EXTREMAL H-COLORINGS OF GRAPHS

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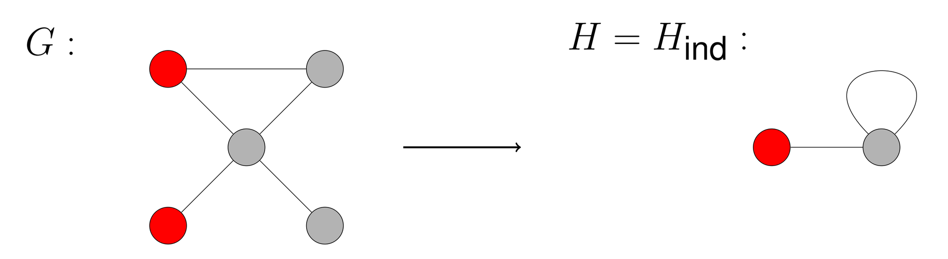
1. Definition and Examples

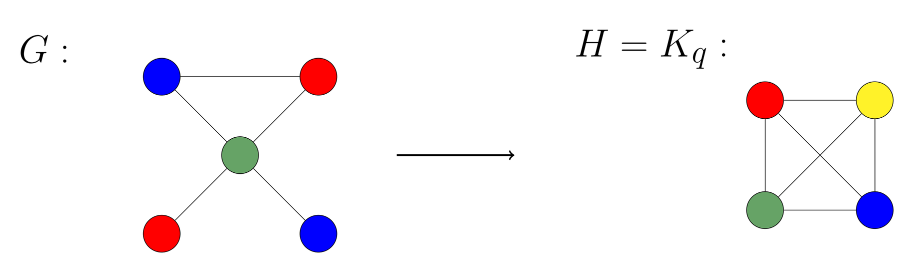
Definition 1.1 *Given graphs G and H, an H-coloring of G (or graph homomorphism) is an edge-preserving map from the vertices of G to the vertices of H.*

Examples:

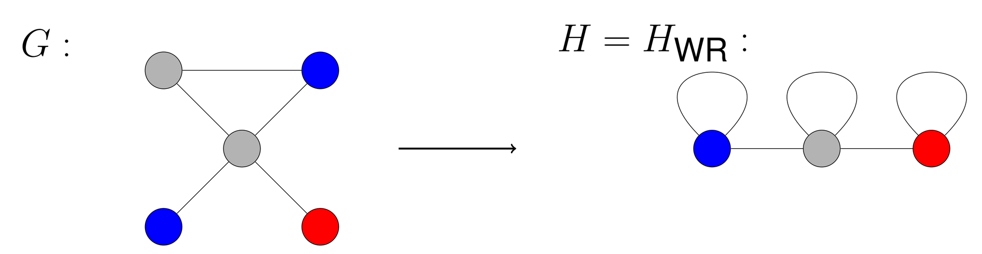
- Independent sets



- Proper q-colorings



- Widom-Rowlinson



- Components, bipartite components

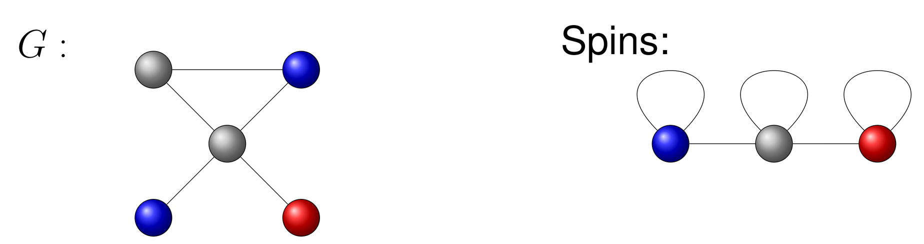
Also:

- H is a blueprint; it encodes the coloring scheme
- Natural to allow H to have loops
- $\hom(G, H)$ = the number of H-colorings of G
  - $\hom(G, H_{\text{comp}}) = 2^{(\text{\# of components of } G)}$
  - $\hom(G, K_2) = \mathbf{1}_{\{G \text{ bipartite}\}} 2^{(\text{\# of bipartite components of } G)}$

2. Statistical Physics Interpretation

Hard Constraint Spin Systems:

- Imagine $V(G)$ = particles, $E(G)$ = adjacency (e.g. spatial proximity)
- Place spins on those particles so that adjacent particles receive 'compatible' spins



