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ADVANCED REVIEW

A Journey from Univariate to Multivariate Functional Time Series: A Comprehensive Review

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Abstract

Functional time series (FTS) analysis has emerged as a potent framework for modeling and forecasting time-dependent data with functional attributes. In this comprehensive review, we navigate through the intricate landscape of FTS methodologies, meticulously surveying the core principles of univariate FTS and delving into the nuances of multivariate FTS. The journey commences with an exploration of the foundational aspects of univariate FTS analysis. We delve into representation, estimation, and modeling, spotlighting the effectiveness of various parametric and nonparametric models at our disposal. The stage then transitions to multivariate FTS analysis, where we confront the intricacies posed by high-dimensional data. We explore strategies for dimensionality reduction, forecasting, and the integration of diverse parametric and nonparametric models within the multivariate realm. We also highlight commonly used R packages for modeling and forecasting FTS and multivariate FTS. Acknowledging the dynamic evolution of the field, we dissect challenges and chart future directions, paving a course for refinement and innovation. Through a fusion of multivariate statistics, functional analysis, and time series forecasting, this review underscores the interdisciplinary essence of FTS analysis. It not only reveals past accomplishments but also illuminates the potential of FTS in unraveling insights and facilitating well-informed decisions across diverse domains.

KEYWORDS
functional time series, forecasting, functional principal component analysis, multivariate functional time series

1 | INTRODUCTION

Functional data pertains to datasets in which each observation represents a function that is defined over a continuous domain. In contrast to conventional scalar or multivariate data, functional data retains the complete functional characteristics of the observations, encompassing their shape, evolution, and variability. This unique data type provides researchers with a flexible and potent framework for analyzing intricate and high-dimensional datasets, facilitating the comprehension and modeling of the underlying functional structures and dynamics inherent in the data. For more comprehensive information on functional data types and application methods, interested readers are directed to the monographs by Ramsay and Silverman (2005), Ferraty (2006), Horváth and Kokoszka (2012) and Kokoszka and Reimherr (2017).

Functional time series (FTS) represents a specialized iteration of functional data with a unique focus on capturing the temporal evolution of functions over a continuous timeline. In FTS, the dataset consists of a sequence of functions indexed by time, encapsulating the dynamic behavior inherent to a functional process. By extending time series analysis principles to functional data, FTS incorporates temporal dependencies and patterns that manifest both within and between these observed functions. Exploring FTS offers researchers valuable insights into temporal dynamics, trends, and interdependencies within functional processes, thus enabling tasks such as forecasting, modeling, and comprehending the evolving behavior of intricate systems across time. FTS methodologies facilitate the identification and extraction of temporal patterns, enabling a deep exploration of

Abbreviations: FTS, functional time series; MFTS, multivariate functional time series; FAR, functional autoregressive; FPCA, functional principal component analysis; FSSA, functional singular spectrum analysis; MFDLM, multivariate functional dynamic linear model; MFSSA, multivariate functional singular spectrum analysis relationships and interactions among functional variables at distinct time instances. A notable advancement in FTS analysis is the consideration of multiple FTS simultaneously, over either the same or distinct domains. Referred to as multivariate FTS (MFTS) analysis, this emerging and critical research domain delves into the analysis, modeling, and forecasting of FTS scenarios when multiple functional variables come into play. This extension addresses the intricate complexities that arise when engaging with the simultaneous consideration of numerous functional processes evolving over time.

The rise of FTS/MFTS analysis comes from the idea that understanding and representing real-world things can be improved by thinking about data as evolving functions over time instead of just regular numbers or sets of numbers. FTS gives us a strong way to study how complicated systems change. There are a few reasons why people are interested in studying FTS/MFTS:

1. Dealing with Complex Data: Many modern datasets are collected in the form of functional data, such as curves, shapes, or images, arising in diverse domains such as finance, economics, biology, environmental sciences, and healthcare. FTS/MFTS provides a specialized framework to model and analyze the unique characteristics of these complex functional datasets, including inherent smoothness, irregular sampling, and high-dimensionality (Ramsay & Silverman, 2005; Ferraty, 2006).

2. Temporal Dependencies and Dynamics: One of the key advantages of FTS/MFTS analysis is its ability to investigate the dynamics between functions while controlling for within-function dynamic changes. This feature allows for a more comprehensive understanding of the relationships and interactions between functional processes over time. By utilizing FTS/MFTS techniques, researchers can disentangle the within-function and between-function dynamics, enabling a deeper exploration of the relationships and dependencies between functional processes. This approach facilitates a more nuanced analysis of complex systems and provides valuable insights into the interplay between multiple functional variables over time (Hörmann & Kokoszka, 2010).

3. Forecasting and Decision-Making: FTS/MFTS offers powerful tools for forecasting future functional observations. The ability to model and predict the future evolution of functional processes is highly valuable in decision-making, resource allocation, and risk management scenarios. FTS/MFTS methods provide reliable forecasts, allowing practitioners to make informed decisions and plan for the future based on the expected behavior of functional systems (see e.g., Hyndman & Ullah, 2007; Ferraty, 2006; Gao & Shang, 2017).

4. Interdisciplinary Applications: FTS/MFTS finds applications in a wide range of disciplines. From economics and finance to environmental sciences, healthcare, and beyond, the study of FTS/MFTS has the potential to enhance our understanding of complex systems and improve decision-making in various domains. FTS/MFTS methods can unveil hidden patterns, detect anomalies, and extract meaningful information from complex functional datasets, leading to novel insights and advancements in interdisciplinary research.

When confronted with datasets characterized by high-resolution measurements, irregular sampling intervals, and intricate, high-dimensional structures, traditional multivariate approaches often encounter limitations. These challenges are intricately linked to the complexity of the data itself. To vividly illustrate this, we have introduced a practical example in Figure 1. In this example, we present a bivariate example that combines intraday hourly mean temperature curves and NDVI images. This illustrative case focuses on a parallelogram-shaped geographical region situated just east of Glacier National Park in Montana, U.S.A. The area spans longitudes from 113.30°E to 113.40°E and latitudes from 48.70°N to 48.77°N. Data collection occurred at intervals of 16 days, commencing on January 1, 2008, and concluding on September 30, 2013. The visual depiction emphasizes the intricate nature of this dataset, underscoring the need for a comprehensive analytical framework, such as FTS or MFTS, to effectively address the intricate complexities inherent to this hybrid dataset. It is evident that the nature of this dataset is inherently multifaceted, making it challenging to analyze effectively using traditional multivariate time series methods.

The subsequent sections of this paper are organized as follows. Section 2 delves into a comprehensive review of the existing literature on FTS, encompassing discussions on various representations, developed theories, dimension reduction approaches, estimation techniques, and forecasting methodologies. In Section 3, our focus shifts to the exploration of MFTS methods, along with an examination of certain extensions of FTS to the multivariate context. The software landscape pertaining to FTS and MFTS is surveyed in Section 4, which highlights the existing software packages in this domain. Lastly, Section 5 concludes the paper by examining current challenges and potential avenues for future advancements in the field of MFTS.

2 | UNIVARIATE FUNCTIONAL TIME SERIES ANALYSIS

Consider a positive integer d and the compact set $\mathcal{T} \subseteq \mathbb{R}^d$. For curve data, $\mathcal{T}$ is typically a continuous interval (i.e., $d = 1$), while for image data, $\mathcal{T}$ can be a compact closed domain in $\mathbb{R}^2$. Let $\mathbb{H} = L^2(\mathcal{T})$ denote the real Hilbert space comprising all squared integrable functions over $\mathcal{T}$. The inner product $\langle x, y\rangle_{\mathbb{H}}$ is defined as the integral of the product of functions $x(\cdot)$ and $y(\cdot)$ over

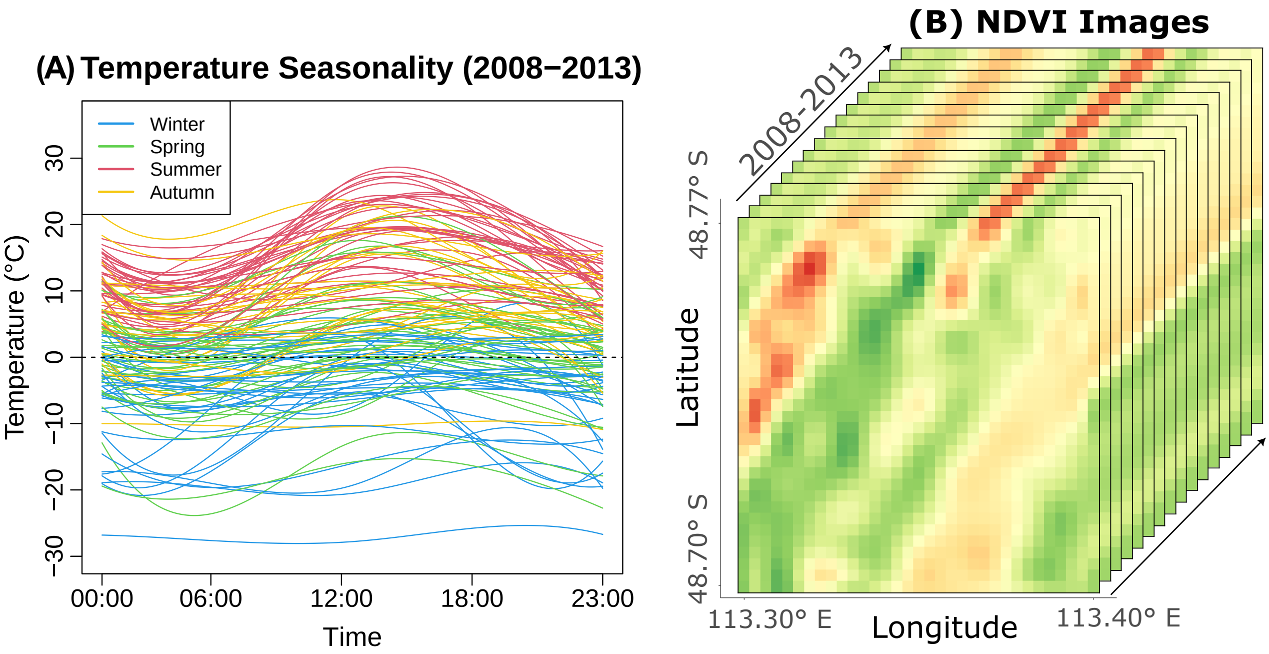


FIGURE 1 An example of MFTS from Jan 2008 to September 2013 - (A): Intraday temperature curves recorded in Saint Mary, Montana, U.S.A.; (B): Time series of NDVI images. Temperature curves and NDVI images are observed every 16 days.

