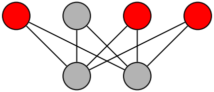
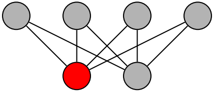


Graph theory — to the extreme!

John Engbers

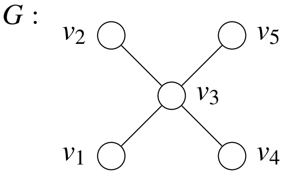
Marquette University
Department of Mathematics, Statistics and Computer Science

Calvin College Colloquium
April 17, 2014



Graph theory - basics

Graph G: set of vertices $V(G)$ and a set edges $E(G)$.



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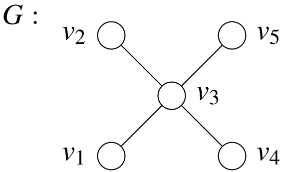

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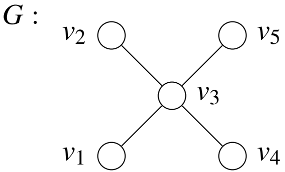

Note: Our graphs will be finite with no loops or multi-edges.

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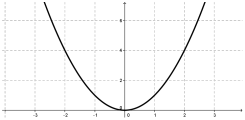
Examples: Facebook pages/friendships; countries/border-sharing; Calvin courses/student taking both courses

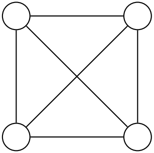
NOT: Graph of $y = f(x) = x^2$:



Main Characters/Examples

Complete graph:

K_4 :



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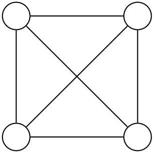

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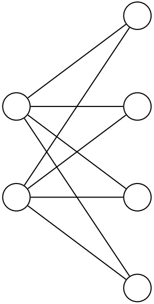

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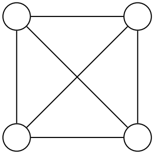

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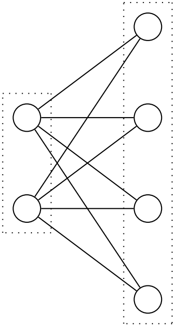

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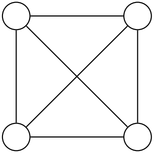

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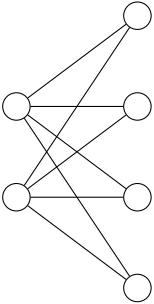

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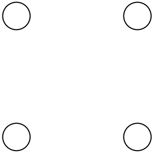

Main technique — Counting!

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Binomial coefficients: If I have n objects, then there are

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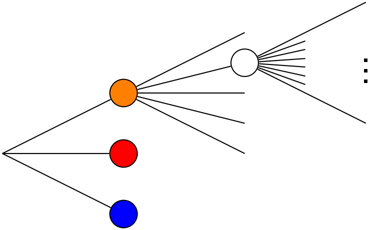
Example: Given a pool of 23 math majors, there are

$$\binom{23}{3} = \frac{23!}{3!\,20!} = 1771$$

different 3-person committees that can be formed.

Main technique — Counting!

Shirts-pants-shoes idea: Suppose that I have 3 different shirts, 5 different pairs of pants, and 8 different pairs of shoes. How many different outfits can I wear?



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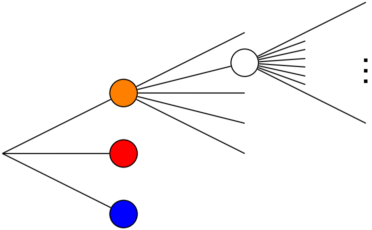

Answer: $3 \cdot 5 \cdot 8 = 120$ different outfits.

Extremal graph theory

Extremal graph theory:

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Extremal graph theory:

Tries to figure out the "most extreme" graph from a family of graphs.

Extremal graph theory

Question (General question)

Fix a family $\mathcal{G}$ of graphs. Out of all of the graphs in $\mathcal{G}$, which has the largest/smallest [insert something here]?

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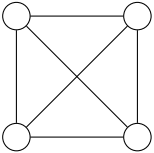

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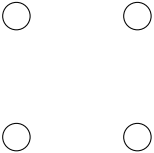

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Solution (Turán 1941): $n = 7$ with no K_4:

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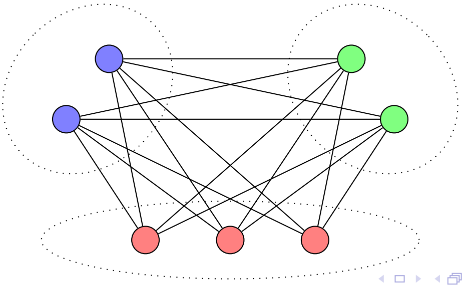
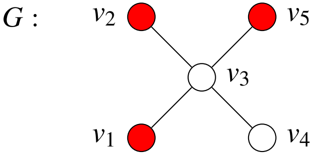


Detour: graph theory — slightly beyond basics

Independent set (of vertices): A set of vertices which are pairwise non-adjacent.



