}}a$

$$f(.5,0)=[1,.5,0,0]a$$

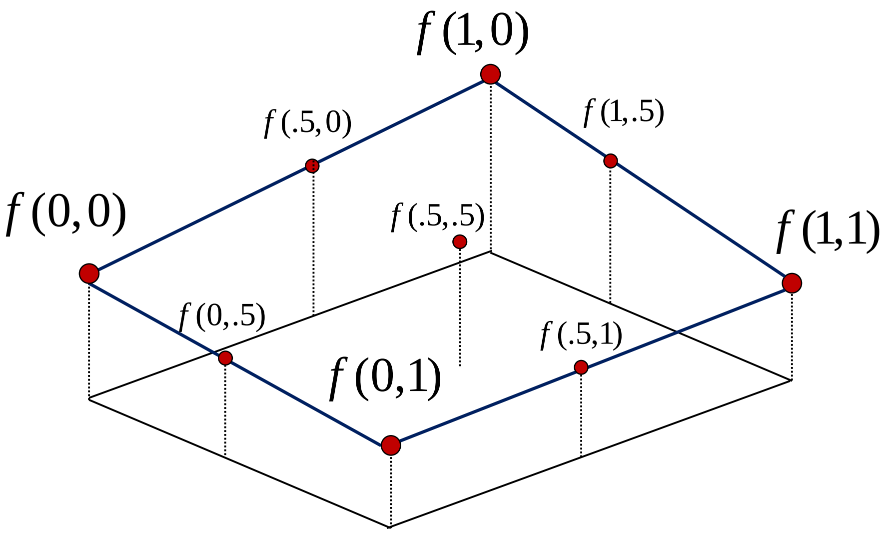
More rows to interpolate more pixels.

$$\begin{bmatrix} f(.5,0) \\ f(1,.5) \\ f(.5,.5) \\ f(0,.5) \\ f(.5,1) \end{bmatrix} = \begin{bmatrix} 1 & .5 & 0 & 0 \\ 1 & 1 & .5 & .5 \\ 1 & .5 & .5 & .25 \\ 1 & 0 & .5 & 0 \\ 1 & .5 & 1 & .5 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$$

5. 2D BiLinear Interpolation

Easiest to draw planes between points and use values within the planes.

Once we've solved for the coefficients, we interpolate.



$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy$$

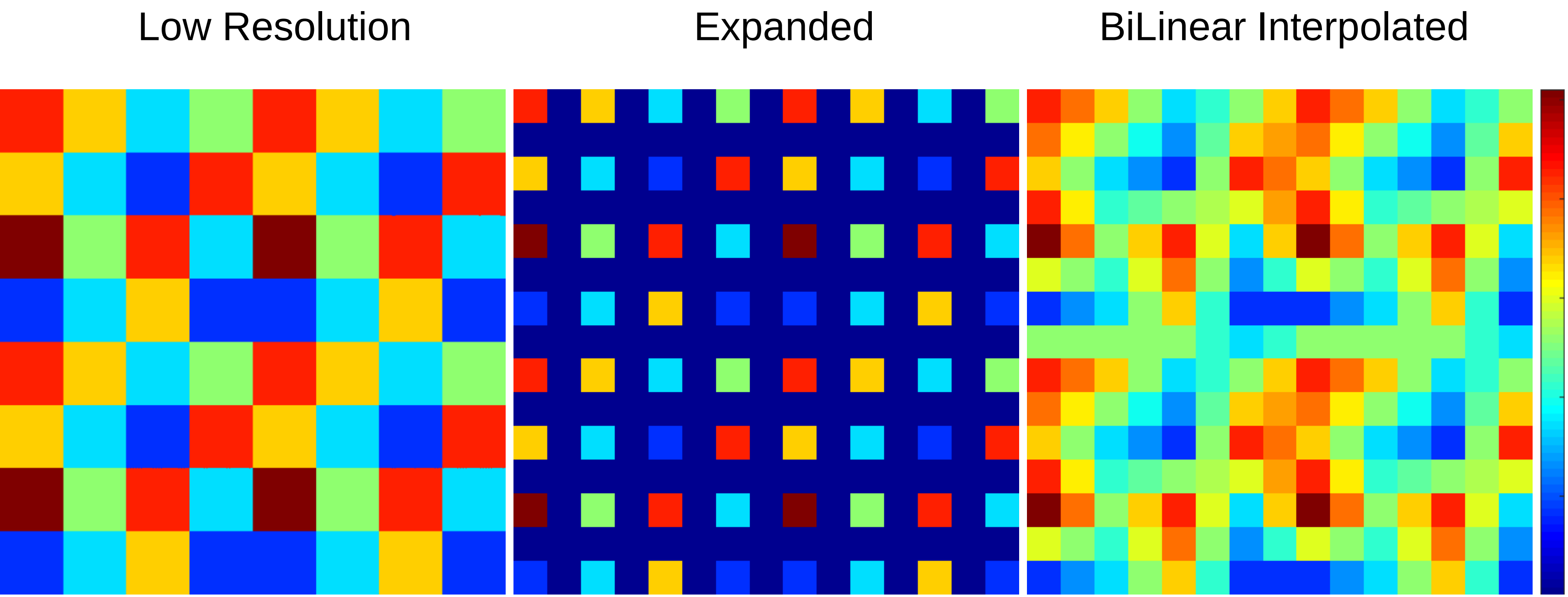
$f(.5,0)$ $\quad f(1,.5)$

$f(.5,.5)$

$f(0,.5)$ $\quad f(.5,1)$

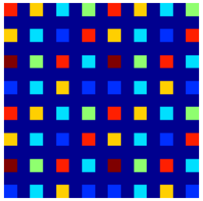
5. 2D BiLinear Interpolation

Example: 8×8 interpolate 1 to 15×15

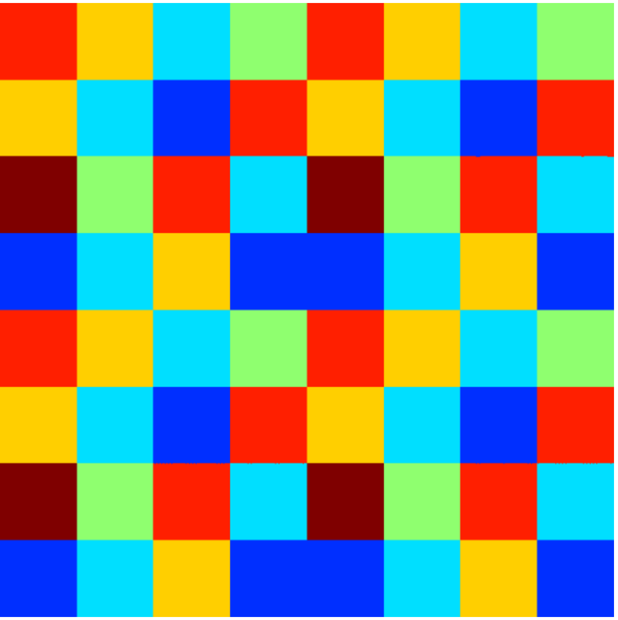


5. 2D BiLinear Interpolation

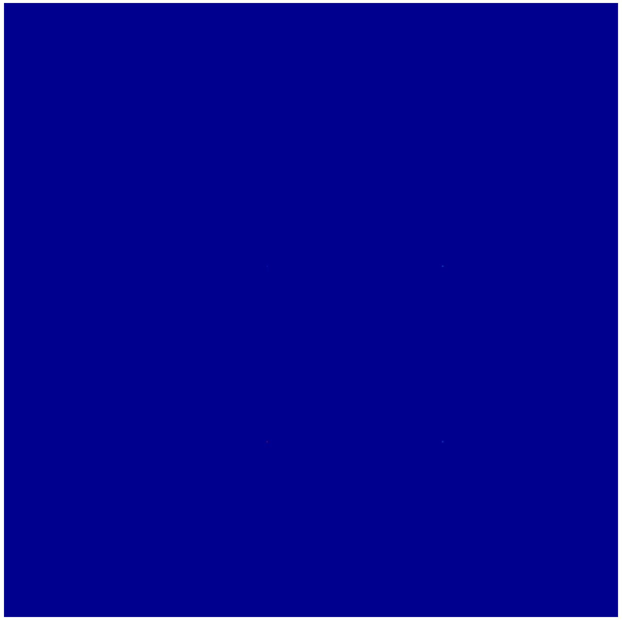
Example: 8×8 interpolate 1001 to 7015×7015


Expanded

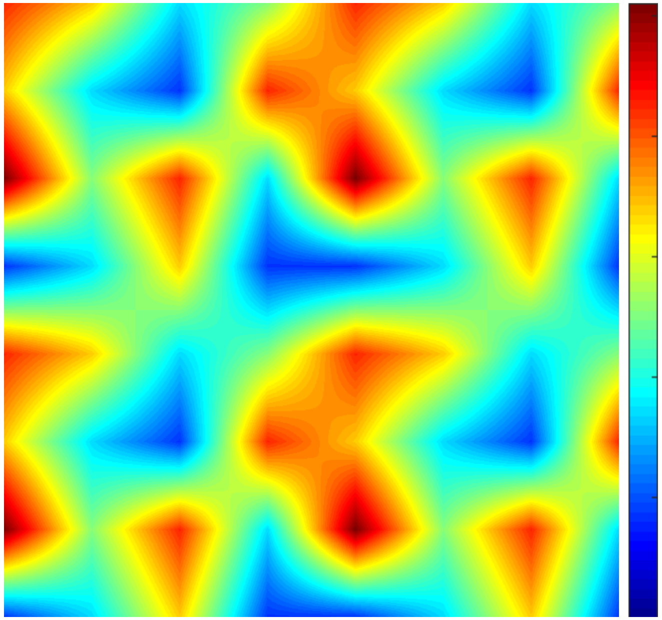
Low Resolution



Expanded


can't see original pixels

BiLinear Interpolated


note "kinks" between patches

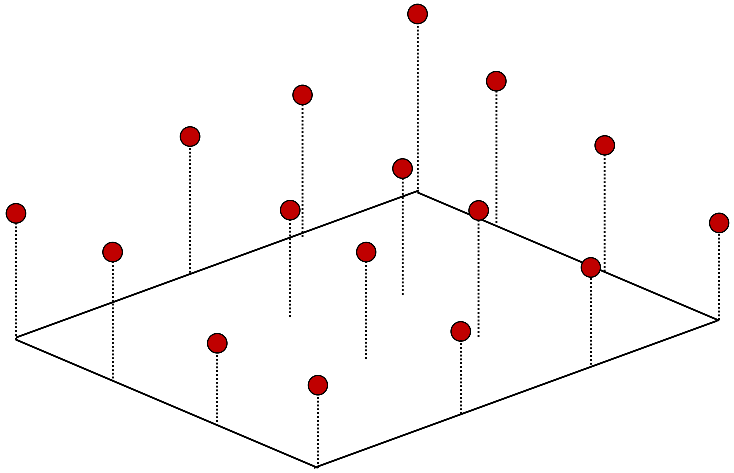
6. 2D BiCubic Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Interpolate:

Sixteen points define a 2D bicubic surface.

