Conway's Solitaire Army Problem

John Engbers

Department of Mathematics
University of Notre Dame

Graduate Student Seminar
March 15, 2010

Outline

Peg Solitaire

What is it?
A Common single-player game played around the world:

Peg Solitaire

What is it?
A Common single-player game played around the world:

Game: make checkers jumps until a single peg remains.

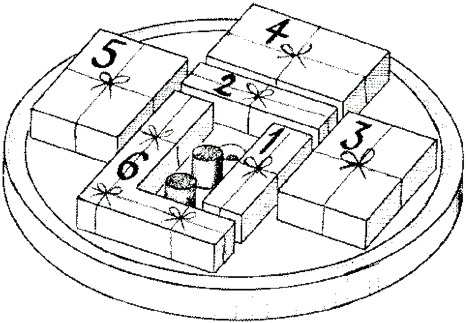
Peg Solitaire

What is it?
A Common single-player game played around the world:

Game: make checkers jumps until a single peg remains.
Spoiler Alert!

Peg Solitaire

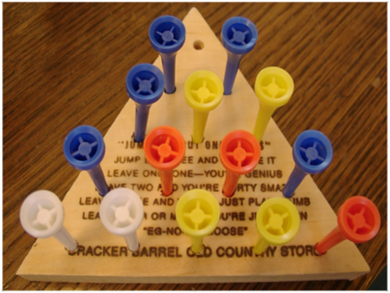
To solve peg solitaire:



Think in terms of 'packaged' moves.

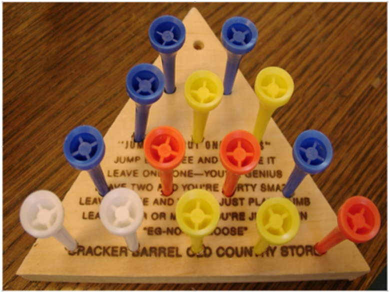
Eg-No-Ra-Moose

A variation on the theme is found at restaurants.



Eg-No-Ra-Moose

A variation on the theme is found at restaurants.



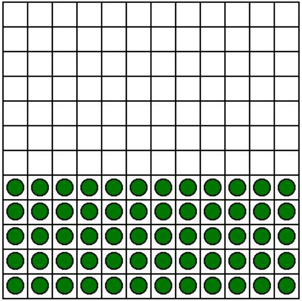
"Leave only one - you're genius...leave four or more'n you're just plain 'eg-no-ra-moose'".

Conway's Soldiers

Now, let's imagine the pegs are army soldiers. Suppose they stand initially on one side of a straight line beyond which is an infinite empty desert.

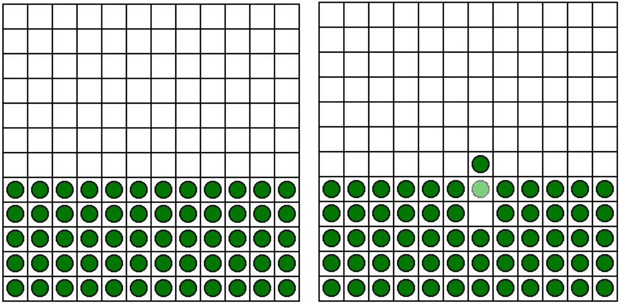
Conway's Soldiers

Now, let's imagine the pegs are army soldiers. Suppose they stand initially on one side of a straight line beyond which is an infinite empty desert.



Conway's Soldiers

Now, let's imagine the pegs are army soldiers. Suppose they stand initially on one side of a straight line beyond which is an infinite empty desert.



How far out can we send a scout from an army with finitely many men?

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	
1	
2	
3	
4	
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	
2	
3	
4	
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	
3	
4	
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	4
3	
4	
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	4
3	8
4	
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	4
3	8
4	20
5	
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	4
3	8
4	20
5	impossible**
⋮	
n	

Conway's Soldiers

Related question: What is the least number of soldiers in the army needed to get our scout k squares into the desert?

Demo

Level	Minimal # Soldiers
0	1
1	2
2	4
3	8
4	20
5	impossible**
⋮	
n	

Unless you are familiar with sequence A014225 in the Online Encyclopedia of Integer Sequences, it is quite remarkable and surprising that level 5 is unreachable!

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Conway's idea: Pagoda Functions/Resource Counting - assign weights to the different squares, and do so in such a way that the legal moves never increase the total weight of all the soldiers.