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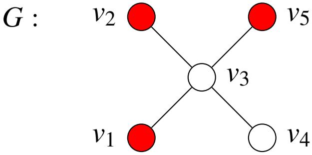

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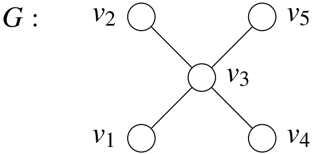

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In the example above,

$$i_0(G) = 1, \quad i_1(G) = 5, \quad i_2(G) = 6, \quad i_3(G) = 4, \quad i_4(G) = 1, \quad i_5(G) = 0$$

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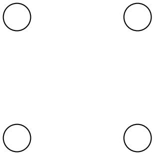

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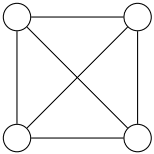

Extremal graph theory

Question (Today's question - different family)

Fix a family $\mathcal{G}$ of graphs. Out of all the graphs in $\mathcal{G}$, which has the largest value of $i_t(G)$ for each t?

Intuition: fewer edges implies more independent sets of a fixed size.

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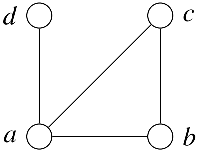
Lex:



$n = 4, m = 4$

ab
ac
ad
bc
bd
cd

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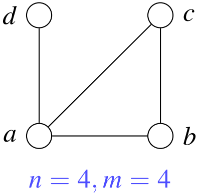

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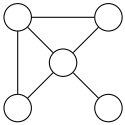
- Is there a more indirect (local) way to force lots of edges?

Extremal graph theory

Minimum degree δ: Smallest number of edges adjacent to a vertex

G :



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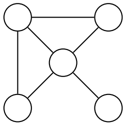
G : 

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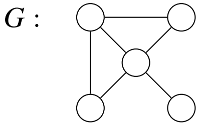

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Out of all the graphs in $\mathcal{G}_n(\delta)$, which has the largest value of $i_t(G)$ for each t?

Remarks (true for *all* graphs on n vertices):

- For all graphs G, $i_0(G) = 1, i_1(G) = n$.

Extremal graph theory

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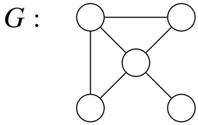

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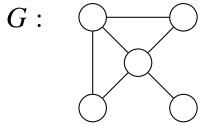

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Guess: extremal graph in $\mathcal{G}_n(\delta)$ should have all degrees equal to δ.

$\delta = 0, 1$

$\mathcal{G}_n(0) \implies$ empty graph maximizes $i_t(G)$. ✓

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$$\mathcal{G}_n(0) \implies \text{empty graph maximizes } i_t(G). \checkmark$$

Question (Today's question, $\delta = 1$)

Out of all the graphs in $\mathcal{G}_n(1)$, which has the largest value of $i_t(G)$ for each t?

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$\mathcal{G}_n(0) \implies$ empty graph maximizes $i_t(G)$. ✓

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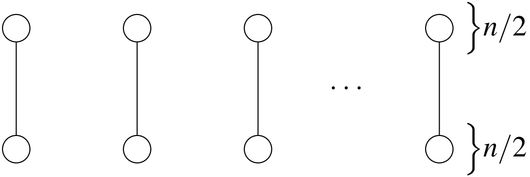
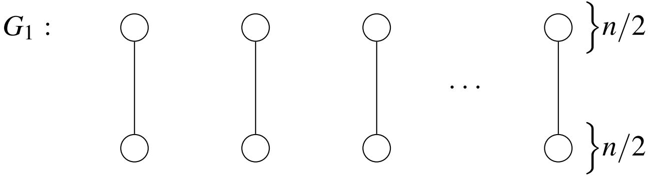
Extremal guess (graph with the least number of edges, n even):



Has largest value of $i_t(G)$ for $t = 0, 1, 2$ and for all G in $\mathcal{G}_n(1)$.