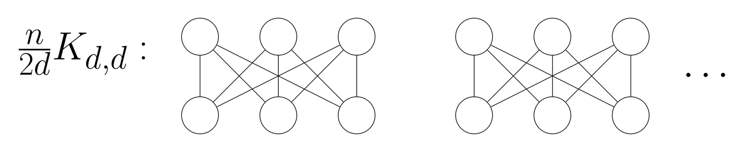
- Spins = colors; a spin configuration is an H-coloring
- This idea generalizes to putting objects (with relationships) into classes with hard rules

3. An Extremal Question

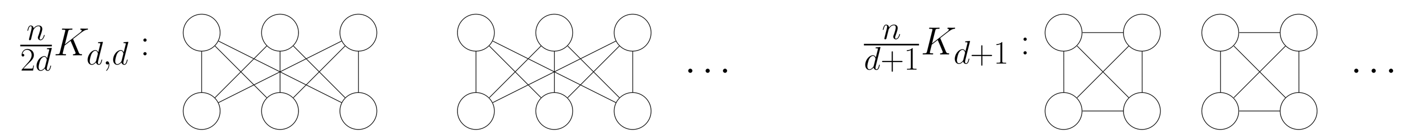
Question 3.1 *Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?*

Various Families $\mathcal{G}$:

- $\mathcal{G}$ = n-vertex graphs
  - $\hom(G, H)$ maximized when $G = E_n$, the empty graph
- $\mathcal{G}$ = n-vertex m-edge graphs
  - Results for $H = H_{\text{ind}}, H_{\text{WR}}$, class of H (e.g. [1]); maximized by one of five graphs G
  - $H = K_q$: various results (e.g. [7]), still open in general
- $\mathcal{G}$ = n-vertex d-regular bipartite graphs
  - $H = H_{\text{ind}}$ (e.g. [6]), generalized to *all* H ([5]); maximized when $G = \frac{n}{2d} K_{d,d}$



- $\mathcal{G}$ = n-vertex d-regular graphs
  - $H = H_{\text{ind}}$ ([8]), various H ([4,9]); maximized when $G = \frac{n}{2d} K_{d,d}$ or $G = \frac{n}{d+1} K_{d+1}$



Conjecture 3.2 *Fix H. For $\mathcal{G}$ = n-vertex d-regular graphs, $\hom(G, H)$ is maximized when $G = \frac{n}{2d} K_{d,d}$ or $G = \frac{n}{d+1} K_{d+1}$.*

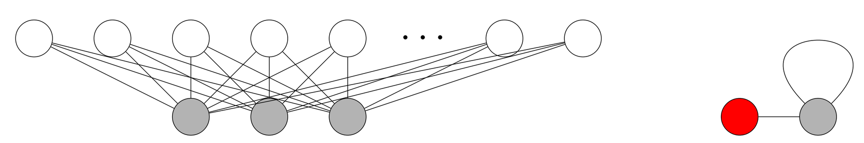
4. Graphs with Fixed Minimum Degree

Notation: $\mathcal{G}(n, \delta)$ = n-vertex graphs with minimum degree δ

Question 4.1 *Fix H. Which $G \in \mathcal{G}(n, \delta)$ maximizes $\hom(G, H)$?*

Independent Sets ($H = H_{\text{ind}}$):

Theorem 4.2 (Galvin, 2011 [3]) *For all $G \in \mathcal{G}(n, \delta)$ and $n \geq 8\delta^2$, $\hom(G, H_{\text{ind}})$ is maximized when $G = K_{\delta, n-\delta}$.*



Degree convention: $d(v)$ is the degree of a vertex (loops count *once*)

Conjecture 4.3 *Fix H. For all $G \in \mathcal{G}(n, \delta)$ and n large enough, $\hom(G, H)$ is maximized when $G = K_{\delta, n-\delta}$, $\frac{n}{2\delta} K_{\delta,\delta}$, or $\frac{n}{\delta+1} K_{\delta+1}$.*

5. Main Theorem

Theorem 5.1 (E., 2013 [2])
- *Conjecture 4.3 is true for $\delta = 1$, $\delta = 2$.*
- *Suppose that H satisfies $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$. Then, for $n > c^\delta$ and $G \in \mathcal{G}(n, \delta)$, $\hom(G, H)$ is maximized when $G = K_{\delta, n-\delta}$.*

Proof Techniques for $\delta = 1$, $\delta = 2$:

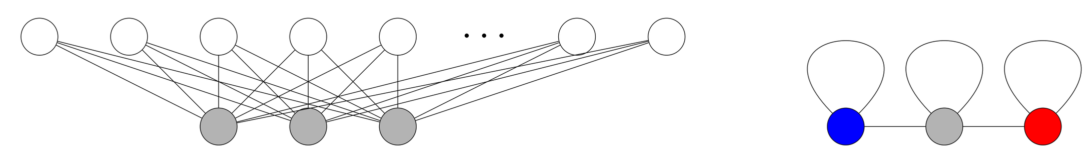
- Analyze structural properties of *edge-critical* graphs G (remove any edge $\implies$ minimum degree drops)