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Conway's idea: Pagoda Functions/Resource Counting - assign weights to the different squares, and do so in such a way that the legal moves never increase the total weight of all the soldiers.

One example:

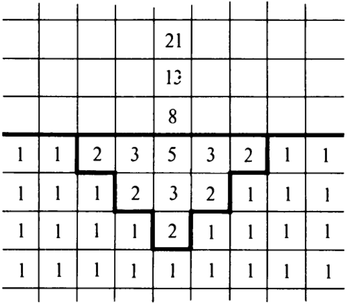
				21				
				13				
				8				
1	1	2	3	5	3	2	1	1
1	1	1	2	3	2	1	1	1
1	1	1	1	2	1	1	1	1
1	1	1	1	1	1	1	1	1

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Conway's idea: Pagoda Functions/Resource Counting - assign weights to the different squares, and do so in such a way that the legal moves never increase the total weight of all the soldiers.

One example:



Total weight = sum of occupied squares.

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Conway's idea: Pagoda Functions/Resource Counting - assign weights to the different squares, and do so in such a way that the legal moves never increase the total weight of all the soldiers.

One example:

				21				
				13				
				8				
1	1	2	3	5	3	2	1	1
1	1	1	2	3	2	1	1	1
1	1	1	1	2	1	1	1	1
1	1	1	1	1	1	1	1	1

Total weight = sum of occupied squares.

Resource counts only show that some things are impossible (usually hard to do).

Resource Counting

We must somehow show that there is no way to get a soldier 5 squares out...

Conway's idea: Pagoda Functions/Resource Counting - assign weights to the different squares, and do so in such a way that the legal moves never increase the total weight of all the soldiers.

One example:

				21				
				13				
				8				
1	1	2	3	5	3	2	1	1
1	1	1	2	3	2	1	1	1
1	1	1	1	2	1	1	1	1
1	1	1	1	1	1	1	1	1

Total weight = sum of occupied squares.

Resource counts only show that some things are impossible (usually hard to do).

Lots of different resource counts!

Resource Counting

Other resource counts:

Place a 1 in every box.

Resource Counting

Other resource counts:

Place a 1 in every box.

No need to be symmetric...

				55				
				34				
				21				
				13				
8	0	8	0	8	5	3	2	1
5	0	5	0	5	3	2	1	1
3	0	3	0	3	2	1	1	1
2	0	2	0	2	1	1	1	1
1	0	1	0	1	1	1	1	1

Resource Counting

Other resource counts:

Place a 1 in every box.

No need to be symmetric...

				55				
				34				
				21				
				13				
8	0	8	0	8	5	3	2	1
5	0	5	0	5	3	2	1	1
3	0	3	0	3	2	1	1	1
2	0	2	0	2	1	1	1	1
1	0	1	0	1	1	1	1	1

What are these good for? Different resource counts give different information.

Resource Counting

We want a resource count that tells us that we can't reach row 5.

Resource Counting

We want a resource count that tells us that we can't reach row 5.

Enter: The Golden Pagoda

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

Resource Counting

We want a resource count that tells us that we can't reach row 5.

Enter: The Golden Pagoda

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

Choose a specific number x so that no move increases the total weight of the occupied squares.

Resource Counting

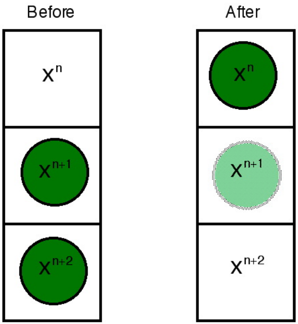
What number x should we choose?

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

Resource Counting

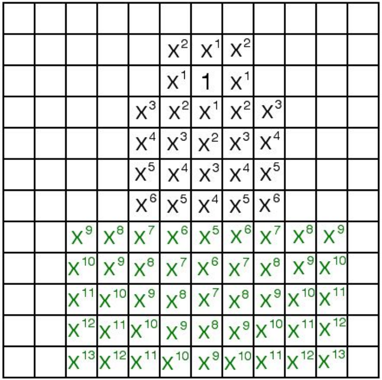
What number x should we choose?

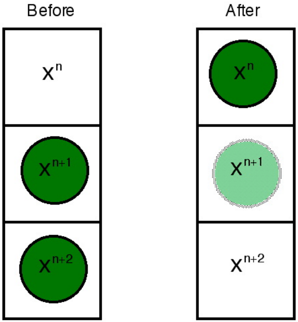
					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	



Resource Counting

What number x should we choose?