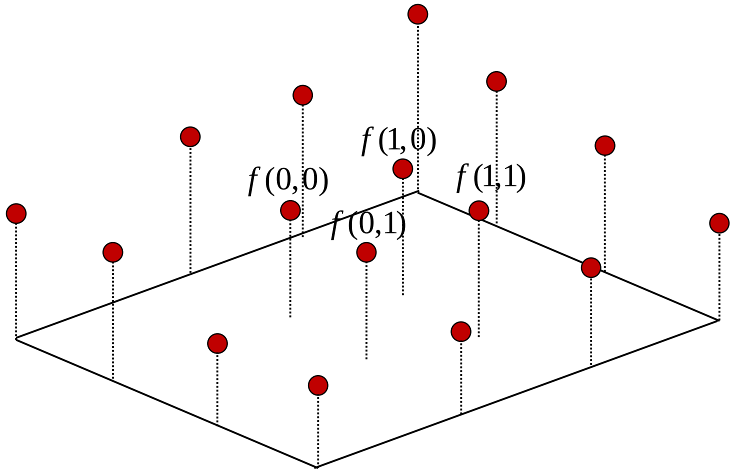
Find the equation of the surface between 4 points using neighbors.



6. 2D BiCubic Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Normalization: $f(0,0),\ f(1,0)$
$f(0,1),\ f(1,1)$



6. 2D BiCubic Interpolation

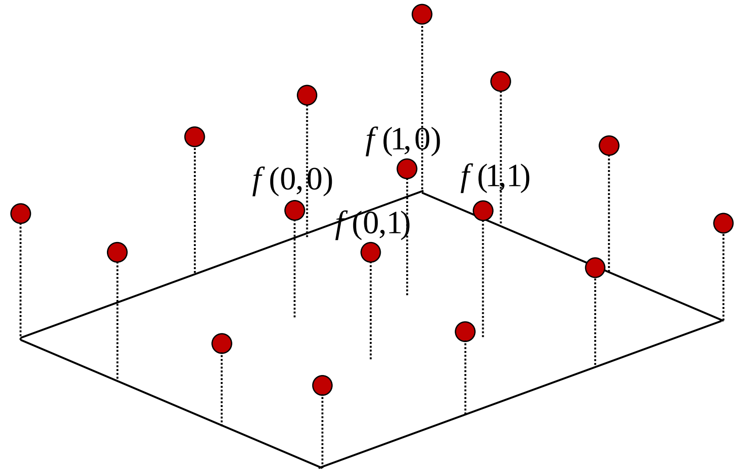
Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Normalization: $f(0,0),\ f(1,0)$

$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{3}\sum_{i=0}^{3}a_{ij}x^i y^j$$

$$x=-1,0,1,2$$



6. 2D BiCubic Interpolation

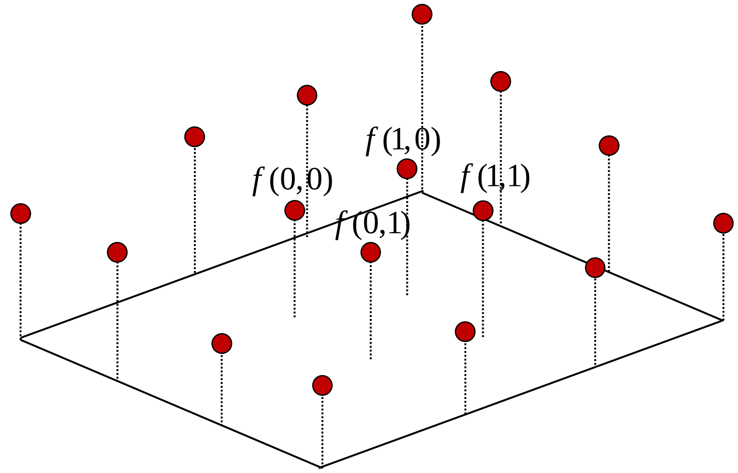
Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Normalization: $f(0,0),\ f(1,0)$

$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{3}\sum_{i=0}^{3}a_{ij}x^i y^j$$

$$x=-1,0,1,2$$



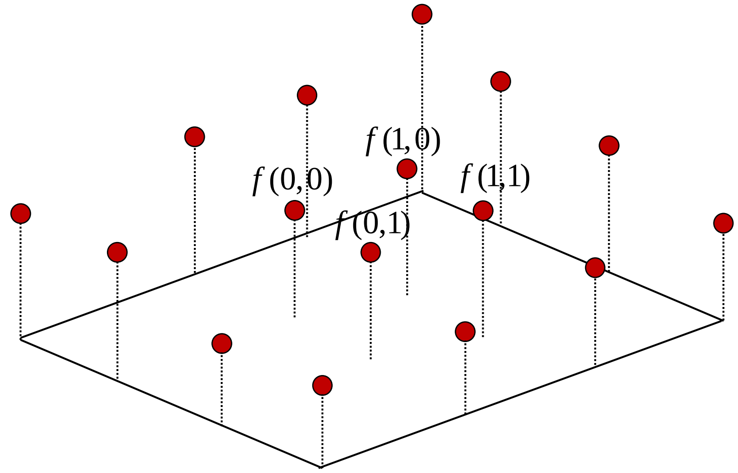
Solve: a_{ij}

6. 2D BiCubic Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Interpolate:

Find the equation of the surface between 4 points using neighbors and and determine a_{ij}'s.

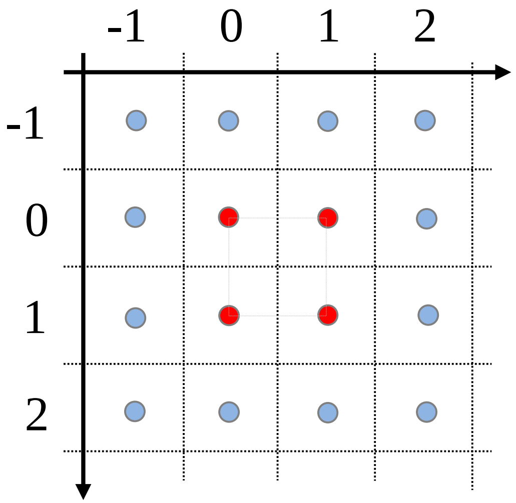


6. 2D BiCubic Interpolation

System of Equations: 16 equations, 16 unknowns

System of Equations $y = Xa$

Image $I(x,y)$



16 equations from $f(x,y)$

$$f(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} x^i y^j$$

x,y=-1,0,1,2

Simply insert all x,y combinations to get 16 equations.

6. 2D BiCubic Interpolation

Values from polynomial.

$$y = Xa$$

$$\begin{bmatrix} f(-1,-1) \\ f(0,-1) \\ f(1,-1) \\ f(2,-1) \\ f(-1,0) \\ f(0,0) \\ f(1,0) \\ f(2,0) \\ f(-1,1) \\ f(0,1) \\ f(1,1) \\ f(2,1) \\ f(-1,2) \\ f(0,2) \\ f(1,2) \\ f(2,2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 2 & 4 & 8 & -1 & -2 & -4 & -8 & 1 & 2 & 4 & 8 & -1 & -2 & -4 & -8 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 1 & 2 & 4 & 8 & 1 & 2 & 4 & 8 & 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 & 2 & -2 & 2 & -2 & 4 & -4 & 4 & -4 & 8 & -8 & 8 & -8 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 8 & 8 & 8 & 8 \\ 1 & 2 & 4 & 8 & 2 & 4 & 8 & 16 & 4 & 8 & 16 & 32 & 8 & 16 & 32 & 64 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{02} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{03} \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j$$

x,y=-1,0,1,2

6. 2D BiCubic Interpolation

Estimate coefficient values.