$\mathcal{T}$, and the norm of a function x is given by $\|x\|_{\mathbb{H}} = \sqrt{\langle x, x\rangle_{\mathbb{H}}}$. Additionally, let $\mathcal{L}(\mathbb{H})$ denote the space of all continuous linear operators from $\mathbb{H}$ to $\mathbb{H}$, equipped with the operational norm $\|\cdot\|_{\mathcal{L}}$. Within $\mathcal{L}(\mathbb{H})$, there exists an important subspace known as the space of Hilbert-Schmidt operators, denoted as $\mathcal{HS}(\mathbb{H})$. This subspace is endowed with a Hilbert space structure featuring an inner product $\langle A, B\rangle_{\mathcal{S}} = \sum_{k=1}^{\infty}\langle A\phi_k, B\phi_k\rangle_{\mathbb{H}}$ and a norm $\|A\|_{\mathcal{S}} = \left(\sum_{k=1}^{\infty}\|A\phi_k\|_{\mathbb{H}}^2\right)^{1/2}$. Here, $\{\phi_k\}$ represents any orthonormal basis defined on $\mathbb{H}$. All random functions are defined within a shared probability space denoted as $(\Omega, \mathcal{A}, P)$. Given a random function X, if $E\|X\|_{\mathbb{H}}^k < \infty$, we denote it as $X \in L_{\mathbb{H}}^k$.

In this paper, a univariate FTS is represented by $\{X_t\}_{t\in\mathbb{Z}}$, where the positive integer indices t correspond to time, and for each t, we assume X_t to be a random element of $\mathbb{H}$. The mean function of an FTS is defined as $\mu_t = \mu_t(\tau)$, $\tau \in \mathcal{T}$, with $\mu_t(\tau) = E(X_t(\tau))$, representing the expected value of $X_t(\tau)$. For fixed integers s and t, we define the autocovariance kernel function as $c_{s,t}(\tau, \sigma) = \mathrm{cov}(X_s(\tau), X_t(\sigma))$, and the autocovariance operator $\mathcal{C}_{s,t}$ is the corresponding integral operator of the kernel function $c_{s,t}(\cdot,\cdot)$ where

$$\mathcal{C}_{s,t}(x)(\tau) := \int_{\mathcal{T}} c_{s,t}(\tau, \sigma)x(\sigma)d\sigma.$$

These representations and estimations form the foundation for analyzing FTS and allow us to explore the dynamics and dependencies within the data using mathematical tools from functional data analysis and time series analysis.

An FTS is considered (weakly) stationary if it satisfies two conditions: (i) The mean function, denoted as μ_t, remains independent of time, i.e., $\mu_t = \mu$. (ii) The autocovariance operator, denoted as $\mathcal{C}_h$, solely depends on the time distance and is represented as $\mathcal{C}_h := \mathcal{C}_{t,t+h} = \mathcal{C}_{0,h}$. An stationary FTS $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ with autocovariance operator C_h^{ε} is called functional white noise if:

(i) $\varepsilon_t \in L_{\mathbb{H}}^2$, $E(\varepsilon_t(\tau)) = 0$, for all $\tau \in \mathcal{T}$,
(ii) $C_h^{\varepsilon} = 0$ for all $h > 0$.

For a finite sample path of a stationary FTS, denoted as $\{X_t\}_{t=1}^N$, we can estimate the sample mean and sample autocovariance kernel of lag h as follows:

$$\hat{\mu}(\tau) = \frac{1}{N}\sum_{t=1}^{N} X_t(\tau) \quad \text{and} \quad \hat{c}_h(\tau, \sigma) = \frac{1}{N}\sum_{t=1}^{N}\left(X_{t+h}(\tau) - \hat{\mu}(\tau)\right)\left(X_t(\sigma) - \hat{\mu}(\sigma)\right).$$

The sample autocovariance operator of lag h can be obtained as the corresponding integral operator of $\hat{c}_h(\tau, \sigma)$:

$$\hat{C}_h(x)(\tau) = \frac{1}{N} \sum_{t=1}^{N} (X_{t+h}(\tau) - \hat{\mu}(\tau)) \langle X_t - \hat{\mu}, x \rangle_{\mathbb{H}}. \tag{1}$$

These statistical measures allow us to characterize the stationary behavior of the functional time series and provide estimations for the sample mean and sample autocovariance kernel, essential for subsequent analysis and modeling. The estimator (1) has been subjected to comprehensive large-sample analysis in Bosq (2000) under mild conditions. Notably, it has been demonstrated that assuming the FAR(1) model and $E\|X_0\|_{\mathbb{H}} < \infty$, the following convergence holds:

$$||\hat{C}_h - C_h||_{\mathcal{S}} = O\left(\left(\frac{LogN}{N}\right)^{1/2}\right) \quad a.s, \quad h = 1, 2, \dots,$$

where $||.||_{\mathcal{S}}$ represents the Hilbert-Schmidt norm.

FTS analysis has witnessed various developments and extensions in recent years. Bosq (2000) introduced a linear model framework and autoregressive process in function space (FAR) for FTS. Additionally, the first order functional autoregressive model, FAR(1), was considered by Besse, Cardot, and Stephenson (2000) to predict the entire annual cycle of sea surface temperature, while Damon and Guillas (2002) incorporated exogenous variables into the FAR model (FARX) for ozone forecasting.

Nonparametric forecasting of FTS was explored by Hyndman and Ullah (2007) using functional principal components (FPCs) through Karhunen-Loève expansion. Extending this work, Hyndman and Shang (2009) introduced Functional Partial Least Squares Regression. Weak dependence in FTS was studied by Hörmann and Kokoszka (2010), under the concept of $L^p - m$-approximability, impacting statistical functional data procedures. Regarding stationarity assumptions, Horváth, Kokoszka, and Rice (2014) extended KPSS statistics to test the null hypothesis of FTS stationarity. In contrast, Panaretos and Tavakoli (2013) and Hörmann, Kidziński, and Hallin (2015) utilized Fourier analysis to introduce dynamic FPCA for stationary FTS. Vector autoregressive (VAR) models for forecasting FTS were introduced by Aue, Norinho, and Hörmann (2015). The estimation of long-run autocovariance operator in FTS was addressed by Rice and Shang (2017) using an asymptotically consistent method. In the work by Aue and Klepsch (2017), an approach was introduced, namely the functional moving average (FMA) processes, for estimating invertible FTS. To address seasonal behavior, González, San Roque, and Perez (2017) extended the standard seasonal ARMAX model to FTS (FSARMAX). In the work of Xu, Li, and Chen (2017), the authors proposed the varying coefficient FAR (VC-FAR) models to address non-stationary FTS. This model incorporates smoothing transitions that depend on time index, and its estimation is accomplished using time-varying kernels. On the other hand, Chen and Li (2017) introduced the adaptive FAR (AFAR) model, which does not assume any specific type of structural change. The estimation of AFAR is carried out using a sequential testing procedure under the local homogeneity assumption. In the study by Chen, Marron, and Zhang (2019), an approach called the warping FAR (WFAR) model was introduced to effectively handle both phase and amplitude variations in FTS with seasonality. The proposed method utilizes warping, a seasonal adjustment technique, to isolate seasonal phase variations in functional curves, and then incorporates the seasonally-adjusted curves into the FAR framework for analysis and modeling. Recently, new methodologies have emerged in FTS analysis. Haghbin, Najibi, Mahmoudvand, Trinka, and Maadooliat (2021) introduced functional singular spectrum analysis (FSSA) as a model-free procedure to decompose FTS into informative trends and components. The assessment of serial correlation across different time lags plays a pivotal role in the identification and evaluation of FTS characteristics, as demonstrated by Mestre, Portela, Rice, San Roque, and Alonso (2021). The authors introduce functional adaptations of the autocorrelation (ACF) and partial autocorrelation functions (PACF) for FTS. These adaptations are founded on the L^2 norm of lagged autocovariance operators inherent to the series. To aid in order selection and the evaluation of FSARMAX model appropriateness, the authors propose diagnostic plots of these functions. Additionally, prediction bounds are established utilizing large sample results for ACF and PACF, which are estimated from a robust functional white noise series. These tools offer a rapid and effective approach for order selection and adequacy assessment within FSARMAX modeling contexts. In the study conducted by Zamani, Haghbin, Hashemi, and Hyndman (2022), the authors proposed the development of seasonal FAR (SFAR) models along with the introduction of a portmanteau test to assess the adequacy of these models. Chang, Chen, Qiao, and Yao (2023) studied high-dimensional FTS with errors, where each functional data point was decomposed into two uncorrelated components: one dynamic and one white noise. They introduced a learning framework leveraging autocovariance-based dimension reduction and block regularized minimum distance estimation. Moreover, Yeh, Rice, and Dubin (2023) proposed a "spherical autocorrelation" measure for FTS to describe the direction of serial dependence and enhance robustness to outliers. In the work by Huang and Shang (2023), an innovative nonlinear approach to the functional autocorrelation function (fACF) was introduced, designed to grasp intricate nonlinear patterns within FTS analysis. Their study showcased that if the temporal dynamics of the process are linear, the insights gained from their novel nonlinear approach align with those from the traditional fACF. However, in cases where the underlying process demonstrates nonlinear temporal relationships, their proposed method outperforms the standard fACF by revealing the intricate nonlinear structures that the latter might mislead. These recent advancements and methodologies have significantly contributed to the robust analysis and forecasting of FTS, opening new avenues for future research.