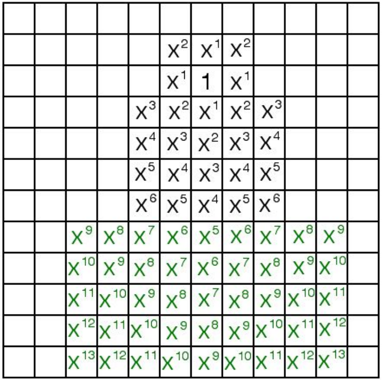


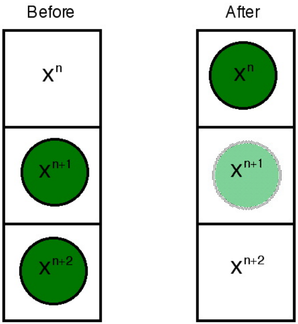


$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$

Resource Counting

What number x should we choose?



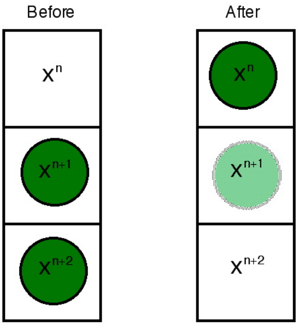


$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$
$$-x^{n+1} - x^n + x^{n+1} = -x^n$$

Resource Counting

What number x should we choose?

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	



$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$
$$-x^{n+1} - x^n + x^{n+1} = -x^n$$
$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Resource Counting

$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$

$$-x^{n+1} - x^n + x^{n+1} = -x^n$$

$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Resource Counting

$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$

$$-x^{n+1} - x^n + x^{n+1} = -x^n$$

$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Find an x with

- $x > 0$

Resource Counting

$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$

$$-x^{n+1} - x^n + x^{n+1} = -x^n$$

$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Find an x with

- $x > 0$
- $x^n(-x^2 - x + 1) = 0.$

Resource Counting

$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$
$$-x^{n+1} - x^n + x^{n+1} = -x^n$$
$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Find an x with

- $x > 0$
- $x^n(-x^2 - x + 1) = 0.$

Now $-x^2 - x + 1 = 0$ implies $x = \frac{\sqrt{5} - 1}{2} = \frac{1}{\phi} \approx .618$.

Resource Counting

$$-x^{n+2} - x^{n+1} + x^n = x^n(-x^2 - x + 1)$$
$$-x^{n+1} - x^n + x^{n+1} = -x^n$$
$$-x^n - x^{n+1} + x^{n+2} = x^n(-1 - x + x^2)$$

Find an x with

- $x > 0$
- $x^n(-x^2 - x + 1) = 0.$

Now $-x^2 - x + 1 = 0$ implies $x = \dfrac{\sqrt{5}-1}{2} = \dfrac{1}{\phi} \approx .618$.

Remark 1: $x^2 = 1 - x$.
Remark 2: Using this x, we see that no move will ever *increase* the weights of all the soldiers in the army.

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

We can add the columns (in green) using infinite series:

$$x^5 + x^6 + x^7 + \cdots = x^5(1 + x + x^2 + \cdots) = \frac{x^5}{1-x} = \frac{x^5}{x^2} = x^3$$

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

We can add the columns (in green) using infinite series:

$$x^5 + x^6 + x^7 + \cdots = x^5(1 + x + x^2 + \cdots) = \frac{x^5}{1-x} = \frac{x^5}{x^2} = x^3$$

$$2(x^6 + x^7 + x^8 + \cdots) = 2x^6(1 + x + x^2 + \cdots) = \frac{2x^6}{1-x} = \frac{2x^6}{x^2} = 2x^4$$