Almost Money Slide

$$a = X^{-1} I$$

$$\begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{02} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{03} \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \frac{1}{36} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -12 & -18 & 36 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 18 & -36 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6 & 18 & -18 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -12 & 0 & 0 & 0 & -18 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & -6 & 0 & 0 \\
4 & 6 & -12 & 2 & 6 & 9 & -18 & 3 & -12 & -18 & 36 & -6 & 2 & 3 & -6 & 1 \\
-6 & 12 & -6 & 0 & -9 & 18 & -9 & 0 & 18 & -36 & 18 & 0 & -3 & 6 & -3 & 0 \\
2 & -6 & 6 & -2 & 3 & -9 & 9 & -3 & -6 & 18 & -18 & 6 & 1 & -3 & 3 & -1 \\
0 & 18 & 0 & 0 & 0 & -36 & 0 & 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -9 & 18 & -3 & 12 & 18 & -36 & 6 & -6 & -9 & 18 & -3 & 0 & 0 & 0 & 0 \\
9 & -18 & 9 & 0 & -18 & 36 & -18 & 0 & 9 & -18 & 9 & 0 & 0 & 0 & 0 & 0 \\
-3 & 9 & -9 & 3 & 6 & -18 & 18 & -6 & -3 & 9 & -9 & 3 & 0 & 0 & 0 & 0 \\
0 & -6 & 0 & 0 & 0 & 18 & 0 & 0 & 0 & -18 & 0 & 0 & 0 & 6 & 0 & 0 \\
2 & 3 & -6 & 1 & -6 & -9 & 18 & -3 & 6 & 9 & -18 & 3 & -2 & -3 & 6 & -1 \\
-3 & 6 & -3 & 0 & 9 & -18 & 9 & 9 & -9 & 18 & -9 & 0 & 3 & -6 & 3 & 0 \\
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 & 3 & -9 & 9 & -3 & -1 & 3 & -3 & 1
\end{bmatrix} \begin{bmatrix} I(-1,-1) \\ I(0,-1) \\ I(1,-1) \\ I(2,-1) \\ I(-1,0) \\ I(0,0) \\ I(1,0) \\ I(2,0) \\ I(-1,1) \\ I(0,1) \\ I(1,1) \\ I(2,1) \\ I(-1,2) \\ I(0,2) \\ I(1,2) \\ I(2,2) \end{bmatrix}$$

recycle matrix

6. 2D BiCubic Interpolation

Estimate coefficient values.

$$a = X^{-1} I$$

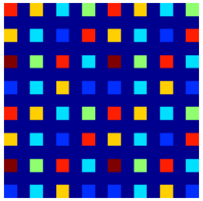
Interpolate pixel values.

$$f(x,y) = \sum_{j=0}^{3}\sum_{i=0}^{3} a_{ij} x^i y^j \qquad 0<x<1,\ 0<y<1$$

Can do biquadratic in corners and linear-quadratic on sides.

6. 2D BiCubic Interpolation

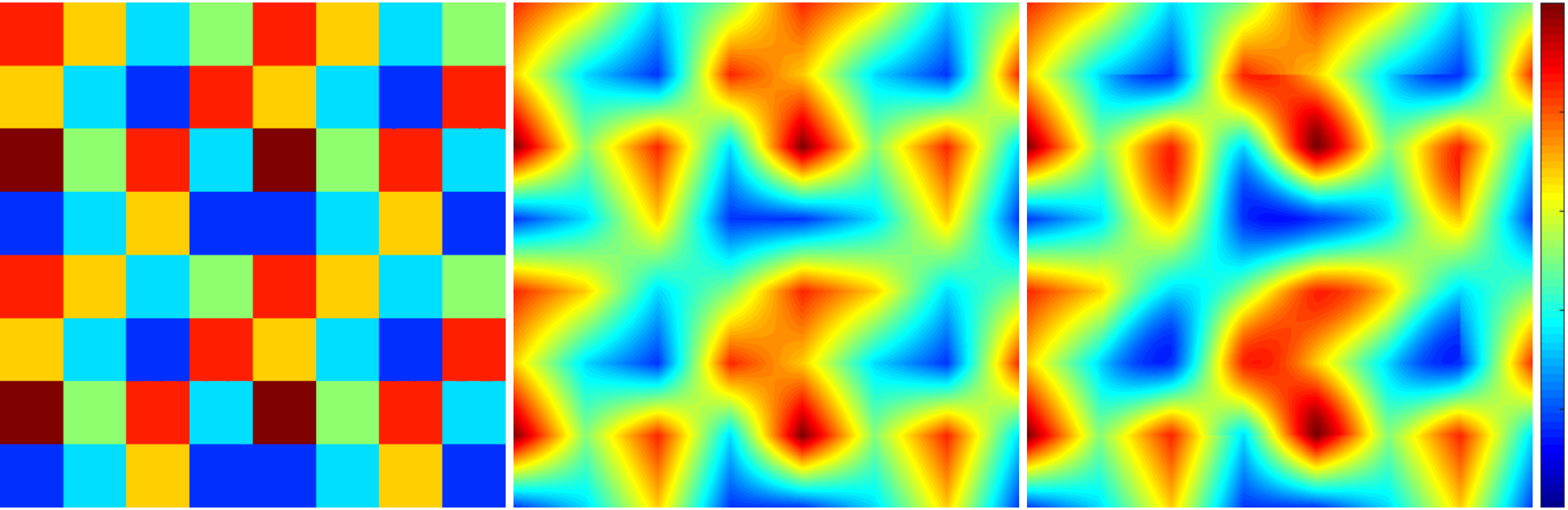
Example: 8×8 interpolate 1001 to 7015×7015



Expanded

* bilinear edges

Low Resolution | Bilinear Interpolated | BiCubic Interpolated



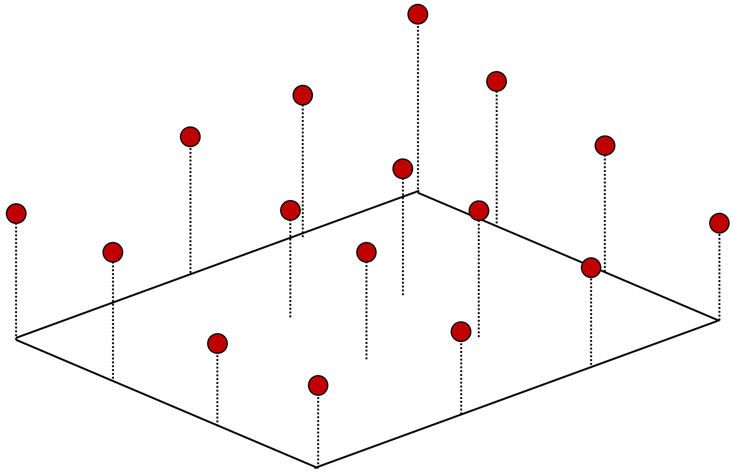
still “kinky” between patches

7. 2D BiCubic Spline Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Interpolate:

Will use 4 points and 12 derivatives at those points to define a bicubic surface.



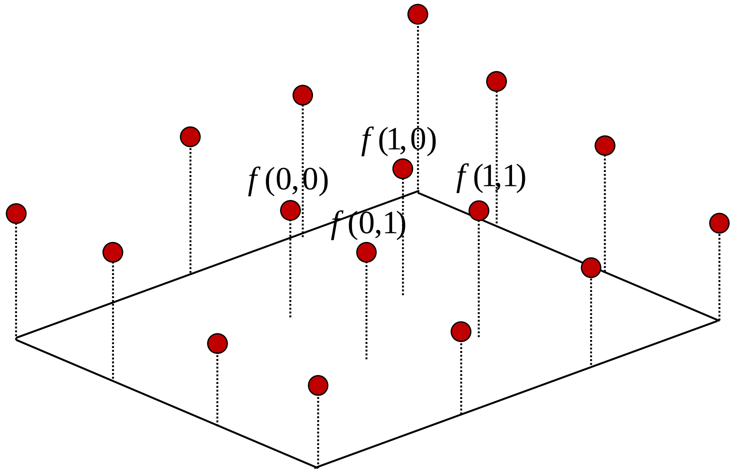
array/image coordinate system

7. 2D BiCubic Spline Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Normalization: $f(0,0),\ f(1,0)$

$f(0,1),\ f(1,1)$



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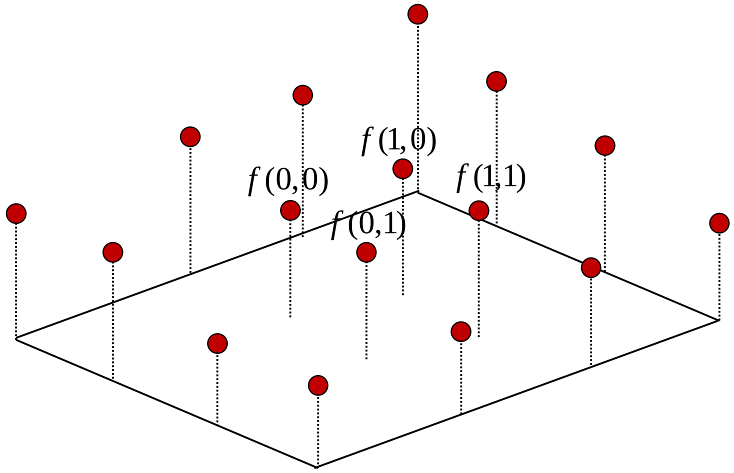
Normalization: $f(0,0),\ f(1,0)$

$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{3}\sum_{i=0}^{3}a_{ij}x^i y^j$$



7. 2D BiCubic Spline Interpolation

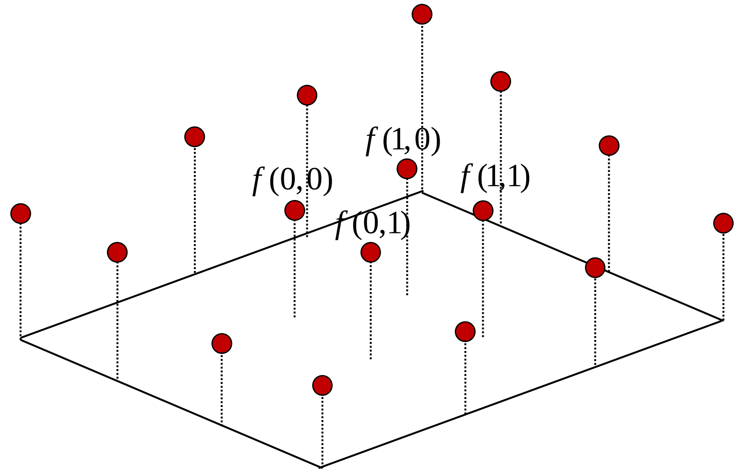
Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Normalization: $f(0,0),\ f(1,0)$