While our primary focus in this review pertains to the analysis of stationary FTS, it is worth noting that some studies have extended their exploration to the realm of non-stationary phenomena within FTS analysis. An illustrative example is the investigation of periodically correlated FTS, wherein data showcases cyclic patterns in both the mean function and the autocovariance operator across time. For a deeper understanding, one can delve into Soltani and Hashemi (2011), which elaborates on the definition of the FAR model for periodically correlated FTS, and also consult Kidziński, Kokoszka, and Jouzdani (2018) for a comprehensive treatment of applying FPCA to such non-stationary FTS data.

2.1 | Functional Principal Component Analysis

Given the autocovariance operator, denoted as $\mathcal{C}_0$, it's straightforward to demonstrate that this operator is a non-negative, self-adjoint Hilbert-Schmidt operator (Weidmann, 1980). As a consequence of these properties, $\mathcal{C}_0$ can be effectively decomposed, as follows:

$$\mathcal{C}_0(x) = \sum_{k=1}^{\infty} \lambda_k \langle \nu_k, x\rangle_{\mathbb{H}} \nu_k, \quad x \in \mathbb{H},$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ are called the eigenvalues, and orthonormal elements $\nu_k(\cdot)$ represent the corresponding eigenfunctions, i.e., $\mathcal{C}_0(\nu_k) = \lambda_k \nu_k$. These eigenfunctions are also commonly referred to as the (static) FPCs of the autocovariance operator $\mathcal{C}_0$. The Karhunen-Loève Theorem further tells us that the FPCs span the entire space $\mathbb{H}$, enabling us to represent each element in this space as a linear combination of these components. Specifically, we express any FTS as:

$$X_t(\tau) = \mu(\tau) + \sum_{k=1}^{\infty} \xi_{k,t} \nu_k(\tau),$$

where $\xi_{k,t} = \langle X_t - \mu, \nu_k\rangle_{\mathbb{H}}$ represents the FPC score associated with the FPC ν_k. To estimate the eigenelements, we introduce the estimated FPCs (EFPCs), which can be obtained by solving the equation:

$$\hat{\mathcal{C}}_0(\hat{\nu}_k) = \hat{\lambda}_k \hat{\nu}_k, \quad k = 1, 2, \ldots, M. \tag{2}$$

In this expression, M represents the number of FPCs possessing positive eigenvalues, and it is constrained such that $M \leq N$. A substantial body of literature exists that addresses the consistency of eigenvalues and EFPCs both in independent and dependent datasets. Within the realm of FTS, notable contributions have been made. For instance, Bosq (2000) established that, under the assumption of the FAR(1) model and $E\|X_0\|_{\mathbb{H}} < \infty$, the following convergence relations are upheld:

$$\sup_{k\geq 1} |\hat{\lambda}_k - \lambda_k| = O\left(\left(\frac{LogN}{N}\right)^{1/2}\right) \quad \text{and} \quad \|\hat{\nu}_k - \nu_k\|_{\mathbb{H}} = O\left(\left(\frac{LogN}{N}\right)^{1/2}\right), \quad a.s.$$

The concept of weak dependence has been approached and formalized through various means. In the realm of time series analysis, measures of dependence based on moments, particularly autocorrelations and cumulants, have gained widespread recognition. A notion related to moment-based concepts of weak dependence is presented by Hörmann and Kokoszka (2012), referred to as L^k-m-approximable dependence. In this context, an element-wise representation can be applied to an FTS $\{X_t\}_{t\in\mathbb{Z}}$ with elements in $L^k_{\mathbb{H}}$. Such an FTS is labeled L^k-m-approximable if each element X_t can be represented as $X_t = f(\varepsilon_t, \varepsilon_{t-1}, \cdots)$. Here, the ε_t elements are i.i.d. and take values within a measurable space S, while f is a measurable function with the form $f : S^\infty \to \mathbb{H}$. Under the assumption of L^4-m-approximable dependence and a descending order of eigenvalues, Hörmann and Kokoszka (2012) demonstrated that the following relationships hold:

$$E\left(|\hat{\lambda}_k - \lambda_k|^2\right) = O\left(N^{-1}\right) \quad \text{and} \quad E\left(\|\hat{c}_k\hat{\nu}_k - \nu_k\|^2_{\mathbb{H}}\right) = O\left(N^{-1}\right),$$

where $\hat{c}_k = \langle \hat{\nu}_k, \nu_k \rangle_{\mathbb{H}}$. The technique of FPCA forms the cornerstone of various fundamental FTS studies. For instance, in the forecasting methods of Hyndman and Ullah (2007) and Aue et al. (2015), stationary test of Horváth et al. (2014), bootstrap prediction bands of FTS by Paparoditis and Shang (2023) .

2.2 Dynamic Functional Principal Component Analysis

As observed in FPCA methodology, the FPCs solely rely on information from the the autocovariance operator at lag 0. However, to leverage information from other lags, the concept of dynamic FPCA was introduced by Hörmann et al. (2015) and Panaretos and Tavakoli (2013). Moreover, the existing FPCA method has been developed for independent observations, which is a serious weakness when we are dealing with FTS data. The serial dependence between the curves is taken into account with the dynamic FPCA. To define dynamic FPCA, let us denote the spectral density operator at frequency θ by $\mathcal{F}_\theta$, defined as Fourier transform of autocovariance operator

$$\mathcal{F}_\theta(x) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathcal{C}_h(x) e^{-\mathrm{i}h\theta}, \quad \theta \in [-\pi, \pi], \tag{3}$$

where i denotes the imaginary unit. Its corresponding kernel is represented by:

$$f_\theta(\tau, \sigma) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} c_h(\tau, \sigma) e^{-\mathrm{i}h\theta}, \quad \theta \in [-\pi, \pi].$$

Under the condition

$$\sum_{h=-\infty}^{\infty} \left(\int_{\mathcal{T}} \int_{\mathcal{T}} |c_h(\tau, \sigma)|^2 d\tau d\sigma \right)^{1/2} < \infty, \tag{4}$$

for every frequency θ, the operator $\mathcal{F}_\theta$ is a non-negative, self-adjoint Hilbert Schmidt operator (Hörmann et al., 2015). Consequently, it admits the spectral decomposition:

$$\mathcal{F}_\theta(x) = \sum_{m=1}^{\infty} \lambda_m(\theta) \langle x, \varphi_m(\theta) \rangle_{\mathbb{H}} \varphi_m(\theta).$$

where $\lambda_m(\theta)$ and $\varphi_m(\theta)$ represent the dynamic eigenvalues and eigenfunctions, respectively. These eigenvalues are arranged in descending order such that $\lambda_1(\theta) \geq \lambda_2(\theta) \geq ... \geq 0$ for all $\theta \in [-\pi, \pi]$, while the eigenfunctions are standardized to satisfy $\|\varphi_m(\theta)\|_{\mathbb{H}} = 1$ for all $m \geq 1$.

Note that for fixed $\theta \in [-\pi, \pi]$, each dynamic eigenfunction $\varphi_m(\theta) \in \mathbb{H}$ can be represented in a function form $\varphi_m(\tau|\theta)$ for $\tau \in \mathcal{T}\swarrow$ Let $\{X_t\}_{t\in\mathbb{Z}}$ be mean-zero stationary FTS satisfying (4) and

$$\phi_{m,h}(\tau) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_m(\tau|\theta) e^{-\mathrm{i}h\theta} d\theta, \quad h \in \mathbb{Z}, m \in \mathbb{N}.$$

Then the m-th dynamic functional principal component score of $\{X_t\}_{t\in\mathbb{Z}}$ is

$$\xi_{m,t} = \sum_{h=-\infty}^{\infty} \langle X_{t-h}, \phi_{m,h} \rangle_{\mathbb{H}}.$$

Then the dynamic Karhunen-Loève expansion of X_t is

$$X_t(\tau) = \sum_{m=1}^{\infty} \sum_{h=-\infty}^{\infty} \xi_{m,t+h} \phi_{m,h}(\tau).$$

In practice, the operator $\mathcal{F}_\theta$ in (3) can be estimated by

$$\hat{\mathcal{F}}_\theta(x) = \sum_{h=-q}^{q} \left(1 - \frac{|h|}{q}\right) \hat{\mathcal{C}}_h(x) e^{-\mathrm{i}h\theta}.$$

Here, the parameter q is subject to the condition $q = q_N \to \infty$ with $q_N/N \to 0$ as $N \to \infty$. Hörmann et al. (2015) have demonstrated that if $q^3 = o(N)$, then the estimator $\hat{\mathcal{F}}_\theta(x)$ is a consistent estimate of $\mathcal{F}_\theta(x)$. For a more comprehensive discussion on typical choices of the tuning parameter q, we direct the reader to Chapters 10-11 in the work by Brockwell and Davis (2009) and to the insights provided by Politis (2011).