$$\vdots$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$x^3 + 2x^4 + 2x^5 + \cdots = x^3 + 2x^4(1 + x + x^2 + \cdots)$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$\begin{aligned} x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\ &= x^3 + 2\frac{x^4}{1-x} \end{aligned}$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$\begin{aligned} x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\ &= x^3 + 2\frac{x^4}{1-x} \\ &= x^3 + 2x^2 \end{aligned}$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$\begin{aligned} x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\ &= x^3 + 2\frac{x^4}{1-x} \\ &= x^3 + 2x^2 \\ &= x^2(x+2) \end{aligned}$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$
\begin{aligned}
x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\
&= x^3 + 2\frac{x^4}{1-x} \\
&= x^3 + 2x^2 \\
&= x^2(x+2) \\
&= (1-x)(2+x)
\end{aligned}
$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$\begin{aligned}
x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\
&= x^3 + 2\frac{x^4}{1-x} \\
&= x^3 + 2x^2 \\
&= x^2(x+2) \\
&= (1-x)(2+x) \\
&= 2 - x - x^2
\end{aligned}$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$\begin{aligned} x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\ &= x^3 + 2\frac{x^4}{1-x} \\ &= x^3 + 2x^2 \\ &= x^2(x+2) \\ &= (1-x)(2+x) \\ &= 2 - x - x^2 \\ &= 2 - 1 \end{aligned}$$

Total Weight

Therefore, the *total* weight of having *every* soldier in the lower half plane is

$$
\begin{aligned}
x^3 + 2x^4 + 2x^5 + \cdots &= x^3 + 2x^4(1 + x + x^2 + \cdots) \\
&= x^3 + 2\frac{x^4}{1-x} \\
&= x^3 + 2x^2 \\
&= x^2(x+2) \\
&= (1-x)(2+x) \\
&= 2 - x - x^2 \\
&= 2 - 1 \\
&= 1.
\end{aligned}
$$

Total Weight

We only start, by assumption, with a finite number of soldiers.

Total Weight

We only start, by assumption, with a finite number of soldiers.

Hence, the weight of the soldiers that we start with is strictly less than 1, so since $(0,5)$ has weight 1 associated to it, we can't reach $(0,5)$ $\implies$ we're done!

					x^2	x^1	x^2				
					x^1	1	x^1				
				x^3	x^2	x^1	x^2	x^3			
				x^4	x^3	x^2	x^3	x^4			
				x^5	x^4	x^3	x^4	x^5			
				x^6	x^5	x^4	x^5	x^6			
		x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8	x^9	
		x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9	x^{10}	
		x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}	x^{11}	
		x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}	x^{12}	
		x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}	x^{13}	

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

  Can't reach row $3d - 1$

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

  Can't reach row $3d - 1$
- (Skew Army) Allow *only* diagonal jumps instead of vertical and horizontal.

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

  Can't reach row $3d-1$
- (Skew Army) Allow *only* diagonal jumps instead of vertical and horizontal.

  Can't reach row 7

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

  Can't reach row $3d - 1$

- (Skew Army) Allow *only* diagonal jumps instead of vertical and horizontal.

  Can't reach row 7

- (Pablito's Army) Use a hexagonal grid on a triangular board (Macalester POW # 860).

Extensions

What are some natural extensions?

- Play the same game on $\mathbb{Z}^d$.

  Can't reach row $3d - 1$

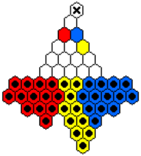
- (Skew Army) Allow *only* diagonal jumps instead of vertical and horizontal.

  Can't reach row 7

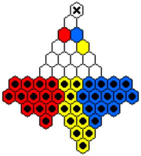
- (Pablito's Army) Use a hexagonal grid on a triangular board (Macalester POW # 860).

Can't reach row 7

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).

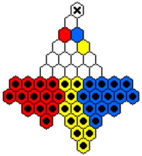


- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



Can't reach row 8

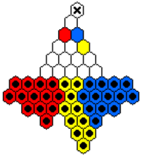
- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).

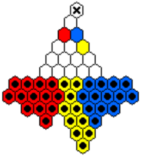


Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

Can't reach row 9

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



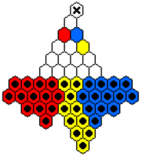
Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

Can't reach row 9

- Minimal number of soldiers/moves?

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



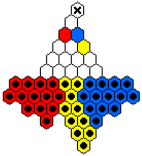
Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

Can't reach row 9

- Minimal number of soldiers/moves?
- Allow 2 soldiers in one square.

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



Can't reach row 8

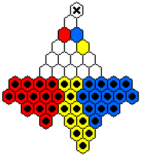
- Conway's army plus diagonal jumps in both directions.

Can't reach row 9

- Minimal number of soldiers/moves?
- Allow 2 soldiers in one square.