Graphs H satisfying $\sum_{v \in V(H)} d(v) < (\Delta_H)^2$:

- $H_{\text{ind}} : \sum d(v) = 3; (\Delta_H)^2 = 4$ ✓
- $K_q : \sum d(v) = q(q-1); (\Delta_H)^2 = (q-1)^2$ ✘ (e.g. $K_2 : \sum d(v) = 2; (\Delta_H)^2 = 1$ ✘)
- $H_{\text{comp}} : \sum d(v) = 2; (\Delta_H)^2 = 1$ ✘
- $H_{\text{WR}} : \sum d(v) = 7; (\Delta_H)^2 = 9$ ✓
- Any* H with looped dominating vertex

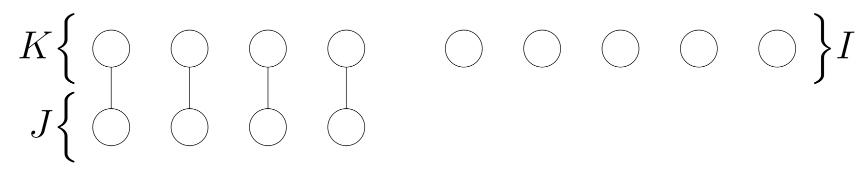
6. Idea of Proof for $H = H_{\text{WR}}$

Goal: $\sum_{v \in V(H)} d(v) < (\Delta_H)^2 \implies \hom(G, H)$ maximized for $G = K_{\delta, n-\delta}$



$$\hom(K_{\delta, n-\delta}, H_{\text{WR}}) \geq 3^{n-\delta}$$

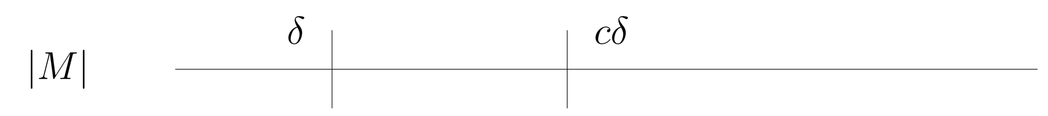
Idea: Partition $\mathcal{G}(n, \delta)$ by the size of a maximum matching M:



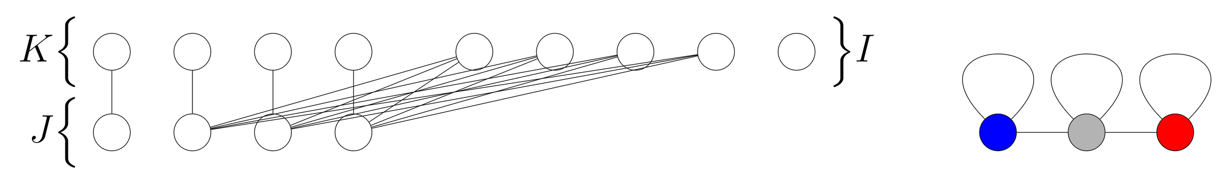
Then:

$$\hom(G, H_{\text{WR}}) \leq 7^{|M|} 3^{n-2|M|} = \left(\frac{7}{3^2}\right)^{|M|} 3^n = \left(\frac{\sum d(v)}{(\Delta_H)^2}\right)^{|M|} 3^n$$

- This implies that any maximizing graph G has $|M| \leq c\delta$



- Graphs with $|M| \leq \delta$ maximized by $K_{\delta, n-\delta}$
- Graphs with $\delta + 1 \leq |M| \leq c\delta$ require analyzing the remaining edges:



  - Show that G contains $K_{\delta, \Omega(n)}$:

$$\hom(G, H_{\text{WR}}) \leq \left(\frac{7}{9}\right) 3^{n-\delta} + \left(\frac{2}{3}\right)^{\Omega(n)} 3^n < 3^{n-\delta}$$

7. Future Directions

- Find a necessary and sufficient blue condition on H for $\hom(G, H) \leq \hom(K_{\delta, n-\delta}, H)$
- Solve Conjecture 4.3 for other values of $\delta \geq 3$
- Find meaningful structural properties of edge-critical graphs when $\delta \geq 3$
- Solve Question 3.1 for $\mathcal{G}$ = n-vertex graphs with min degree δ, max degree at most Δ

8. References

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