A new contender



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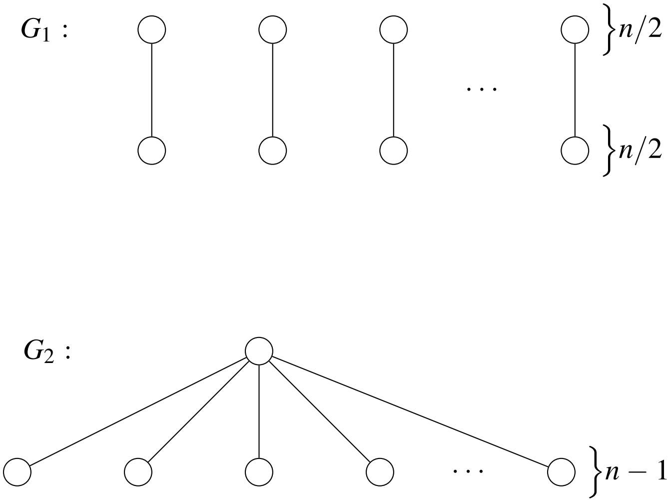

A new contender

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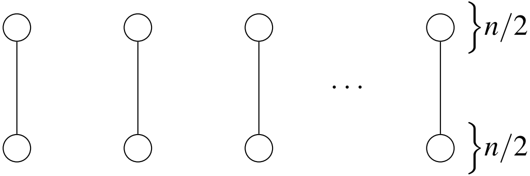
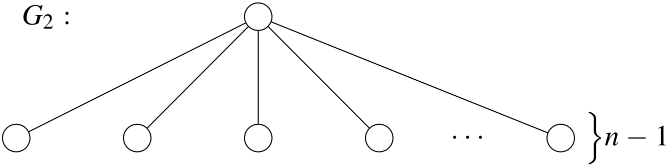


$$i_3(G_1) = \frac{n(n-2)(n-4)}{3!}$$

G_2 :



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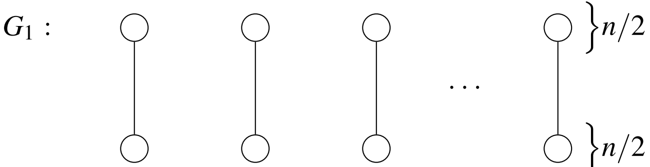

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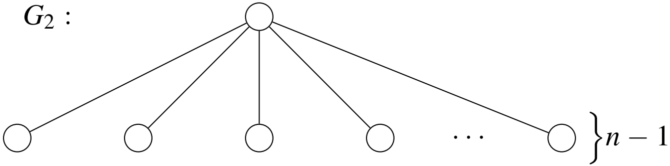

$$i_3(G_2) = \frac{(n-1)(n-2)(n-3)}{3!} > i_3(G_1)$$

$\delta = 1$

Old thought: extremal graph has fewest number of edges.

New thought: extremal graph has largest maximal independent set (for all $t \geq 3$).

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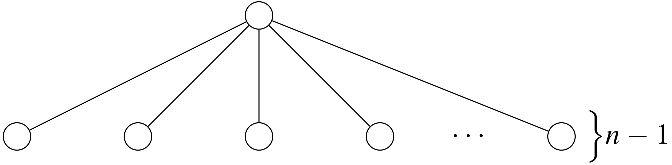
Theorem (Galvin, 2011)

For each $3 \leq t \leq n-1$*, any graph* $G \in \mathcal{G}_n(1)$ *has*

$$i_t(G) \leq i_t(K_{1,n-1}) = \binom{n-1}{t}.$$



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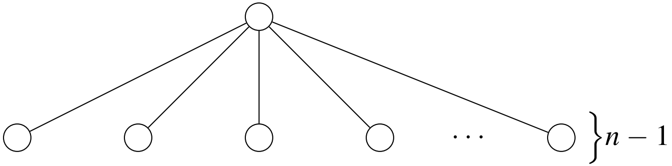
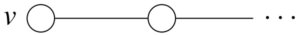


Why?

Proof idea, $t = 3$

Let's start by looking at size $t = 3$: Let G be any graph with minimal degree at least 1.

- Suppose that G has a vertex v as pictured:



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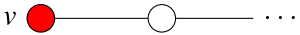

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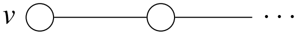

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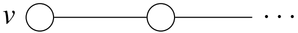

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- **Goal:** show if we don't have this situation, then we have at most $\binom{n-1}{3}$ independent sets of size 3.