Two soldiers in *any* square ⇒ row 5 reachable

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

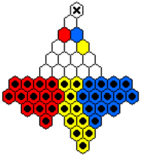
Can't reach row 9

- Minimal number of soldiers/moves?
- Allow 2 soldiers in one square.

Two soldiers in *any* square $\Rightarrow$ row 5 reachable

Or...

- Use on a hexagonal grid on an infinite board (equivalent to Conway plus one diagonal jump).



Can't reach row 8

- Conway's army plus diagonal jumps in both directions.

Can't reach row 9

- Minimal number of soldiers/moves?
- Allow 2 soldiers in one square.

Two soldiers in *any* square $\Rightarrow$ row 5 reachable

Or...theoretically, there is enough weight on the board if we place soldiers at every point in the lower half plane...

Conway's Infinite Army

Suppose that we have a soldier on every point (x, y) with $y \leq 0$. Our previous argument shows that to get a soldier to $(0, 5)$, we'll need to use every soldier on the board.

Conway's Infinite Army

Suppose that we have a soldier on every point (x, y) with $y \leq 0$. Our previous argument shows that to get a soldier to $(0, 5)$, we'll need to use every soldier on the board.

We'll need to make infinitely many moves, and no move can decrease the board weight.

Conway's Infinite Army

Suppose that we have a soldier on every point (x, y) with $y \leq 0$. Our previous argument shows that to get a soldier to $(0, 5)$, we'll need to use every soldier on the board.

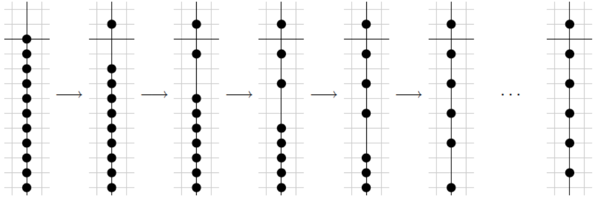
We'll need to make infinitely many moves, and no move can decrease the board weight.

What kind of moves can we make?

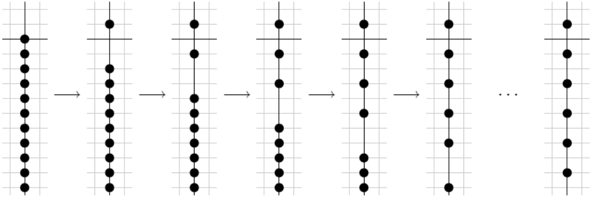
Moves in parallel: $(x, -1)$ jumps $(x, 0)$ at the same time.

More interestingly, let's do a sequence of moves.

More interestingly, let's do a sequence of moves.

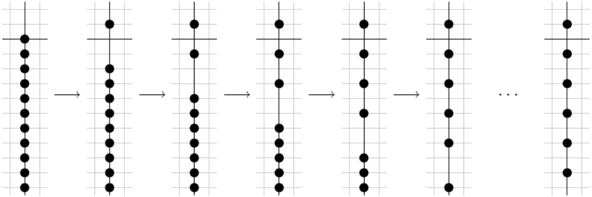


More interestingly, let's do a sequence of moves.



To do this, we should introduce a time factor, so the moves take place at times $\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\}$.

More interestingly, let's do a sequence of moves.



To do this, we should introduce a time factor, so the moves take place at times $\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\}$.

We can still make more moves!

Infinite Model

To make sense of this, we should come up with a model for our situation.

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time
- Only legal peg solitaire moves are allowed

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \rightarrow \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time
- Only legal peg solitaire moves are allowed
- Each point is involved in a (uniformly) bounded number of moves

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time
- Only legal peg solitaire moves are allowed
- Each point is involved in a (uniformly) bounded number of moves
- Starting Position: $f(x, y, t_{start}) = 1$ if $y \leq 0$ and 0 otherwise.

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time
- Only legal peg solitaire moves are allowed
- Each point is involved in a (uniformly) bounded number of moves
- Starting Position: $f(x, y, t_{start}) = 1$ if $y \leq 0$ and 0 otherwise.
- Ending Position: $f(0, 5, t_{end}) = 1$.

Infinite Model

To make sense of this, we should come up with a model for our situation.

Our moves are described by a function $f : \mathbb{Z}^2 \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}$ where $f(x, y, t)$ gives the state of (x, y) at time t such that:

- All changes of state occur by means of a move
- Only one move can happen at a time
- Only legal peg solitaire moves are allowed
- Each point is involved in a (uniformly) bounded number of moves
- Starting Position: $f(x, y, t_{start}) = 1$ if $y \leq 0$ and 0 otherwise.
- Ending Position: $f(0, 5, t_{end}) = 1$.