Dynamic FPCA assumes a central role in various methods and applications, as illustrated by its utilization in the following examples: foreign exchange forecasts through dynamic FTS (Shang & Kearney, 2022), dynamic principal component regression (Shang, 2019), surface time series models (Martínez-Hernández & Genton, 2023), as well as high-dimensional analyses undertaken by Gao, Shang, and Yang (2019), Tang and Shi (2021), Martínez-Hernández, Gonzalo, and González-Farías (2022), and Hallin, Nisol, and Tavakoli (2023).

2.3 | Functional Autoregressive Models

The FAR(1) model for a zero-mean FTS X_t is defined by the equation:

$$X_t = \Psi(X_{t-1}) + \varepsilon_t, \tag{5}$$

where Ψ is a bounded linear operator and ε_t represents a zero-mean, independent and identically distributed (i.i.d.) functional white noise process. We introduce the linear operator norm $\|.\|_{\mathcal{L}}$ and consider the following conditions:

C_0: There exists an integer j_0 such that $\|\Psi^{j_0}\|_{\mathcal{L}} < 1$.
C_1: For every $j \geq 0$, there exist $a > 0$ and $0 < b < 1$ satisfying $\|\Psi^j\|_{\mathcal{L}} \leq ab^j$.

As discussed in Bosq (2000) and Hörmann and Kokoszka (2012), these conditions are equivalent, and if either of them holds, there exists a unique strictly stationary causal solution to (5), given by $X_t = \sum_{j=0}^{\infty} \Psi^j \varepsilon_{t-j}$.

Under the stationary condition and using the first q EFPCs, an estimation of the operator Ψ can be obtained as follows:

$$\hat{\Psi}(x) = \frac{1}{N-1} \sum_{k=1}^{N-1} \sum_{j=1}^{q} \sum_{i=1}^{q} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle_{\mathbb{H}} \langle X_k, \hat{\nu}_j \rangle_{\mathbb{H}} \langle X_{k+1}, \hat{\nu}_i \rangle_{\mathbb{H}} \hat{\nu}_i. \tag{6}$$

Here, it must be assumed that $q \leq N$ and to establish the consistency of this estimator, assumed $q = q_N$ is a function of the sample size N. Bosq (2000) established sufficient conditions for the consistency of this estimator.

More generally, the higher-order FAR(q) model can be define as

$$X_t = \Psi_1(X_{t-1}) + \Psi_2(X_{t-2}) + \ldots + \Psi_p(X_{t-q}) + \varepsilon_t,$$

As showed in Bosq (2000), the above equation has unique stationary solution if $\sum_{j=1}^{q} \|\Psi_j\|_{\mathbb{H}_j} < 1$.

FAR models hold a pivotal position in the realm of FTS analysis, representing a cornerstone assumption that has significantly shaped various research endeavors over the years. This modeling framework, rooted in the autoregressive paradigm, has proven crucial for capturing the temporal dependencies inherent in FTS data. The versatility and effectiveness of FAR models have led them to be adopted as a foundational tool in many scholarly works spanning different disciplines. Researchers have harnessed the FAR model's ability to characterize the complex interplay between time points and functional observations, enabling the development of sophisticated forecasting techniques, dimension reduction methods, and predictive models. Its enduring importance is exemplified by its recurrent use as a fundamental assumption in numerous literature pieces, attesting to its instrumental role in advancing the understanding and analysis of FTS data.

2.4 | Functional Singular Spectrum Analysis

The FSSA model is introduced by Haghbin et al. (2021), stands as a nonparametric approach aimed at dissecting a time-evolving ensemble of FTS data. Through the utilization of functional singular value decomposition (FSVD), FSSA discerns distinctive components within the data, encompassing elements like mean, seasonal variations, trends, and noise. The algorithm encapsulating FSSA unfolds across four pivotal steps: embedding, decomposition, grouping, and reconstruction. Consider a given FTS sequence $\{X_t\}_{t=1}^N$, and designate a fixed parameter, window length, $L \in \{2, 3, \ldots, N/2\}$.

1. Embedding Step: The initial phase involves defining the trajectory operator, embodied as a matrix operator $\boldsymbol{\mathcal{X}} : \mathbb{R}^K \to \mathbb{H}^L$. This operator is structured as follows:

$$\boldsymbol{\mathcal{X}}(\mathbf{a}) = \sum_{j=1}^{K} a_j \boldsymbol{x}_j, \qquad \mathbf{a} = (a_1, \dots, a_K)^\top \in \mathbb{R}^K.$$

   Here, $\boldsymbol{x}_j = (y_j, y_{j+1}, \dots, y_{j+L-1})^\top$ corresponds to the jth vector, incorporating L-lagged functions within $\mathbb{H}^L$ associated with the FTS, and where $K = N - L + 1$.
2. Decomposition Step: Within this phase, we undertake the disentanglement of the trajectory operator, $\boldsymbol{\mathcal{X}}$, into a collection of $r \leq \min\{L, K\}$ rank-one operators through the application of the FSVD theorem:

$$\boldsymbol{\mathcal{X}}\mathbf{a} = \sum_{i=1}^{r} \sqrt{\lambda_i} \langle \mathbf{v}_i, \mathbf{a} \rangle_{\mathbb{R}^K} \boldsymbol{\psi}_i, \quad \text{for all } \mathbf{a} \in \mathbb{R}^K, \tag{7}$$

   Here, $\{\boldsymbol{\psi}_i\}_{i=1}^r$ symbolize the orthonormal components of $\mathbb{H}^L$, and $\{\mathbf{v}_i\}_{i=1}^r$ stand as orthonormal vectors in $\mathbb{R}^K$. Introducing $\boldsymbol{\mathcal{X}}_i : \mathbb{R}^K \to \mathbb{H}^L$ as the rank-one operator defined by $\boldsymbol{\mathcal{X}}_i = \sqrt{\lambda_i} \mathbf{v}_i \otimes \boldsymbol{\psi}_i$ for $i = 1, \dots, r$, we proceed to utilize equation (7), which leads us to the decomposition of $\boldsymbol{\mathcal{X}}$ as

$$\boldsymbol{\mathcal{X}} = \sum_{i=1}^{r} \boldsymbol{\mathcal{X}}_i. \tag{8}$$

3. Grouping Step: The grouping procedure involves the reconfiguration and segmentation of the elementary operators denoted as $\boldsymbol{\mathcal{X}}_i$ in equation (8). Establishing a partition, denoted as $\{I_1, I_2, \dots, I_m\}$, within the index set $\{1, \dots, r\}$, a distinctive operator named $\boldsymbol{\mathcal{X}}_{I_q} : \mathbb{R}^K \to \mathbb{H}^L$ is defined, structured as $\boldsymbol{\mathcal{X}}_{I_q} = \sum_{i \in I_q} \boldsymbol{\mathcal{X}}_i$. This formulation leads to the comprehensive expression

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_{I_1} + \boldsymbol{\mathcal{X}}_{I_2} + \cdots + \boldsymbol{\mathcal{X}}_{I_m}. \tag{9}$$

   The underlying objective of this grouping procedure hinges on the strategic delineation of each operator, $\boldsymbol{\mathcal{X}}_{I_q} \in \mathbb{H}^{L \times K}$, to encapsulate a distinct facet of variation intrinsic to the original FTS. These variations might encompass characteristics like mean, periodic patterns, or trends, corresponding to $q = 1, \dots, m$. In the fourth step, the primary aim revolves around distilling a FTS that aligns with each $\boldsymbol{\mathcal{X}}_{I_q}$.
4. Reconstruction Step: Within this phase, the operator $\boldsymbol{\mathcal{T}}^{-1} : \mathbb{H}_H^{L \times K} \to \mathbb{H}^N$ is instrumental in reverting each operator, $\boldsymbol{\mathcal{X}}_{I_q}$ as depicted in equation (9), back into its original FTS manifestation, namely $\tilde{\mathbf{y}}_N^q$ within the space $\mathbb{H}$. To accomplish this, a process akin to off-diagonal averaging and Hankelization, as elaborated in Haghbin et al. (2021), is executed.

Parameter Selection: The FSSA procedure involves two fundamental parameters: the window length (L) and the grouping parameters. The selection of appropriate values for these parameters is crucial as it influences the quality of reconstruction and the reliability of subsequent analyses. The optimal parameter choice is contingent on the intrinsic characteristics of the data and the specific study objectives. However, there exist several guidelines and rules that are effective across a wide range of scenarios. It is advisable to set the window length parameter, L, as a relatively large integer, which should be a multiple of the periodicities present in the FTS but should not exceed half of the total sample size, i.e., $N/2$. Moreover, several tools are available to aid in the effective grouping of the data. These tools encompass the examination of the periodogram, paired plots of the right singular vectors, scree plots of singular values, and w-correlation plots. For more comprehensive details on these utilities, please refer to Haghbin, Trinka, Najibi, and Maadooliat (2023).

While FPCA itself does not explicitly account for the dependence structure inherent in FTS, it serves as the cornerstone for both dynamic FPCA and FSSA methodologies. Dynamic FPCA extends the Karhunen-Loève expansion to various lags, while FSSA essentially performs multivariate FPCA using lagged forms of a single FTS variable. Notably, the strength of FSSA-based approaches lies in their model-independent nature and their ability to operate without stationarity assumptions.