$f(0,1),\ f(1,1)$

Model: $$f(x,y)=\sum_{j=0}^{3}\sum_{i=0}^{3}a_{ij}x^i y^j$$

Solve: a_{ij}

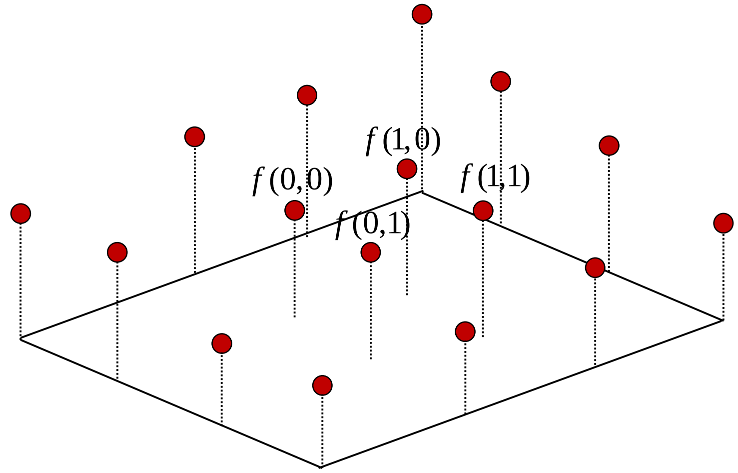


7. 2D BiCubic Spline Interpolation

Need to use adjacent points, estimate surfaces, and use values within the surfaces.

Interpolate:

Will use 4 points and 12 derivatives to define a bicubic splined surface and and determine a_{ij}'s.

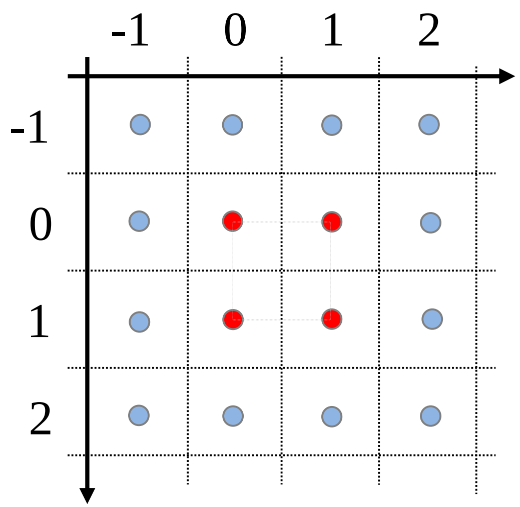


7. 2D BiCubic Spline Interpolation

System of Equations: 16 equations, 16 unknowns

System of Equations $y = Xa$

Image $I(x,y)$



4 equations from $f(x,y)$

$$f(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} x^i y^j$$

x,y=0,1

$$f(0,0) = a_{00}$$

$$f(1,0) = a_{00} + a_{10} + a_{20} + a_{30}$$

$$f(0,1) = a_{00} + a_{01} + a_{02} + a_{03}$$

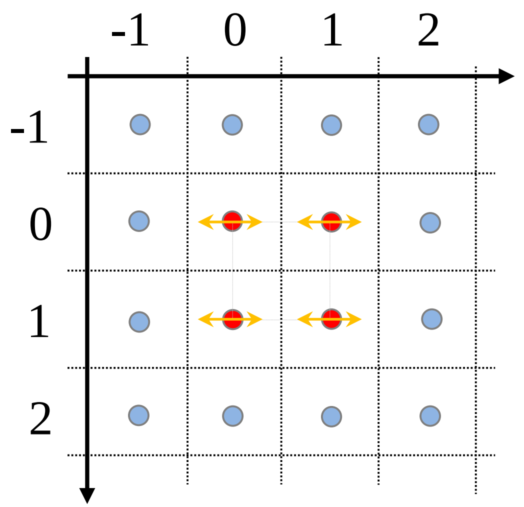
$$f(1,1) = a_{00} + a_{10} + a_{01} + a_{11}$$

7. 2D BiCubic Spline Interpolation

System of Equations: 16 equations, 16 unknowns

System of Equations $y = Xa$

Image $I(x,y)$



4 equations from $f_x(x,y) = \dfrac{\partial}{\partial x} f(x,y)$

$$f_x(x,y) = \sum_{j=0}^{3}\sum_{i=1}^{3} a_{ij} i x^{i-1} y^j$$

x,y=0,1

$$f_x(0,0) = a_{10}$$

$$f_x(1,0) = 1a_{10} + 2a_{20} + 3a_{30}$$

$$f_x(0,1) = a_{10} + a_{11} + a_{12} + a_{13}$$

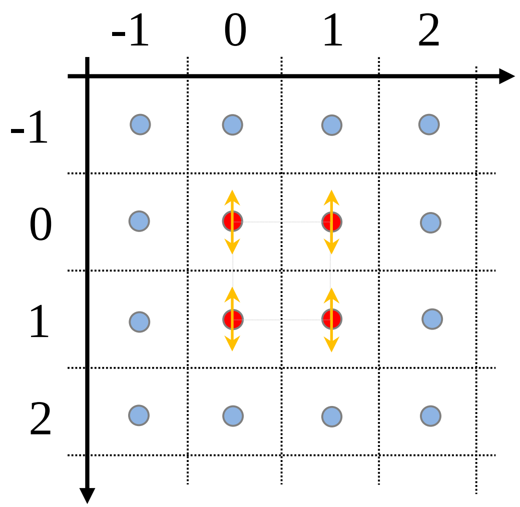
$$\begin{aligned} f_x(1,1) \quad &= \quad 1a_{10} + 2a_{20} + 3a_{31} + 1a_{11} + 2a_{21} + 3a_{31} \\ &+ \quad 1a_{12} + 2a_{22} + 3a_{32} + 1a_{13} + 2a_{23} + 3a_{33} \end{aligned}$$

7. 2D BiCubic Spline Interpolation

System of Equations: 16 equations, 16 unknowns

System of Equations $y = Xa$

Image $I(x,y)$



4 equations from $f_y(x,y) = \dfrac{\partial}{\partial y} f(x,y)$

$$f_y(x,y) = \sum_{j=1}^{3}\sum_{i=0}^{3} a_{ij}\, j x^i y^{j-1}$$

x,y=0,1

$$f_y(0,0) = a_{01}$$

$$f_y(1,0) = a_{01} + a_{11} + a_{21} + a_{31}$$

$$f_y(0,1) = 1a_{01} + 2a_{02} + 3a_{03}$$

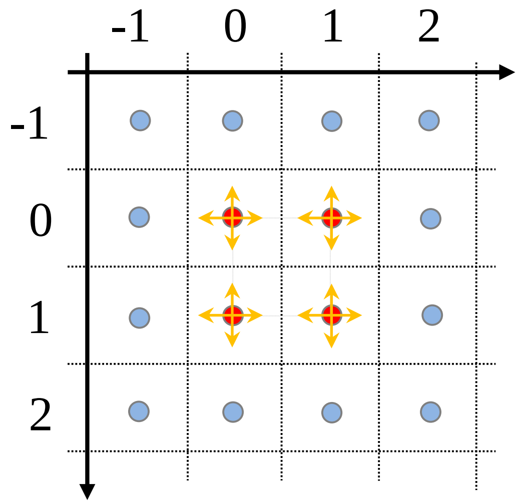
$$\begin{aligned} f_y(1,1) &= 1a_{01} + 1a_{11} + 1a_{21} + 1a_{31} \\ &+ 2a_{02} + 2a_{12} + 2a_{22} + 2a_{32} \\ &+ 3a_{03} + 3a_{13} + 3a_{23} + 3a_{33} \end{aligned}$$

7. 2D BiCubic Spline Interpolation

System of Equations: 16 equations, 16 unknowns

System of Equations $y = Xa$

Image $I(x,y)$



4 equations from $f_{xy}(x,y) = \dfrac{\partial^2}{\partial y \partial x} f(x,y)$

$$f_{xy}(x,y) = \sum_{j=0}^{3}\sum_{i=0}^{3} a_{ij} i j x^{i-1} y^{j-1}$$

$x,y=0,1$

$$f_{xy}(0,0) = a_{11}$$

$$f_{xy}(1,0) = 1a_{11} + 2a_{21} + 3a_{31}$$

$$f_{xy}(0,1) = 1a_{11} + 2a_{12} + 3a_{13}$$

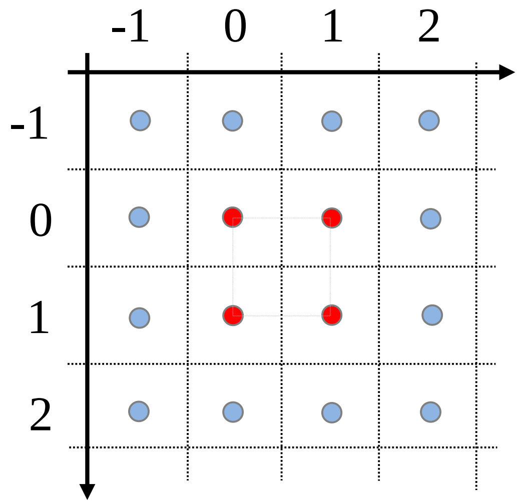
$$\begin{aligned} f_{xy}(1,1) &= 1a_{11} + 2a_{21} + 3a_{31} \\ &+ 2a_{12} + 4a_{22} + 6a_{32} \\ &+ 3a_{13} + 6a_{23} + 9a_{33} \end{aligned}$$