We call f a *valid solution* in this case.

Backwards Moves

Our definition allows us to do infinite moves...in reverse!

Backwards Moves

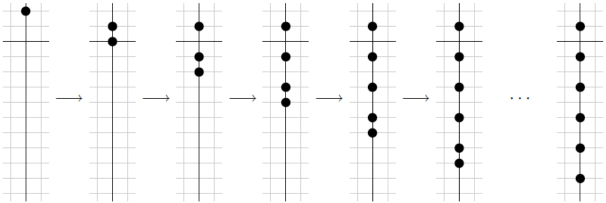
Our definition allows us to do infinite moves...in reverse!

These are easiest to describe backwards (i.e., adding soldiers, Leibniz's favorite way to play peg solitaire):

Backwards Moves

Our definition allows us to do infinite moves...in reverse!

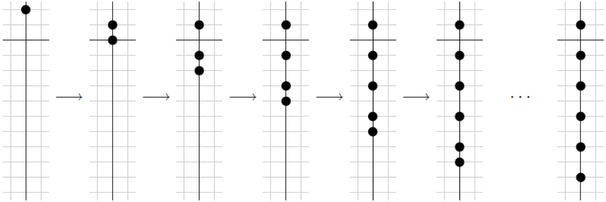
These are easiest to describe backwards (i.e., adding soldiers, Leibniz's favorite way to play peg solitaire):



Backwards Moves

Our definition allows us to do infinite moves...in reverse!

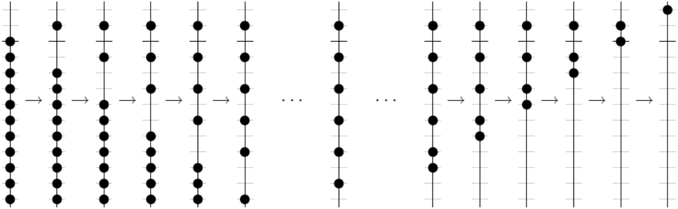
These are easiest to describe backwards (i.e., adding soldiers, Leibniz's favorite way to play peg solitaire):



So, we can think of these moves taking place at times $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$

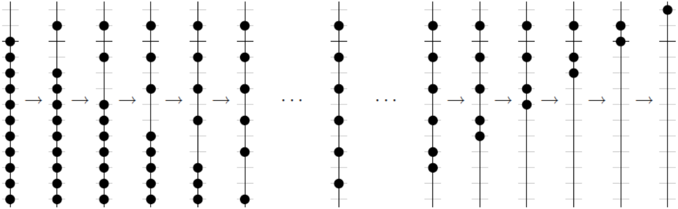
Whoosh

Putting the two moves together, we get a package called the "whoosh":



Whoosh

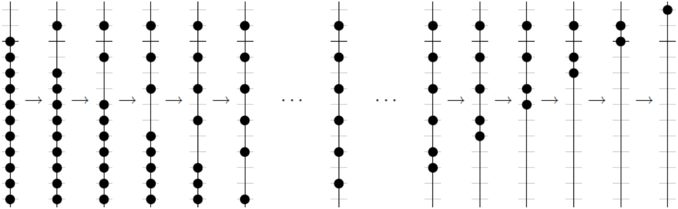
Putting the two moves together, we get a package called the "whoosh":



Wait a second...

Whoosh

Putting the two moves together, we get a package called the "whoosh":



Wait a second...

Theorem

Any function f representing a valid solution has a strictly increasing infinite sequence and a strictly decreasing infinite sequence of moves.

Another Package?

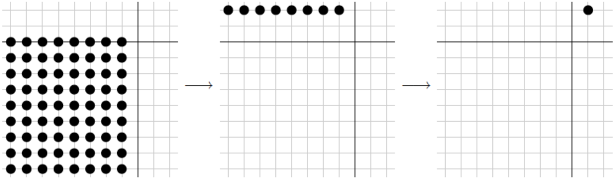
Whoosh: takes a semi-infinite line of pegs and turns it into a single peg.

Another Package?

Whoosh: takes a semi-infinite line of pegs and turns it into a single peg.

2-dimensions: take quarter-plane, turn it into a single peg.

One possible way: Whoosh columns up, then whoosh row over.