2.5 | Forecasting Functional Time Series

Various forecasting methods have been developed for predicting stationary FTS data. We present several such methods below:

TABLE 1 Univariate FTS models

Model Name	Abbreviation	Reference
Functional/Hilbertian Autoregressive	FAR	Bosq (2000)
Functional Autoregressive with Exogenous variables	FARX	Damon and Guillas (2002)
Functional Moving Average	FMA	Aue and Klepsch (2017)
Varying Coefficient Functional Autoregressive	VC-FAR	Xu et al. (2017)
Adaptive Functional Autoregressive	AFAR	Chen and Li (2017)
Warping Functional Autoregressive	WFAR	Chen, Marron, and Zhang (2019)
Functional Singular Spectrum Analysis	FSSA	Haghbin et al. (2021)
Seasonal Functional Autoregressive	SFAR	Zamani et al. (2022)

1. Model-based: These methods typically involve considering a model for FTS and then estimating this model to perform predictions. To facilitate a better understanding of the landscape of FTS developed models, we have compiled a table summarizing these models and references them in the Table 1. The forecasting method studied by Bosq (2000) involves considering the conditional expectation $\mathcal{P}(x) = \mathbb{E}(X_{n+1}|X_n = x)$ for one-step-ahead prediction. If the FTS is assumed to follow the FAR(1) model in (5), then $\hat{X}_{n+1} = \hat{\Psi}(X_n)$, where $\hat{\Psi}$ is defined by Equation (6). It should be noted that this conditional expectation definition implies that the operator $\mathcal{P}$ is not necessarily linear. In Besse et al. (2000), a nonlinear kernel regression was used to estimate $\hat{X}_{n+1}$. Aue and Klepsch (2017) introduced FMA processes for estimating FTS and utilized a functional innovations algorithm for predicting FTS. Additionally, this reference developed a model selection test to determine the FMA order. González et al. (2017) introduced a Seasonal FARMA model with exogenous variables. In their models, the functional parameters are represented as integral operators with kernels modeled as linear combinations of sigmoid functions and the optimization of these parameters is achieved using a multi-layer perceptron (MLP) neural network. Zamani et al. (2022) considered seasonal FAR (SFAR) model and presented best linear predictor for this model.

2. Bayesian Forecasting: Bayesian nonparametric methodologies for modeling and forecasting a specific class of FTS have been explored in works such as (Canale & Ruggiero, 2016; Canale & Vantini, 2016; Rossini & Canale, 2019; King, Canale, & Ruggiero, 2019). These approaches encompass a category of Bayesian dynamic models characterized by autoregressive behavior and offer a comprehensive inferential framework for predicting time series composed of piecewise-constant non-decreasing functions across diverse time horizons. These models, initially inspired by their application in the natural gas market, conceptualize functional time series using a latent particle system. To conduct predictive inference, these models combine principles from Markov chain Monte Carlo and approximate Bayesian computation (ABC), a likelihood-free approach to Bayesian inference centered on quantifying disparities between real and simulated data.
3. Predictive Factor: The predictive factor technique, introduced by Kargin and Onatski (2008), involves deriving a reduced-rank approximation of the autoregressive operator. This approximation is designed to minimize the expected squared norm of the prediction error in a specific direction. The goal is to capture the predictive factors that encapsulate crucial information about the future behavior of the functions. Further insights into the predictive factor method can also be found in other sources such as Didericksen, Kokoszka, and Zhang (2012) and Horváth and Kokoszka (2012).

4. Time Series of FPC Scores: Nonparametric forecasting of FTS was first explored by Hyndman and Ullah (2007). In this method, the observed FTS $\{X_t\}_{t=1}^N$ is decomposed using EFPCs:

$$X_t(\tau) \approx \hat{\mu}(\tau) + \sum_{k=1}^{K} \hat{\xi}_{k,t}\hat{\nu}_k(\tau), \quad K \leq M. \tag{10}$$

For each $k = 1, \ldots, K$, a univariate time series is fitted to the FPC scores $\{\hat{\xi}_{k,t}\}_{t=1}^N$, and h-step ahead prediction is performed using univariate time series techniques to predict the FPC scores of lag $N + h$, $\hat{\xi}_{k,N+h}$, for all $k = 1, \ldots, K$. The predicted FPC scores of lag $N + h$ are then transformed back to the functional form through using the estimated Karhunen-Loève expansion in (10):

$$\hat{X}_{N+h} = \hat{\mu}(\tau) + \sum_{k=1}^{K} \hat{\xi}_{k,N+h}\hat{\nu}_k(\tau).$$

To determine the appropriate order K for the model, Hyndman and Ullah (2007) have introduced the integrated squared forecast error (ISFE), defined as:

$$ISFE_N(h) = \|X_{N+h} - \hat{X}_{N+h}\|^2_{\mathbb{H}}.$$

Let t_0 represents the minimum number of observations required to fit the model. The model is fitted to the data up to time t, where $t_0 \leq t \leq N - h$, and predictions are made for the subsequent K periods to obtain $ISFE$ values for $h = 1, 2, ..., K$. The order K is then chosen by minimizing the cumulative $ISFE$ as:

$$K = \arg\min \sum_{h=1}^{K} \sum_{t=t_0}^{N-h} ISFE_t(h).$$

The Hyndman and Ullah approach was extended by Hyndman and Shang (2009) where the weighted functional partial least squares regression was used instead of FPCs. Aue et al. (2015) considered vector autoregressions to FPC scores instead of fitting them separately in this method.

5. FSSA Forecasting: FSSA can also be used for forecasting FTS. The forecasting method based on FSSA involves decomposing the observed FTS into informative trends, seasonal/cyclical patterns, and noise components using the FSSA method. Once the decomposition is obtained, the forecasted functions for each component can be generated separately (using recurrent-FSSA, and vector-FSSA prediction procedures), and then combined to obtain the final forecast for the original FTS. For more details about FSSA forecasting see Trinka (2021) and Trinka, Haghbin, Lin Shang, and Maadooliat (2023).

These forecasting methods provide valuable tools for predicting functional time series data and have been applied in various domains to capture the dynamic behavior and dependencies within the functional processes.

3 | MULTIVARIATE FUNCTIONAL TIME SERIES ANALYSIS

In this section, we delve into the fundamental concepts, methodologies, and recent developments in multivariate functional time series. We explore how the integration of functional data analysis and time series techniques provides valuable insights into understanding the dynamic behavior and dependencies among multiple functional processes. Furthermore, we investigate the challenges and opportunities presented by MFTS and discuss the application of a Bayesian method and other advanced techniques to facilitate effective analysis and forecasting. The section aims to provide researchers and practitioners with a comprehensive overview of the tools and approaches available for analyzing complex and interrelated FTS data.

The univariate FTS models can be extended to MFTS framework which can also be referred as multiple groups (each sequence of FTS is referred as one group), exhibit both serial dependence and cross dependence among each other. Kowal, Matteson, and Ruppert (2017) proposed an innovative Bayesian approach to model MFTS data. They extended hierarchical dynamic linear models for multivariate time series to the functional data setting, providing a powerful framework for analyzing complex and high-dimensional MFTS data. On the other hand, Chen, Chua, and Härdle (2019) introduced a novel bivariate FTS model called the vector FAR (VFAR) model of order p. They conducted a comprehensive investigation of the model's estimation and asymptotic properties. Gao and Shang (2017) focused on analyzing mortality rates as a MFTS data. To effectively handle the high dimensionality of the data, they employed FPCA for dimension reduction. Furthermore, they utilized a vector error correction model (VECM) to jointly forecast the FPC scores as multivariate time series, providing valuable insights into the temporal dynamics of the MFTS. Multivariate extension of FSSA (MFSSA) method has been developed by Trinka, Haghbin, and Maadooliat (2022). The authors discussed two extensions of MFSSA, namely horizontal MFSSA (HMFSSA) and vertical MFSSA (VMFSSA), and demonstrated that VMFSSA yields equivalent decomposition results to MFSSA due to isomorphic vector spaces. Wang and Cao (2023) introduced a nonlinear prediction (NOP) method specifically designed for FTS and MFTS data. Leveraging neural network-based approaches, the NOP method presents an efficient alternative that circumvents the need for computing covariance functions. This advancement not only facilitates online estimation and prediction but also offers promising opportunities for enhanced forecasting accuracy in the context of MFTS analysis.