7. 2D BiCubic Spline Interpolation

System of Equations: 16 equations, 16 unknowns

Need Image and Derivatives

Image $I(x,y)$



$$f(x,y) = I(x,y)$$

$$f_x(x,y) = [I(x+1,y) - I(x-1,y)]/2$$

$$f_y(x,y) = [I(x,y+1) - I(x,y-1)]/2$$

$$f_{xy}(x,y) = [I(x+1,y+1) - I(x-1,y) - I(x,y-1) - I(x,y)]/4$$

x,y=0,1

Use the graph to reason out the derivatives.
Only using surrounding points for derivatives.

i.e. at (0,0)

$$f_x(0,0) = [I(1,0) - I(-1,0)]/2$$

$$f_y(0,0) = [I(0,1) - I(0,-1)]/2$$

$$f_{xy}(0,0) = [I(1,1) - I(-1,0) - I(0,-1) - I(0,0)]/4$$

7. 2D BiCubic Spline Interpolation

Values from polynomial.

$$y = Xa$$

$$\begin{bmatrix} f(0,0) \\ f(1,0) \\ f(0,1) \\ f(1,1) \\ f_x(0,0) \\ f_x(1,0) \\ f_x(0,1) \\ f_x(1,1) \\ f_y(0,0) \\ f_y(1,0) \\ f_y(0,1) \\ f_y(1,1) \\ f_{xy}(0,0) \\ f_{xy}(1,0) \\ f_{xy}(0,1) \\ f_{xy}(1,1) \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&2&3&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&2&0&0&0&3&0&0&0 \\ 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&2&3&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&2&0&0&0&3&0&0 \\ 0&0&0&0&0&1&2&3&0&2&4&6&0&3&6&9 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{02} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{03} \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j$$

$$f_x(x,y) = \sum_{j=0}^{3} \sum_{i=1}^{3} a_{ij} i x^{i-1} y^j$$

$$f_y(x,y) = \sum_{j=1}^{3} \sum_{i=0}^{3} a_{ij} j x^i y^{j-1}$$

$$f_{xy}(x,y) = \sum_{j=0}^{3} \sum_{i=0}^{3} a_{ij} ij x^{i-1} y^{j-1}$$

$x,y=0,1$

7. 2D BiCubic Spline Interpolation

Values from image.

$$y = DI$$

$$\begin{bmatrix} f(0,0) \\ f(1,0) \\ f(0,1) \\ f(1,1) \\ f_x(0,0) \\ f_x(1,0) \\ f_x(0,1) \\ f_x(1,1) \\ f_y(0,0) \\ f_y(1,0) \\ f_y(0,1) \\ f_y(1,1) \\ f_{xy}(0,0) \\ f_{xy}(1,0) \\ f_{xy}(0,1) \\ f_{xy}(1,1) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I(-1,-1) \\ I(0,-1) \\ I(1,-1) \\ I(2,-1) \\ I(-1,0) \\ I(0,0) \\ I(1,0) \\ I(2,0) \\ I(-1,1) \\ I(0,1) \\ I(1,1) \\ I(2,1) \\ I(-1,2) \\ I(0,2) \\ I(1,2) \\ I(2,2) \end{bmatrix}$$

$$f(x,y) = I(x,y)$$

$$f_x(x,y) = [I(x+1,y) - I(x-1,y)]/2$$

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$$f_{xy}(x,y) = [I(x+1,y+1) - I(x-1,y) - I(x,y-1) - I(x,y)]/4$$

$x,y=0,1$

7. 2D BiCubic Spline Interpolation

Estimate coefficient values.

The Money Slide

$$a = X^{-1}DI$$

$$
\begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{02} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{03} \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}
= \frac{1}{16}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 & -40 & 32 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -8 & 24 & -24 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & -20 & 0 & 8 & -4 & -4 & 0 & 0 & -24 & 16 & -4 & 0 & 0 & 0 & 0 \\
0 & -20 & 12 & 0 & -4 & 0 & 4 & 0 & 0 & 16 & -12 & 4 & 0 & 0 & 0 & 0 \\
0 & 16 & 0 & 0 & 0 & -40 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 8 & 0 & 0 & 32 & -4 & -24 & 0 & -20 & -4 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & -64 & 40 & 0 & -64 & 96 & -68 & 24 & 40 & -68 & -16 & -16 & 0 & 24 & -16 & 0 \\
0 & 40 & -24 & 0 & 32 & -52 & 52 & -24 & -20 & 40 & 16 & 16 & 0 & -16 & 12 & 0 \\
0 & -8 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & -24 & 0 & 0 & 0 & 8 & 0 & 4 \\
0 & -4 & 0 & 0 & -20 & 0 & 16 & 0 & 12 & 4 & -12 & 0 & 0 & 0 & 4 & -4 \\
0 & 32 & -20 & 0 & 40 & -52 & 40 & -16 & -24 & 52 & -52 & 12 & 0 & -24 & 16 & -4 \\
0 & -20 & 12 & 0 & -20 & 28 & -32 & 16 & 12 & -32 & 40 & -12 & 0 & 16 & -12 & 4
\end{bmatrix}
\begin{bmatrix} I(-1,-1) \\ I(0,-1) \\ I(1,-1) \\ I(2,-1) \\ I(-1,0) \\ I(0,0) \\ I(1,0) \\ I(2,0) \\ I(-1,1) \\ I(0,1) \\ I(1,1) \\ I(2,1) \\ I(-1,2) \\ I(0,2) \\ I(1,2) \\ I(2,2) \end{bmatrix}
$$

recycle matrix

7. 2D BiCubic Spline Interpolation

Estimate coefficient values.

$$a = X^{-1}DI$$

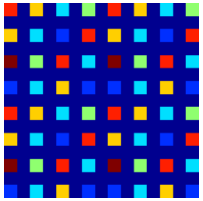
Interpolate pixel values.

$$f(x,y) = \sum_{j=0}^{3}\sum_{i=0}^{3} a_{ij} x^i y^j \qquad 0<x<1,\ 0<y<1$$

Can do biquadratic in corners and linear-quadratic on sides.

7. 2D BiCubic Spline Interpolation

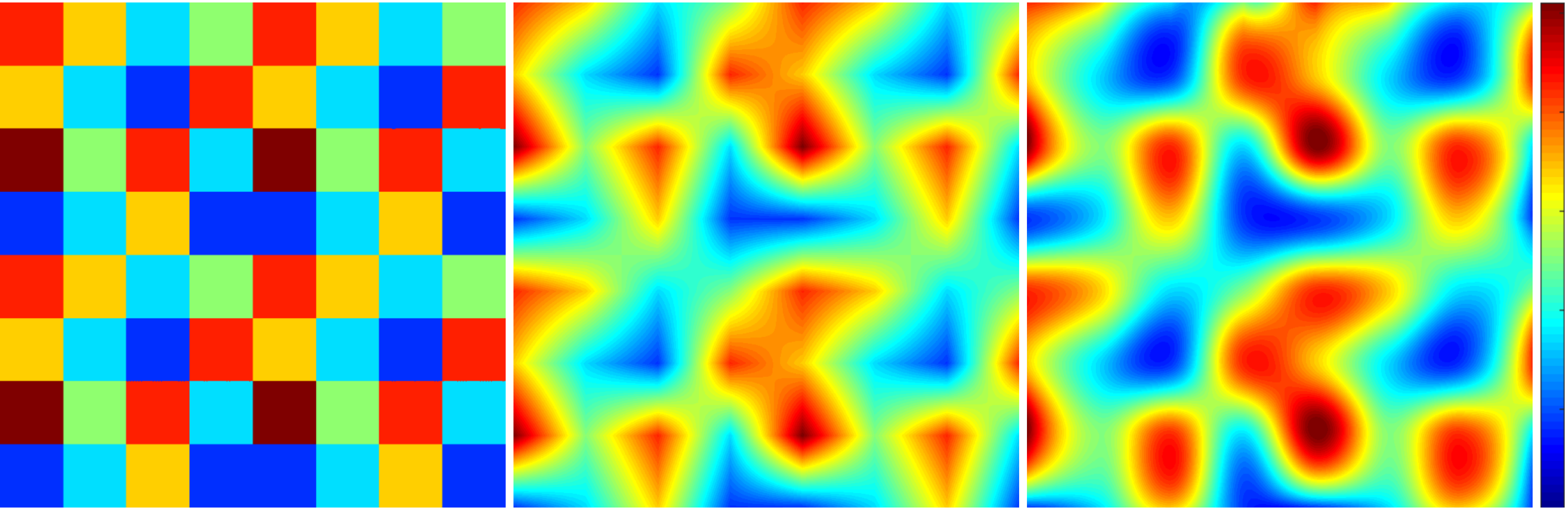
Example: 8×8 interpolate 1001 to 7015×7015