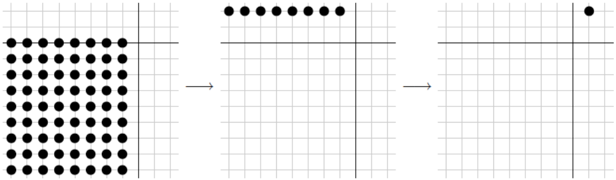


Another Package?

Whoosh: takes a semi-infinite line of pegs and turns it into a single peg.

2-dimensions: take quarter-plane, turn it into a single peg.

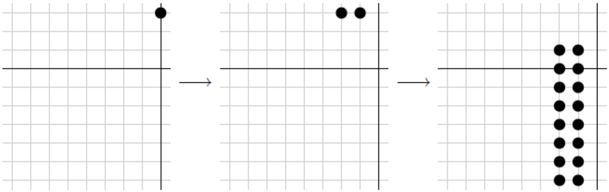
One possible way: Whoosh columns up, then whoosh row over.



Why is this bad?

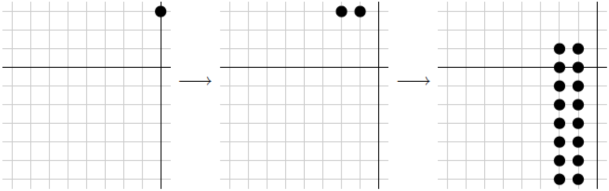
Megawhoosh

A better way: the package called the "megawhoosh," best described in reverse:

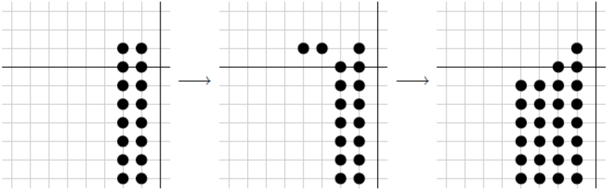


Megawhoosh

A better way: the package called the "megawhoosh," best described in reverse:

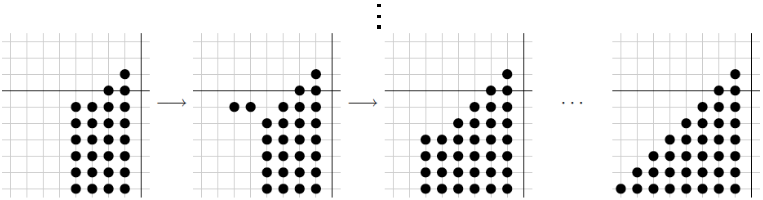


Repeat:

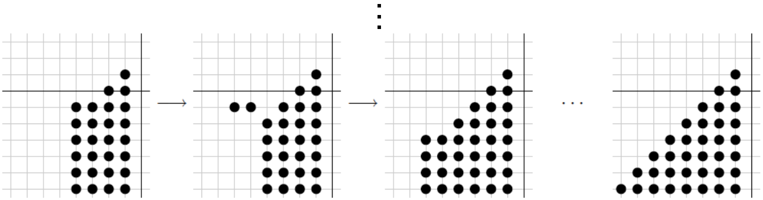


⋮

Megawhoosh

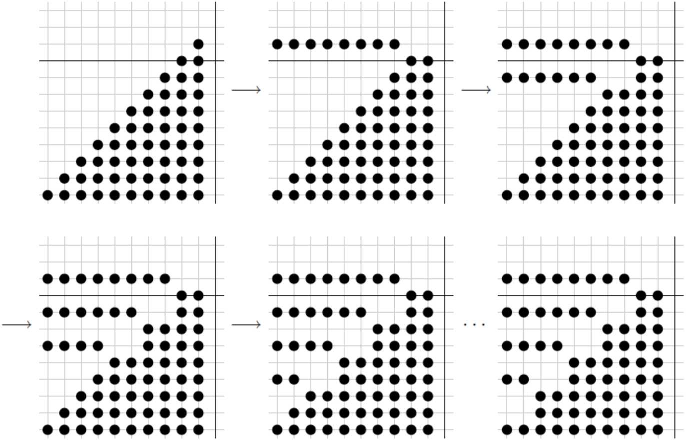


Megawhoosh

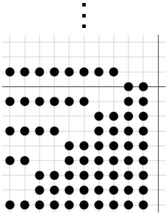


Next, we'll whoosh left with every other peg:

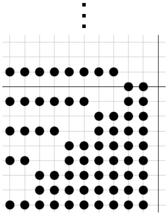
Megawhoosh



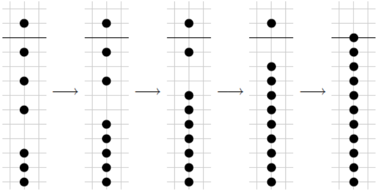
Megawhoosh



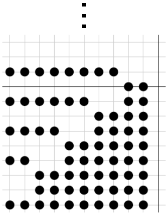
Megawhoosh



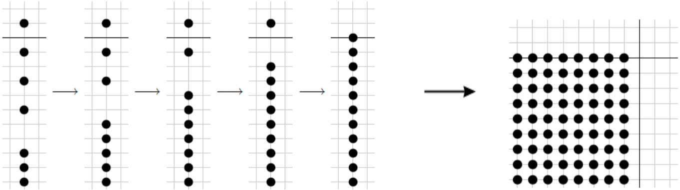
Finally, use downward moves in each column:



Megawhoosh



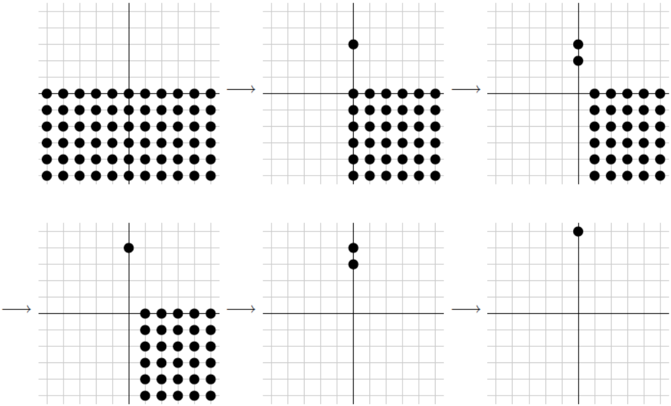
Finally, use downward moves in each column:



This gets us to our quarter-plane. The "megawhoosh" is all of these moves, in reverse.

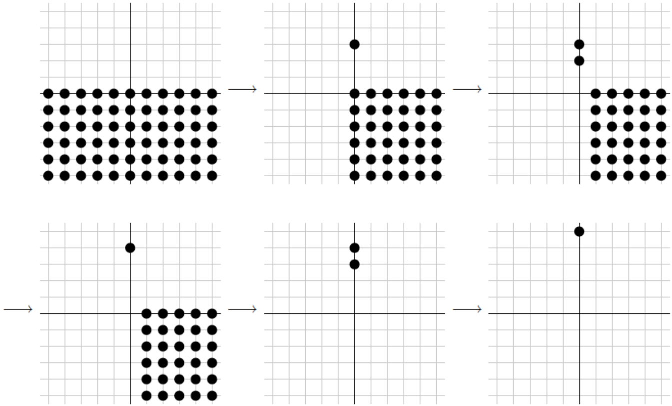
Solution

We can finally put together the entire solution, using the whoosh and megawhoosh packages:



Solution

We can finally put together the entire solution, using the whoosh and megawhoosh packages:



As predicted, all soldiers are used to reach $(0, 5)$.

References

Thanks!

References:

E. Berlekamp, J. H. Conway, R. Guy, *Winning Ways for Your Mathematical Plays Vol. 4*, AK Peters, 2004. Ch 23.

(⋆)J. Beasley, *The Ins & Outs of Peg Solitaire*, Oxford Univ. Press, 1985.

(⋆)S. Tatham, G. Taylor, "Reaching Row 5 in Solitaire Army," *http://tartarus.org/gareth/maths/stuff/solarmy.pdf*.

Internet Resources:

- http://www.cut-the-knot.org/proofs/checker.shtml
- (⋆)http://home.comcast.net/∼gibell/pegsolitaire/army/
- http://www.chiark.greenend.org.uk/∼sgtatham/solarmy/
- (⋆)http://polymathematics.typepad.com/polymath/trekking-into-the-desert.html

(⋆) indicates that figures came from these sources

Uniform Bound?

What happens if we relax the uniform bound? We create a peg out of nothing...idea: recursively eliminate the peg from the origin.

Uniform Bound?

What happens if we relax the uniform bound? We create a peg out of nothing...idea: recursively eliminate the peg from the origin.