3.1 | Representation of Multivariate Functional Time Series

For each $j = 1, 2, \cdots, p$, consider $\mathcal{T}_j$ as a compact subset of $\mathbb{R}^{d_j}$, and $\boldsymbol{\mathcal{T}} = \mathcal{T}_1 \times \ldots \times \mathcal{T}_p$, where p denotes the number of variables, $d_j \in \mathbb{N}^+$ and define $\mathbb{H}_j := L^2(\mathcal{T}_j)$. Additionally, let $\{X_t^{(j)}\}_{t\in\mathbb{Z}}$ be stationary FTS with the mean function $\mu^{(j)}$ and autocovariance operator $\mathcal{C}_h^{(j)}$. To provide a framewok for MFTS we first consider space of multivariate functional data following Happ and Greven (2018). The cartesian product space $\mathcal{H} = \mathbb{H}_1 \times \mathbb{H}_2 \times \cdots \times \mathbb{H}_p$ is a Hilbert space equipped with the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{H}} = \sum_{j=1}^p \langle x_j, y_j \rangle_{\mathbb{H}_j}$. An MFTS $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}}$ is a multivariate functional data, where for each t we have $\boldsymbol{X}_t = \left(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(p)}\right)^\top \in \mathcal{H}$. To specify this process, we have to define mean and autocovariance operator as well. The mean function is define as $\boldsymbol{\mu} = \left(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(p)}\right)^\top$ which is an element of $\mathcal{H}$. To define autocovariance, let us first define cross-covariance kernel between two FTS $\{X_t^{(i)}\}_{t\in\mathbb{Z}}$ and $\{X_t^{(j)}\}_{t\in\mathbb{Z}}$ as $c_h^{(i,j)}(\tau, \sigma) = \operatorname{cov}\left(X_t^{(i)}(\tau), X_{t+h}^{(j)}(\sigma)\right)$. The associated cross-covariance operator between these two FTS is an integral operator with respect to the kernel function $c_h^{(i,j)}(\cdot, \cdot)$; i.e.,

$$\mathcal{C}_h^{(i,j)}(f)(\sigma) := \int_{\mathcal{T}_j} c_h^{(i,j)}(\tau, \sigma) f(\tau) d\tau, \; f \in \mathbb{H}_i.$$

Then the autocovariance operator of MFTS process $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}}$ can de defined by

$$\boldsymbol{\mathcal{C}}_h(\boldsymbol{f})(\boldsymbol{\tau}) := \begin{bmatrix} \sum_{j=1}^p \mathcal{C}_h^{(1,j)}(f_j)(\tau_1) \\ \vdots \\ \sum_{j=1}^p \mathcal{C}_h^{(p,j)}(f_j)(\tau_p) \end{bmatrix}, \tag{11}$$

where $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p) \in \boldsymbol{\mathcal{T}}$ and $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathcal{H}$. The multivariate FPCA (MFPCA) of the operator $\boldsymbol{\mathcal{C}}_0$ in (11) was purposed by Happ and Greven (2018).

3.2 | FPCA-based Strategies for Multivariate Functional Time Series

In the realm of MFTS forecasting, the task at hand holds paramount importance across numerous applications, seeking to foresee forthcoming functional observations by leveraging historical data. To address this challenge and deliver precise predictions for each functional variable within the MFTS data, a range of forecasting methodologies have emerged. Notably, despite the underpinning role of the FAR model in most FTS methods, the majority of proposed forecasting techniques for MFTS pivot towards dimension reduction strategies.

For instance, Gao and Shang (2017) present a comprehensive four-step approach for forecasting MFTS data:

1. **Smoothing Data:** The initial step involves the independent smoothing of observed data within each marginal FTS.
2. **Dimension Reduction:** In the second step, dimension reduction is employed individually for each population via FPCA. This entails the representation of the i-th FTS as:

$$X_t^{(i)}(\tau) \approx \sum_{k=1}^{K(i)} \xi_{k,t}^{(i)} \nu_k^{(i)}(\tau), \quad i = 1, \ldots, p, \tag{12}$$

where $\xi_{k,t}^{(i)} = \langle X_t^{(i)}, \nu_k^{(i)} \rangle_{\mathbb{H}_i}$, and the $K(i)$-dimensional FPC score vectors $\boldsymbol{\xi}_t^{(i)}$ are defined for $t = 1, 2, \ldots, N$.
3. **Multivariate Time Series Modeling:** Subsequently, a multivariate time series model such as VAR or VECM is tailored to the multivariate time series of i-th FPC scores, $\{\boldsymbol{\xi}_t^{(i)}\}_{i=1}^N$.
4. **Generating Forecasts:** The final step involves generating h-step forecasts of the FPC score vectors. This is executed through the utilization of the fitted multivariate time series model. Employing the Karhunen-Loève Theorem to approximate the i-th FTS, forecasts are produced in the form:

$$\hat{X}_{N+h}^{(i)}(\tau) = \sum_{k=1}^{K(i)} \hat{\xi}_{k,N+h}^{(i)} \nu_k^{(i)}(\tau), \quad i = 1, \ldots, p.$$

wherein $\hat{\xi}_{k,N+h}^{(i)}$ represent the estimated k-th FPC score for the i-th variable at time point $N + h$.

This forecasting approach, incorporating dimension reduction techniques, caters to the complexities of MFTS data, offering a structured strategy to achieve accurate predictions across multiple functional variables.

3.3 | Multivariate Functional Dynamic Linear Model

A Bayesian approach for modeling MFTS is proposed by Kowal et al. (2017) and is known as the multivariate functional dynamic linear model (MFDLM). Given an observed p-dimensional path of a MFTS denoted as $\{\boldsymbol{X}_t\}_{t=1}^N$, the MFDLM is represented by the following system of equations:

$$\begin{cases} \boldsymbol{X}_t(\tau) = \boldsymbol{F}(\tau)\beta_t + \boldsymbol{\epsilon}_t, & \boldsymbol{\epsilon}_t|\boldsymbol{E}_t \sim N(\boldsymbol{0}, \boldsymbol{E}_t), \\ \beta_t = \boldsymbol{X}_t\boldsymbol{\theta}_t + \boldsymbol{\nu}_t, & \boldsymbol{\nu}_t|\boldsymbol{V}_t \sim N(\boldsymbol{0}, \boldsymbol{V}_t), \\ \boldsymbol{\theta}_t = \boldsymbol{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, & \boldsymbol{\omega}_t|\boldsymbol{W}_t \sim N(\boldsymbol{0}, \boldsymbol{W}_t), \end{cases}$$

where $\boldsymbol{F}(\tau)$ is a $p \times Kp$ block matrix with $1 \times K$ diagonal blocks $[f_1^{(i)}(\tau), \dots, f_K^{(i)}(\tau)]$ for $i = 1, \dots, p$, representing the factor loading curves evaluated at a specific $\tau \in \mathcal{T}$. The parameter K denotes the number of factors per outcome. The matrix $\boldsymbol{\beta}_t$ is a Kp-dimensional vector of factors that serve as the time-dependent weights on the factor loading curves. The known matrix $\boldsymbol{X}_t$ is of size $Kp \times M$ and contains covariates at time t, where M is the total number of covariates. The vector $\boldsymbol{\theta}_t$ is the M-dimensional vector of regression coefficients associated with $\boldsymbol{X}_t$, and $\boldsymbol{G}_t$ is the $M \times M$ evolution matrix governing the behavior of the regression coefficients $\boldsymbol{\theta}_t$ at time t. The error vectors $\boldsymbol{\epsilon}_t$, $\boldsymbol{\nu}_t$, and $\boldsymbol{\omega}_t$ are mutually independent and follow normal distributions with variance matrices $\boldsymbol{E}_t$, $\boldsymbol{V}_t$, and $\boldsymbol{W}_t$, respectively. Kowal et al. (2017) employed a Markov chain Monte Carlo (MCMC) approach to estimate the factor loadings of the MFDLM. This model was subsequently used for predicting MFTS data.

3.4 | Multivariate Functional Singular Spectrum Analysis

As previously discussed,FSSA serves as a nonparametric approach within the realm of FTS analysis, facilitating the dissection of FTS into comprehensible segments encompassing aspects like mean, periodic, and trend components. An extension of this method to a multivariate context was introduced by Trinka et al. (2022).

Similar to its univariate counterpart, the multivariate FSSA (MFSSA) method involves four fundamental stages: Embedding, Decomposition, Grouping, and Reconstruction. However, within the initial embedding phase of MFSSA, a novel emphasis emerges, wherein the construction of a multivariate trajectory operator takes precedence. This operator's range is assembled from elements that effectively encapsulate MFTS behavior across distinct time sub-intervals. Corresponding to the MFTS observed path $\{\boldsymbol{X}_t\}_{t=1}^N$, the formulation of L-lagged multivariate function vectors within $\boldsymbol{\mathcal{H}}^L$ takes the form:

$$\boldsymbol{Y}_k = (\boldsymbol{X}_k, \boldsymbol{X}_{k+1}, \dots, \boldsymbol{X}_{k+L-1}), \quad k = 1, \dots, K.$$

For the attainment of the multivariate trajectory operator, a linear operator defined by $\boldsymbol{Y}_k$ vectors is established. This operator, denoted as $\boldsymbol{\mathcal{X}} : \mathbb{R}^K \to \boldsymbol{\mathcal{H}}^L$, is expressed for a given vector $\mathbf{a} = [a_1, a_2, \dots, a_K]^\top \in \mathbb{R}^K$ as follows:

$$\boldsymbol{\mathcal{X}}(\mathbf{a}) = \sum_{k=1}^{K} a_k \boldsymbol{Y}_k.$$

While most of the subsequent steps in MFSSA closely mirror FSSA, a notable difference emerges in the substitution of the univariate space $\mathbb{H}$ with the multivariate space $\boldsymbol{\mathcal{H}}$. Beyond its proficiency in revealing inherent components, MFSSA's utility extends to predictive modeling. Through the separate forecasting of individual components, MFSSA facilitates the generation of forecasts for diverse patterns such as mean, trend, and periodicity. These individual forecasts can be aggregated to provide comprehensive predictions for the entire MFTS dataset (see Trinka, 2021for more details).