Expanded

* biquadratic corners, linear-quadratic sides

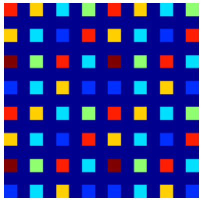
Low Resolution Bilinear Interpolated BiCubic Spline Interpolated



smooth between patches

7. 2D BiCubic Spline Interpolation

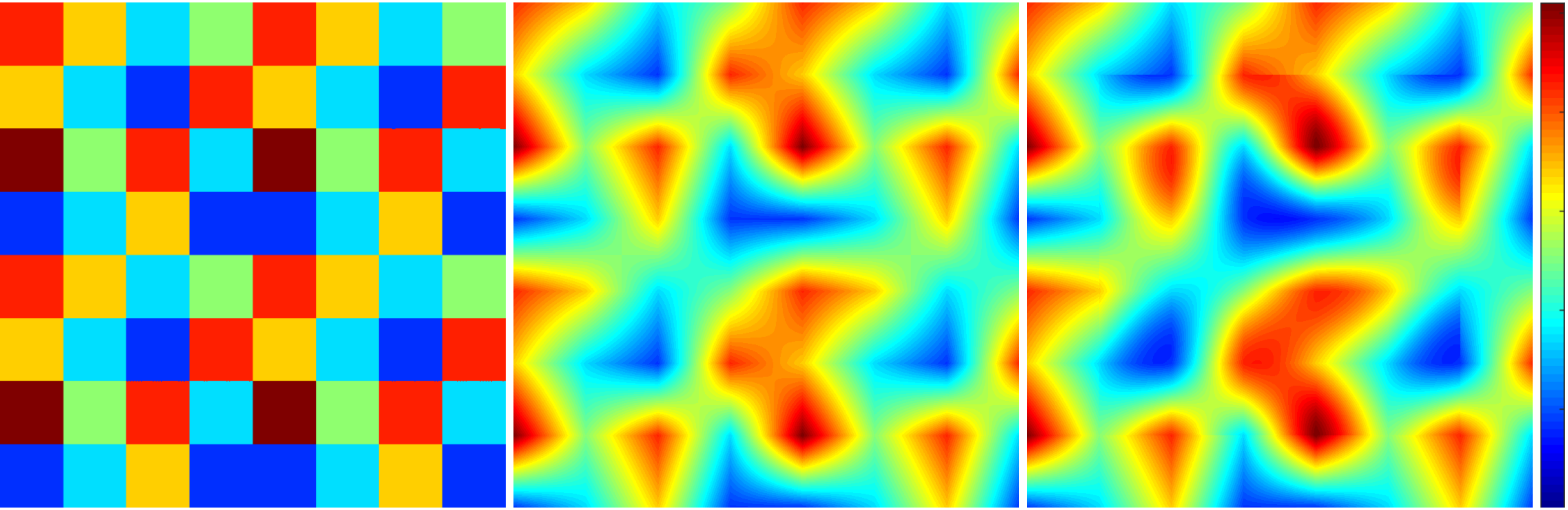
Example: 8×8 interpolate 1001 to 7015×7015