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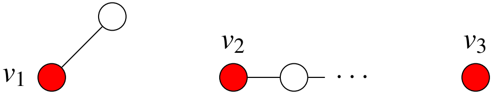

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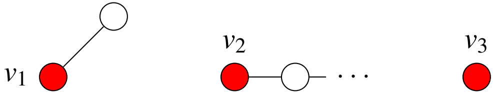

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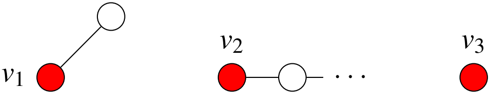

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$$\frac{n(n-2)(n-4)}{3!} < \frac{(n-1)(n-2)(n-3)}{3!} = \binom{n-1}{3}$$

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Proof idea, $t > 3$

We can use ordered independent sets to obtain the result for any $t > 3$:

How many ordered independent sets of size 4 can G in $\mathcal{G}_n(1)$ have?

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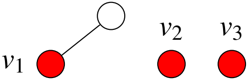

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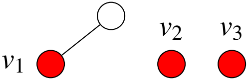

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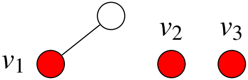

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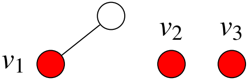

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- This argument works for any $t \geq 4$.

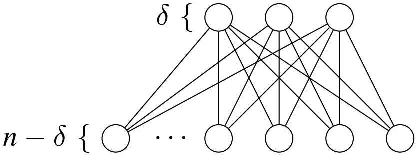
General Result

Larger minimum degrees? Tweaking the previous argument gives:

Theorem (E., Galvin 2014)

Fix $\delta \geq 2$, size $\delta + 1 \leq t \leq n - \delta$, and n large enough. If $G \in \mathcal{G}_n(\delta)$, then

$$i_t(G) \leq i_t(K_{\delta,n-\delta}) = \binom{n-\delta}{t}$$



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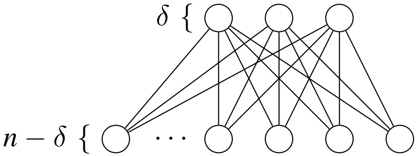

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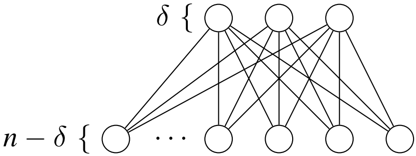

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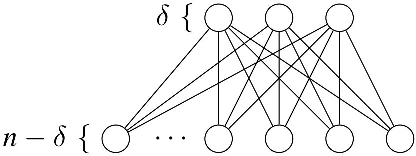

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- Gan-Loh-Sudakov [2014+] Conjecture true(!) (for all $n \ge 2\delta$)

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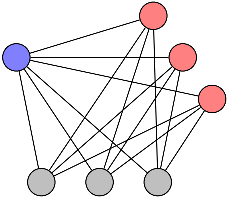

Does this graph maximize $i_t(G)$ ($t \geq 3$) when $n < 2\delta$? [True if $n - \delta | \delta$]

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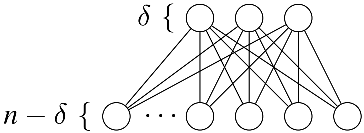

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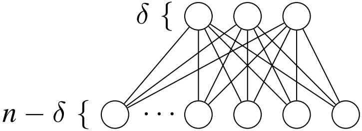

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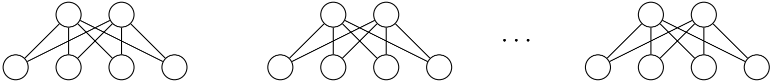

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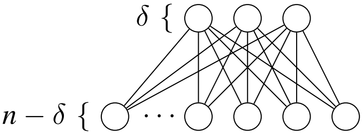

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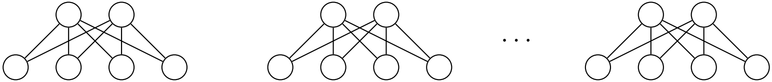

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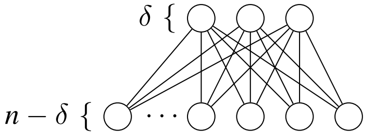

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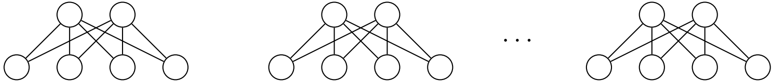

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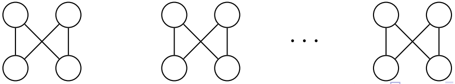

Thank you!

Slides available on my website:
www.mscs.mu.edu/∼engbers/