3.5 | High-Dimensional Functional Time Series

In the realm of MFTS, the issue of high dimensionality arises when the number of variables, denoted as p, exceeds the length of the observed MFTS path, denoted as N. Considering the p-variate MFTS $\{\boldsymbol{X}_t\}_{t=1}^N$, existing literature (e.g., Guo, Wang, and Yao

(2016) and Guo and Qiao (2023)) often employs the following factor model for dimension reduction:

$$\boldsymbol{X}_t \approx \boldsymbol{A}\boldsymbol{F}_t,$$

where $\boldsymbol{A}$ is a $r \times r$ loading matrix ($r \leq p$), and $\{\boldsymbol{F}_t\}_{t=1}^{N}$ represents a set of r-dimensional uncorrelated multivariate functions. To achieve improved accuracy, Gao et al. (2019) propose a twofold dimension reduction technique for modeling high-dimensional MFTS. They employ dynamic FPCA on each set of FTS along with factor analysis on the scores. The long-run covariance operator, denoted as $\mathcal{C}^{(i)}$, is defined as the summation of all autocovariance operators for FTS i:

$$\mathcal{C}^{(i)} = \sum_{h=-\infty}^{\infty} \mathcal{C}_h^{(i,i)}. \tag{13}$$

As a symmetric, non-negative, and self-adjoint Hilbert-Schmidt operator, $\mathcal{C}^{(i)}$ admits an eigendecomposition:

$$\mathcal{C}^{(i)}(x) = \sum_{k=1}^{\infty} \lambda_k^{(i)} \langle \nu_k^{(i)}, x\rangle_{\mathbb{H}_i} \nu_k^{(i)},$$

where $(\lambda_k^{(i)} : i \geq 1)$ represents the eigenvalues of $\mathcal{C}^{(i)}$ in descending order, and $\nu_k^{(i)}$ corresponds to the normalized eigenfunctions. Utilizing the Karhunen-Loève Theorem, the i-th FTS can be approximated using the first K dynamic FPCs as:

$$X_t^{(i)}(\tau) \approx \sum_{k=1}^{K} \xi_{k,t}^{(i)} \nu_k^{(i)}(\tau),$$

where $\xi_{k,t}^{(i)} = \langle X_t^{(i)}, \nu_k^{(i)}\rangle_{\mathbb{H}_i}$. To perform dimension reduction, $\boldsymbol{\xi}_{k,t} := \left(\xi_{k,t}^{(1)}, \dots, \xi_{k,t}^{(p)}\right)^\top$, the p-dimensional dynamic FPC score vectors for each $k = 1, \dots, K$ and $t = 1, 2, \dots, N$ are considered. A factor model is then employed:

$$\boldsymbol{\xi}_{k,t} = \boldsymbol{A}_k \boldsymbol{f}_{k,t} + \boldsymbol{e}_{k,t}, \tag{14}$$

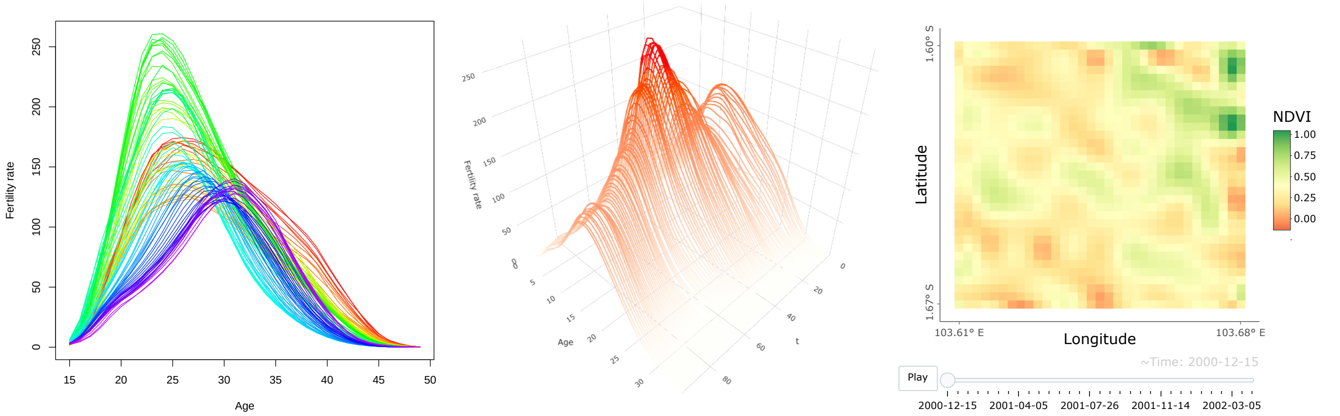
where $\boldsymbol{f}_{k,t}$ is an $r \times 1$ unobserved factor time series; $\boldsymbol{A}_k$ represents an $p \times r$ unknown constant factor loading matrix, and $r \leq N$. This approach allows fitting a scalar time series model to each factor and generating forecasts, which can be used to construct forecast functions. Through a two-fold dimension reduction process, the serial correlation information is encapsulated in the factors $\boldsymbol{f}_{k,t}$. To estimate the long-run covariance operator of (13), the following equation is utilized:

$$\hat{\mathcal{C}}^{(i)} = \sum_{h=-q}^{q} \left(1 - \frac{h}{q}\right) \hat{\mathcal{C}}_h^{(i,i)}. \tag{15}$$

In this context, the bandwidth parameter, denoted as $q = q_N$, is dependent on the sample size N. To ensure a consistent estimator in (15), it is recommended by Gao et al. (2019)that the condition $q^3 = o(N/p)$ is satisfied as $q \to \infty$. Using the estimated long-run covariance in (15), an estimated version of the factor model in (14) can be derived. To forecast the MFTS, an alternative approach involves making predictions on the estimated factors. This can be accomplished using scalar or vector time series models, such as autoregressive moving average (ARMA) models, since the factors are mutually uncorrelated. The prediction of the functions for each $i \in 1, \dots, p$ can then be computed as:

$$\hat{X}_{N+h}^{(i)}(\tau) = \sum_{k=1}^{K} \left(\hat{\boldsymbol{A}}_k \hat{\boldsymbol{f}}_{k,N+h}\right) \nu_k^{(i)}(\tau).$$

where $\hat{X}_{N+h}^{(i)}$ is the h-step-ahead forecast at time t, and h denotes a forecast horizon. Alternatively, Tang and Shi (2021) transform the original observed MFTS into a discrete form and then apply a factor model.



(a) A visualization from '**rainbow**' package for one dimensional-FTS (b) The 'plot_funts(.)' function of the '**Rfssa**' package for one dimensional-FTS (c) The 'plot_funts(.)' function of the '**Rfssa**' package for two dimensional-FTS

FIGURE 2 An example for comparison of R Packages in visualization of various types of FTS datasets.

4 | SOFTWARE PACKAGES

Several R packages (R Core Team, 2023) have been developed to facilitate the analysis of FTS and MFTS data. These tools provide researchers and practitioners with a range of functionalities to preprocess, model, visualize, and forecast functional data. Below is a list of notable software packages designed for FTS and MFTS analysis:

1. **rainbow**: The '**rainbow**' package in R (Shang & Hyndman, 2022), offers a set of tools for visualizing and analyzing functional data. Notable features include rainbow plots, bagplots, and boxplots designed to detect outliers within the data. Outliers are identified based on criteria such as having either the lowest depth or the lowest density, depending on the specific plot utilized. While the package also supports independent and identically distributed (i.i.d) functional data, its visualizations are particularly useful for observing how curves evolve over time through a progression of rainbow colors. The following R code exemplifies how to generate a plot within the **rainbow** package:

```
require(rainbow)
x <- fts(x = 15:49,
        y = Australiasmoothfertility$y,
        xname = "Age",
        yname = "Fertility_rate")
plot(x)
```

   The results of this code are presented in Figure 2(a) for reference.
2. **ftsa:** Notably, this package accommodates FTS data in the form of fts class objects, akin to the objects within the **rainbow** package, while also being compatible with discrete observed datasets. A distinctive characteristic of this package is it non-reliance on basis expansion techniques, differing from the well-known 'fd' objects featured in the **fda** package.
3. **Rfssa:** The '**Rfssa**' package in R (Haghbin et al., 2023) implements the FSSA and MFSSA method, providing visualization and functionalities for decomposition, grouping, and forecasting of FTS and MFTS data. An appealing feature of this package is to efficiently store FTS and MFTS datasets, accommodating both one-dimensional and two-dimensional domains. One of the standout capabilities is the provision of engaging 3D visualizations for FTS data. This 3D representation is particularly useful for discerning meaningful components that induce structural changes in the data over time. Moreover, the package seamlessly handles various classes of functional data types. To illustrate the capabilities of the **Rfssa** package, we offer a practical example and display the resulting plots in Figure 2 (b-c):

```
require(rainbbow)
require(Rfssa)
plotly_funts(Australiafertility, type = '3Dline') # one-dimensional FTS.
plotly_funts(Montana[,2])  # two-dimensional FTS.
```

4. **fdaACF:** The ‘**fdaACF**’ package in R (Marcos, González, Rice, Roque, & Pérez, 2020) offers functional ACF and PACF for one-dimensional domain FTS, as proposed in Mestre et al. (2021).
5. **freqdom.fda:** This R package (Hörmann & Kidziński, 2022) offers implementation of dynamic FPCA for analyzing one-dimensional domain FTS datasets as it discussed in Hörmann et al. (2015). It also includes graphic tools and methods based on frequency domain analysis of FTS.
6. **fpcb:** Within R, the ‘**fpcb**’ package (Hernández, Cugliari, & Jacques, 2022) is dedicated to estimating the spectral density operator for FTS data. It enables a comparative analysis of spectral density operators across two FTS, offering a means to detect disparities not only in frequency but also along the length of the curve. The ‘**fpcb**’ package employs Reproducing Kernel Hilbert Spaces to represent functional data. It employs kernel functions to approximate each curve, allowing for the representation of functional data through smooth functions.
7. **ftsspec:** The R package ‘**ftsspec**’ (Tavakoli, 2015) is designed to estimate the spectral density operator of FTS and enables a comparative analysis of the spectral density operators of two FTS. This functionality facilitates the identification of differences in spectral density across various frequencies and along the curve length.
8. **pcdpca:** The R package ‘**pcdpca**’ (Kidzinski, Jouzdani, & Kokoszka, 2017) introduces an extension of multivariate and dynamic FPCA techniques to handle periodically correlated MFTS. This package is particularly useful for calculating accurate dynamic FPCs in the presence of periodic patterns. It follow implementation guidelines as described inKidziński et al. (2018) in FPCA of periodically correlated MFTS.
9. **STFTS:** The R package ‘**STFTS**’ (Chen & Pun, 2021) offers a collection of statistical hypothesis tests tailored for FTS data. Key tests provided by this package include functional stationarity test, functional trend stationarity test, and functional unit root test.
10. **wwntests:** The R package ‘**wwntests**’ (Kim & Petoukhov, 2022) is a comprehensive resource for white noise hypothesis testing specifically designed for functional data. This package includes a variety of tests based on autocovariance operator norms, developed under both strong and weak white noise assumptions. Additionally, it offers tests based on spectral density operators and FPC dimensional reduction, all developed under strong white noise assumptions.