Expanded

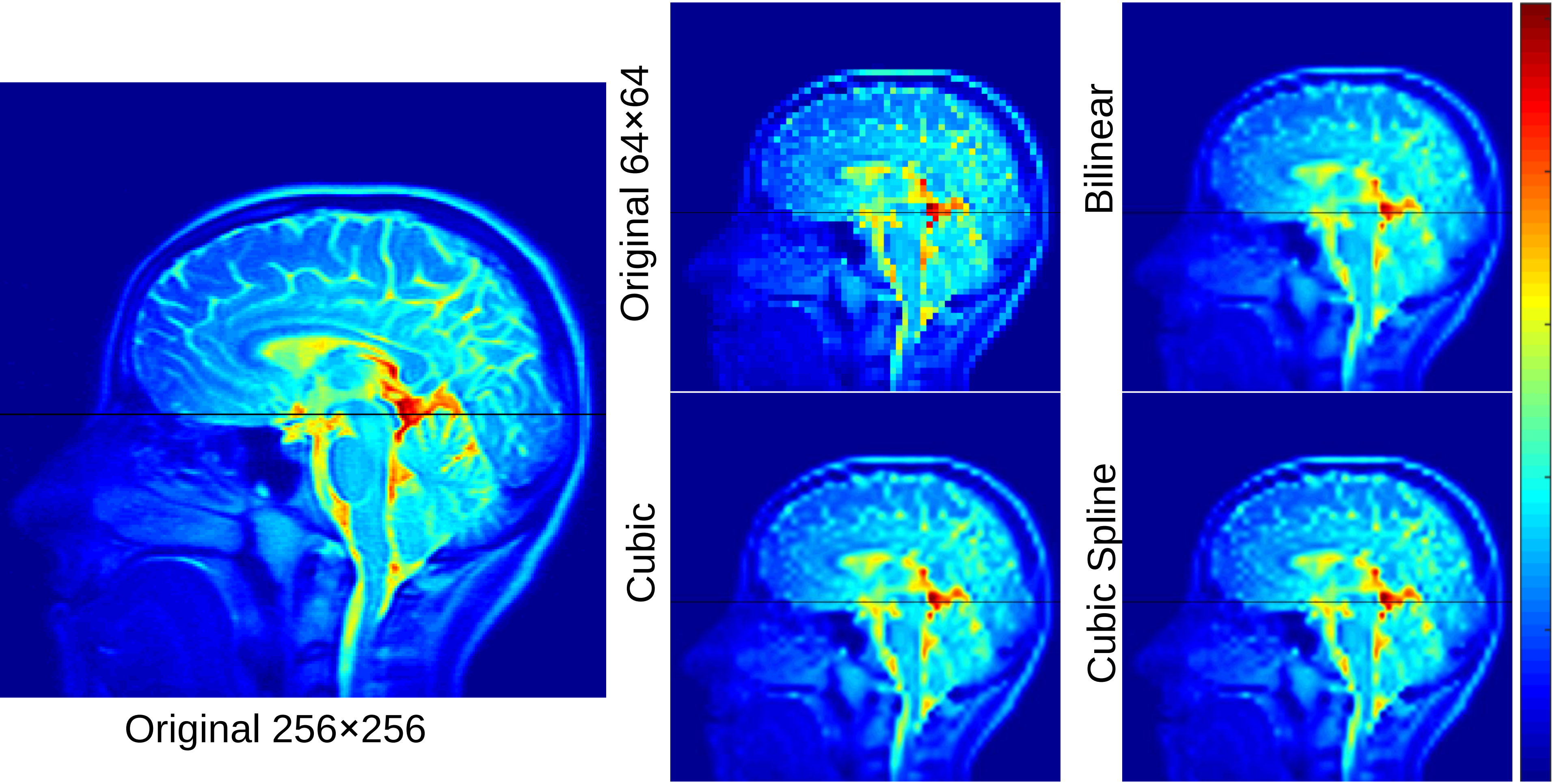
* bilinear edges

Low Resolution Bilinear Interpolated BiCubic Interpolated

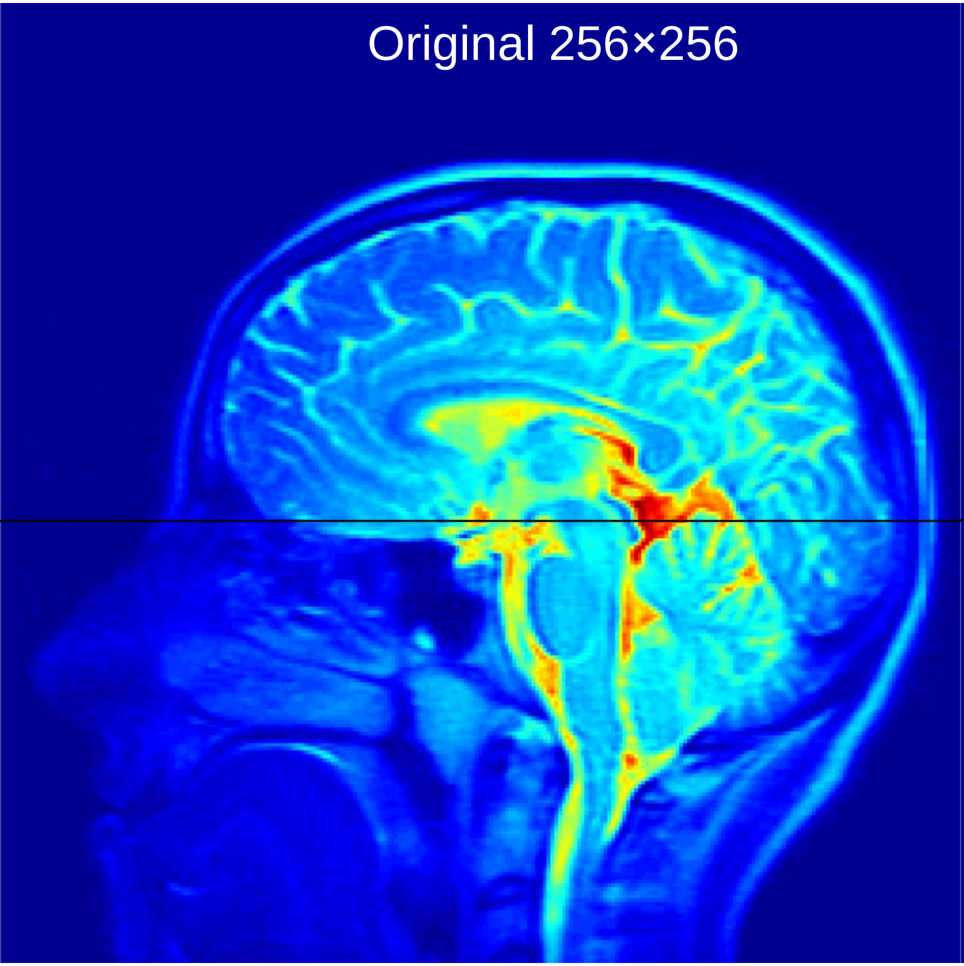


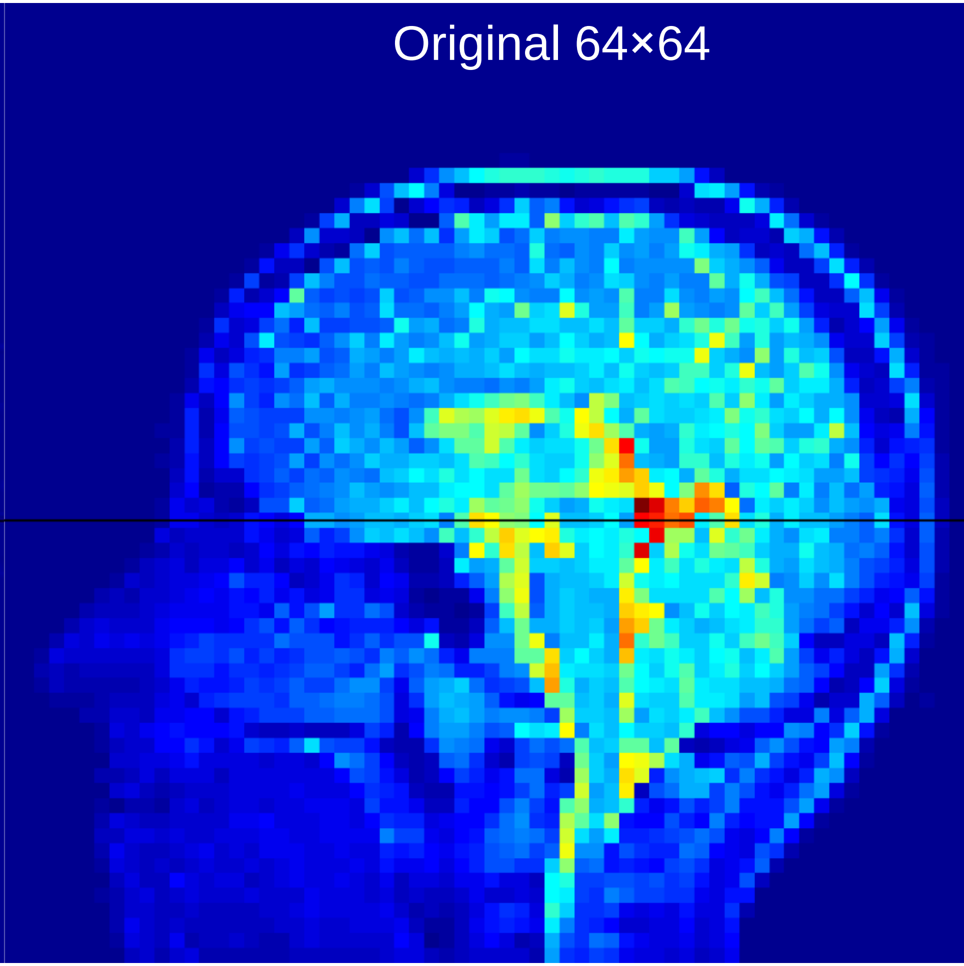
still “kinky” between patches

8. MRI Example

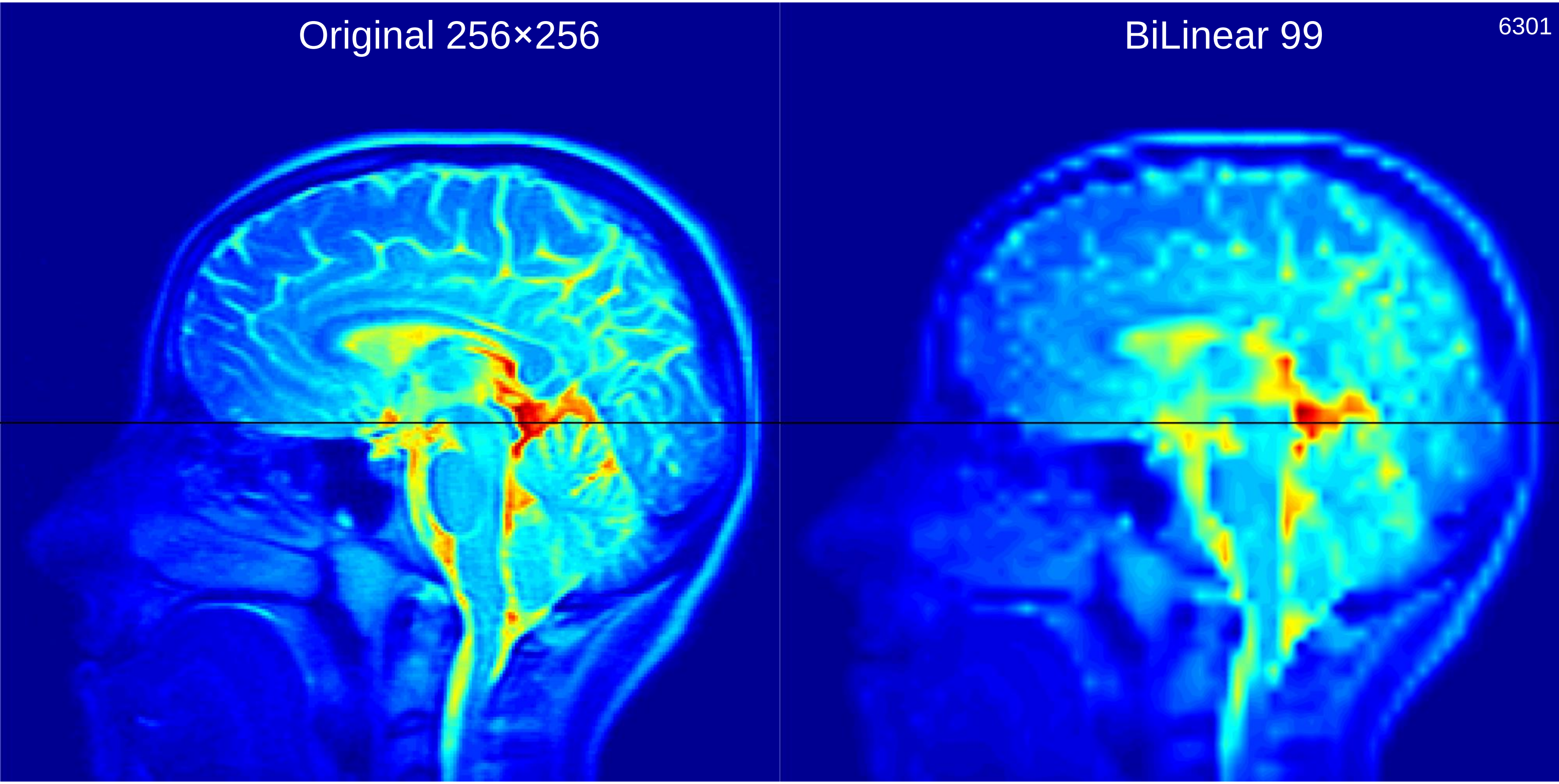


8. MRI Example

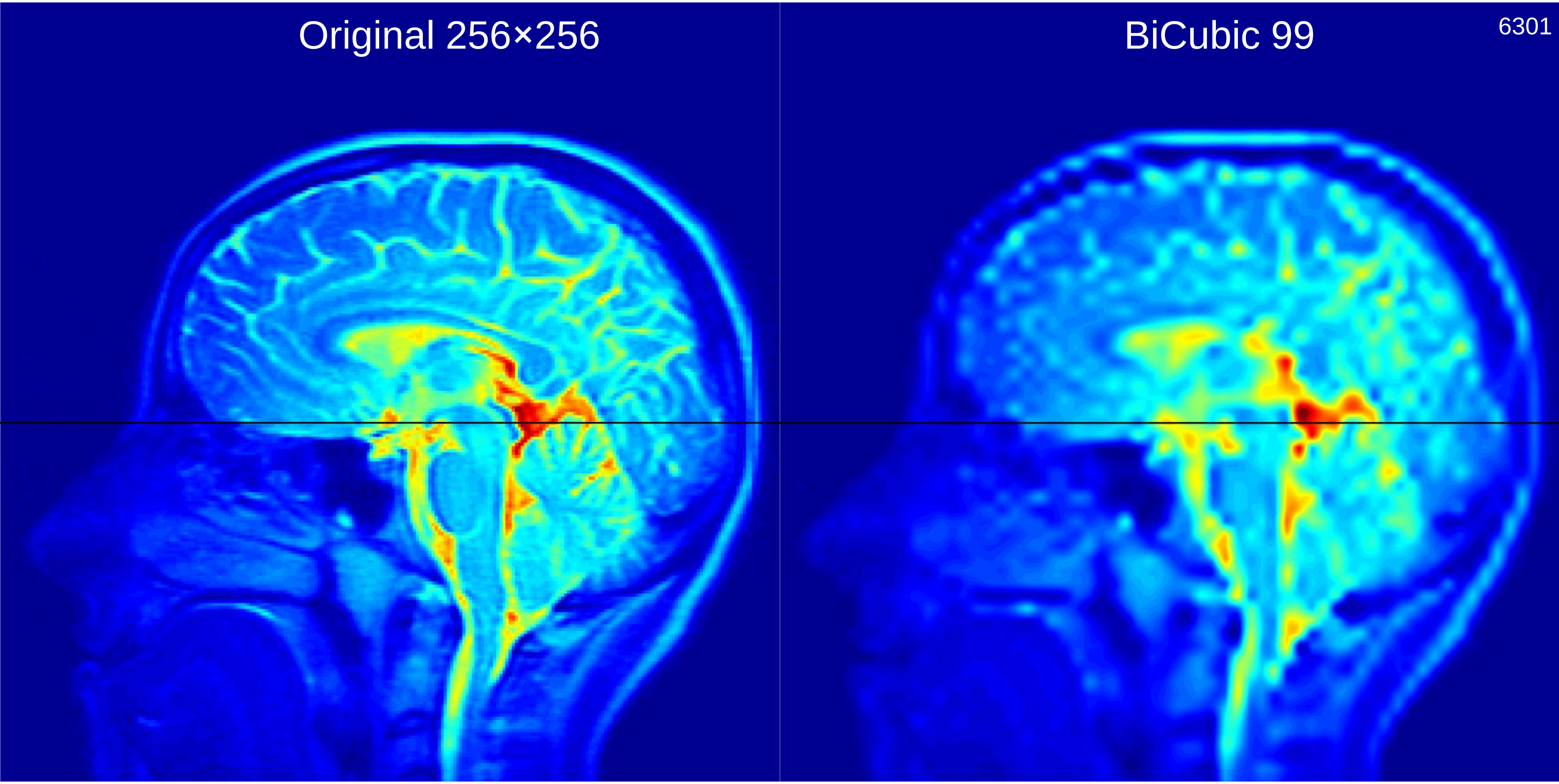




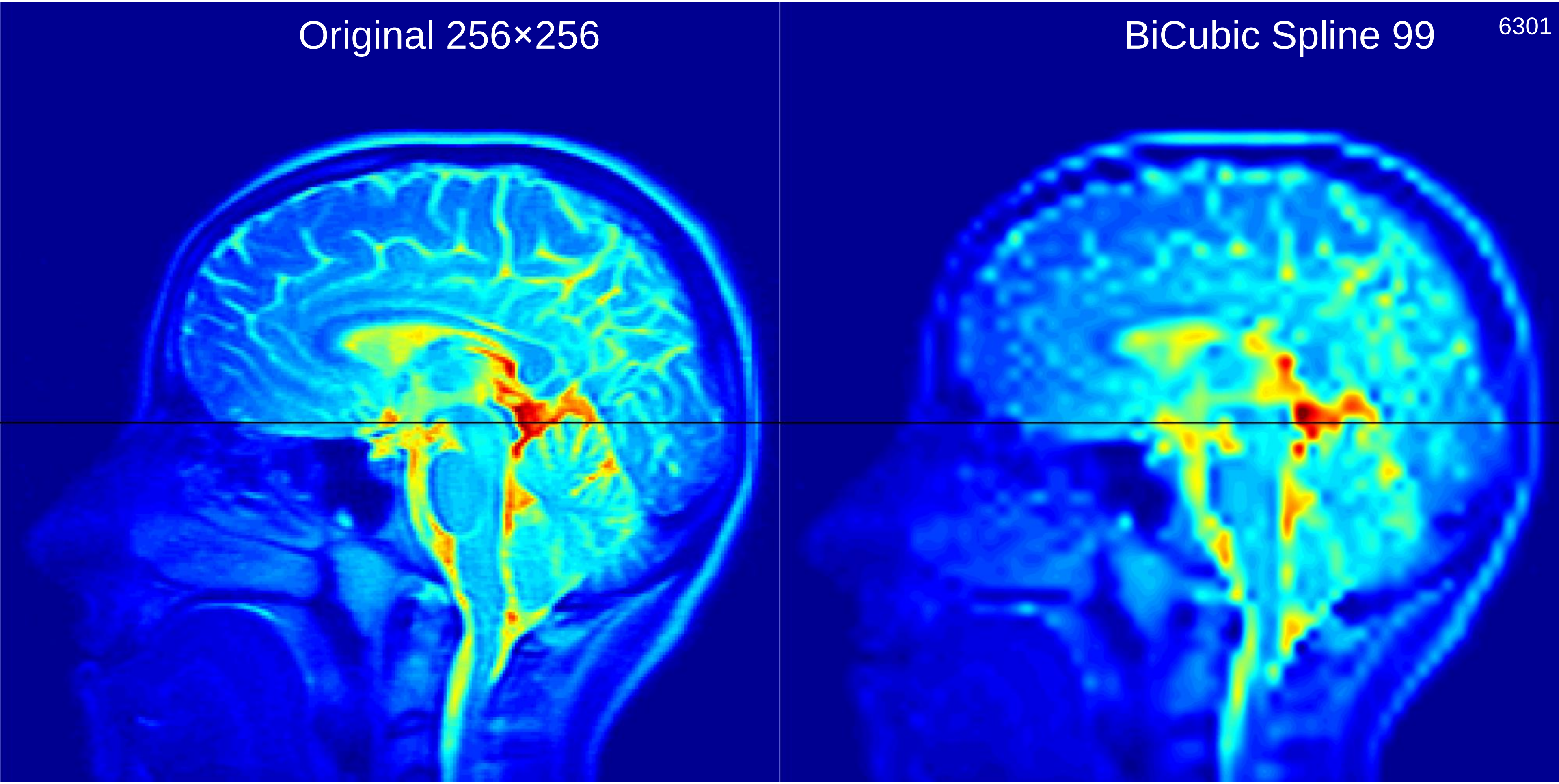
8. MRI Example



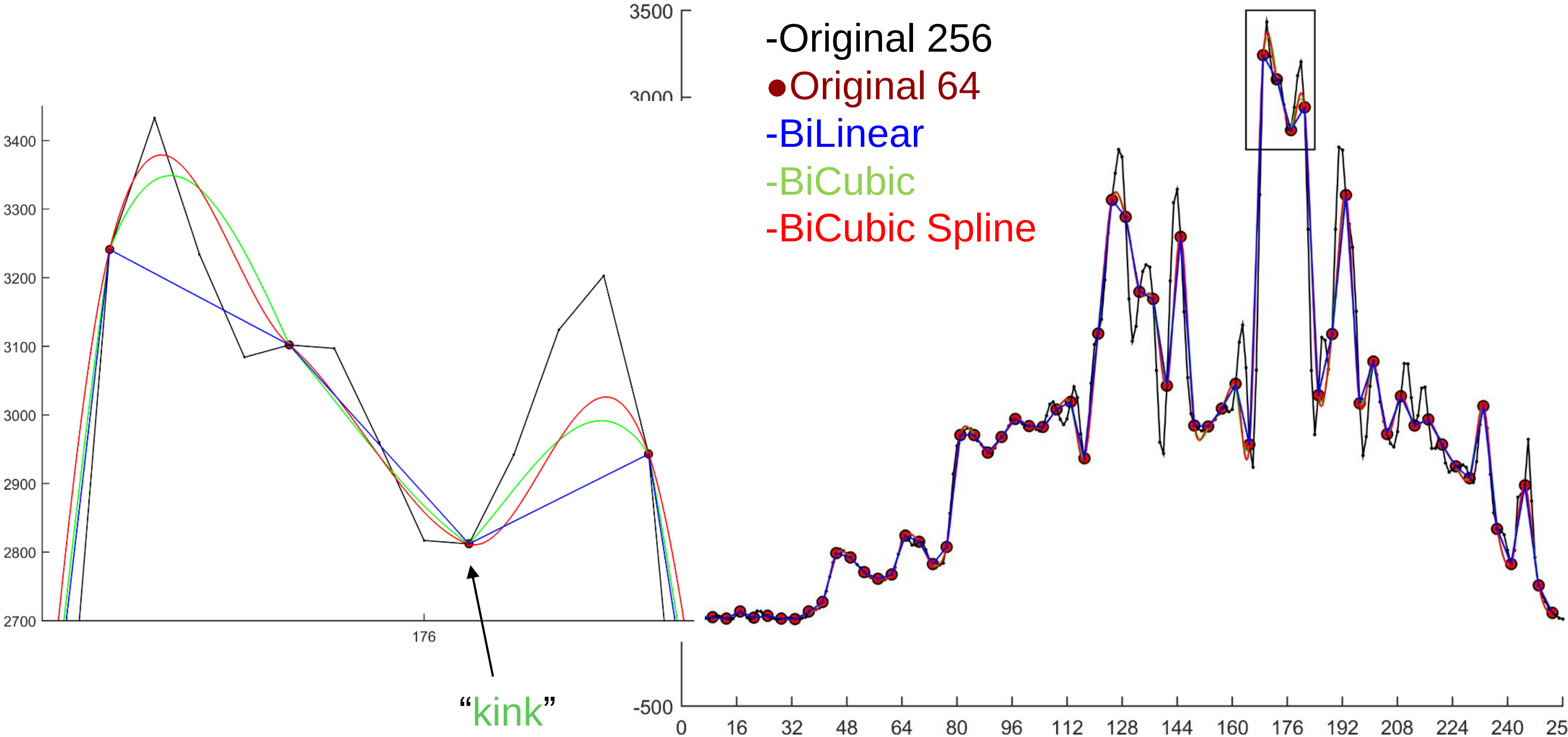
8. MRI Example



8. MRI Example



8. MRI Example



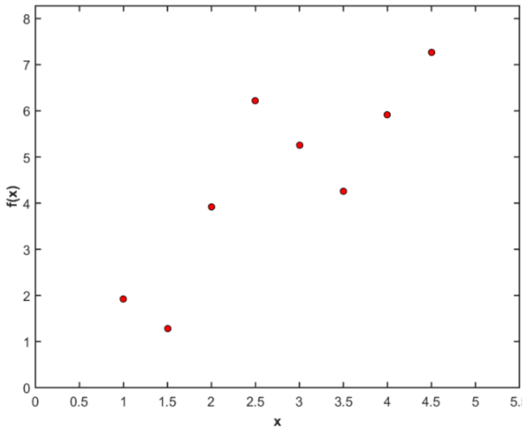
BiLinear, Bicubic, & BiCubic Spline Interpolation:

- To estimate between known pixel values
- BiLinear fits a linear polynomial with cross term.
- BiCubic fits a third order piecewise polynomial.
- BiCubic Spline fits third order smooth polynomial using discrete derivatives
- BiCubic Spline captures curvature through pixels.

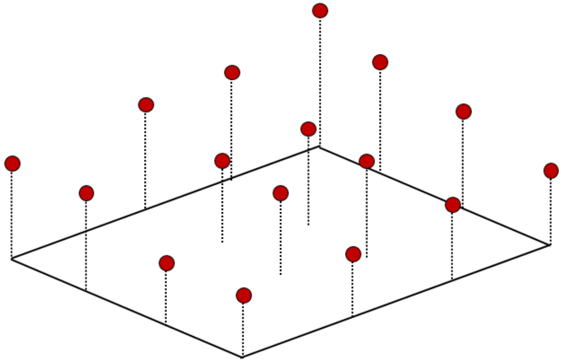
Thank You!

Questions?

1D



2D



Example