You can view a comparison of the aforementioned packages in Table 2. This table aims to provide a basis for comparing the various packages based on specific criteria. We consider whether the packages accommodate MFTS data (first column), work with two-dimensional data (second column), utilize basis expansion techniques (third column), employ kernel smoothing for raw data (fourth column), or have the capability to directly handle raw data (last column). It’s essential to note that making definitive judgments about these packages is challenging because each of them serves different purposes and targets unique objectives. The methods discussed in the previous two sections are mostly rooted in the mathematical framework of infinite-dimensional Hilbert space. However, when it comes to practical implementation for real-world applications, certain adjustments are often necessary. A commonly employed technique involves the utilization of a finite basis function expansion, as discussed in Ramsay and Silverman (2005). In this approach, a collection of predefined basis functions is employed within the function space $\mathbb{H}$. Each sample function can thus be represented as a finite linear combination of these basis functions, which need not necessarily be orthogonal. An illustrative example of such an approach can be observed in Figure 3, which demonstrates a technique utilized in the ‘**Rfssa**’ package to describe a bivariate FTS (e.g., curves and images as given in Figure 1) through basis expansion. The core concept involves capturing the MFTS data at predefined grid points denoted as argvals (a list of grid points), utilizing a

TABLE 2 Comparing some FTS/MFTS R packages

Packages Name	Support MFTS	Support 2D domain data	Deal with basis expansion	Deal with kernel	Take discrete observation
rainbow					✔
ftsa					✔
Rfssa	✔	✔	✔		✔
fdaACF					✔
freqdom.fda			✔	✔	
fpcb				✔	✔
ftsspec	✔		✔	✔	✔
pcdpca	✔		✔		
STFTS				✔	✔
wwntests				✔	✔

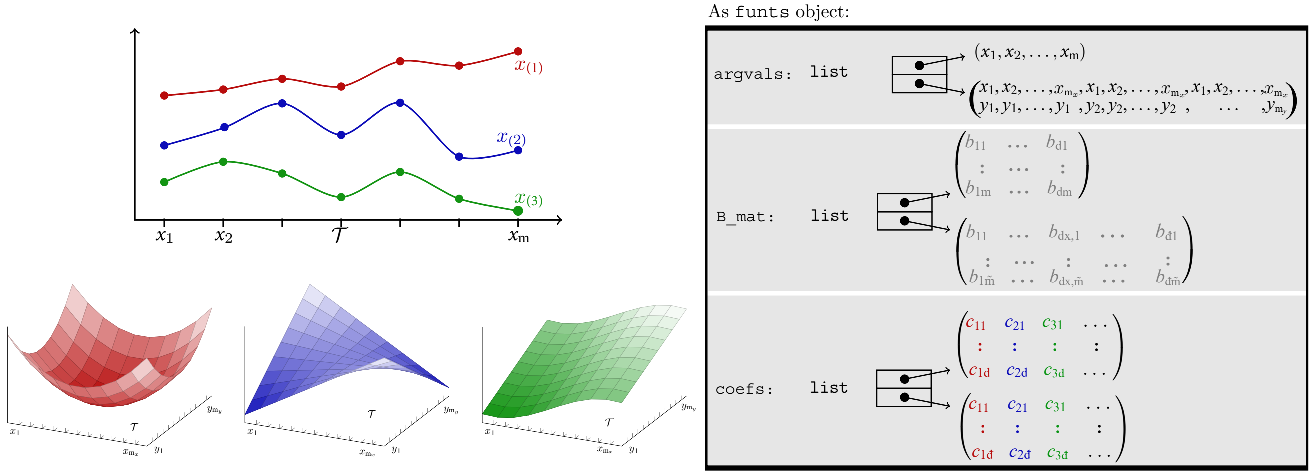


FIGURE 3 Representation of a bivariate FTS (sequence of curves and images) through basis expansion in the '**Rfssa**' package.

pre-selected basis indicated as B_mat (a list of basis functions), and subsequently characterizing the MFTS through of their respective coefficient matrices denoted as coefs (a list of coefficient matrices).

5 | CHALLENGES AND FUTURE DIRECTIONS

5.1 | Extensions Utilizing Multivariate Functional Principal Component Analysis

As observed from the comprehensive literature review presented in the preceding sections, the majority of extensions from FTS to MFTS have employed univariate FPCA or dynamic FPCA as their core techniques for decomposition and dimension reduction. Notably, up until the current state of knowledge, no methodology has been developed that specifically relies on the principles of MFPC.

The theoretical framework and practical implementation of MFPCA have been substantially advanced by several researchers, particularly concerning the dimension reduction of multivariate functional data. In the context of a p-dimensional MFTS denoted as $\{\boldsymbol{X}_t\}_{t=1}^{N}$ and characterized by its autocovariance (11), the application of the multivariate version of Mercer's Theorem yields an eigendecomposition for the zero-lag autocovariance operator as follows:

$$\mathcal{C}_0(\boldsymbol{x}) = \sum_{k=1}^{\infty} \lambda_k \langle \boldsymbol{\psi}_k, \boldsymbol{x} \rangle_{\mathcal{H}} \boldsymbol{\psi}_k.$$

Here, $(\lambda_k : \ k \geq 1)$ represent the eigenvalues of $\mathcal{C}_0$, while $(\boldsymbol{\psi}_k : k \geq 1)$ correspond to the associated normalized eigenfunctions. These eigenfunctions satisfy the conditions $\mathcal{C}_0(\boldsymbol{\psi}_k) = \lambda_k \boldsymbol{\psi}_k$ and $\|\boldsymbol{\psi}_k\|_{\mathcal{H}} = 1$. Consequently, the Karhunen-Loève expansion of $\boldsymbol{X}_t$ can be expressed as:

$$\boldsymbol{X}_t = \sum_{k=1}^{\infty} \xi_{t,k} \boldsymbol{\psi}_k, \tag{16}$$

where $\xi_{t,k} = \langle \boldsymbol{\psi}_k, \boldsymbol{X}_t \rangle_{\mathcal{H}}$. The works of researchers such as Berrendero, Justel, and Svarc (2011), Jeng-Min, Yu-Ting, and Ya-Fang (2014), and Happ and Greven (2018) offer detailed insights into the methodologies and nuances of MFPCA.

It is important to highlight that the dynamic FPC scores in (12) and the FPC scores in equation (14) consist of p-dimensional vectors corresponding to p variables. In contrast, the MFPC scores in equation (16) are scalar values. The latter provides a more concise and efficient representation for dimension reduction within the context of MFTS. Additionally, it simplifies and enhances the dimension reduction steps outlined in (12) (when utilizing marginal FPC scores) or in (14) (in cases involving high-dimensional FTS), thereby contributing to an improved forecasting accuracy of MFTS.

5.2 | Multivariate Model-Based Forecasting Approaches

Model-based forecasting approaches in MFTS are currently in their early stages of development due to the inherent complexities arising from the combination of multivariate and functional aspects of the data. While model-based forecasting has been well-established in the context of univariate FTS, extending these methods to MFTS introduces additional challenges. The joint consideration of multiple functional variables over time necessitates the formulation of complex and computationally intensive models that capture both temporal and functional dependencies simultaneously in multivariate world. As a result, researchers are still exploring novel methodologies and tackling the technical difficulties associated with MFTS modeling and forecasting. However, the potential benefits of model-based forecasting in MFTS are substantial, as it offers the possibility of accounting for intricate relationships among multiple functional variables, leading to more accurate and interpretable predictions. As the field of MFTS continues to evolve, we can expect further advancements in model-based forecasting techniques tailored to address the unique characteristics of MFTS data. Although there have been recent advancements in this research domain, an excessive emphasis on the utilization of deep neural networks in the last decade (Wang & Cao, 2023) could potentially hinder the advancement of model-based approaches.

6 | CONCLUSION

In this comprehensive review, we have delved into various aspects of FTS analysis, exploring both univariate and multivariate scenarios. We embarked on our journey by introducing the foundational concepts, establishing the groundwork for understanding the complexities of FTS data. Through the lens of univariate FTS analysis, we examined the intricacies of representation, estimation, and modeling. FPCA emerged as a powerful tool for dimensionality reduction, with dynamic FPCA further enriching our ability to capture time-varying patterns.

In our exploration of FAR models, we witnessed how these models extend the traditional autoregressive framework to the functional domain, providing a mechanism for forecasting. Moving beyond the conventional, we embarked on a journey through FSSA, unraveling time-dependent components that pave the way for better understanding and forecasting.

Transitioning to the MFTS analysis, we encountered the challenges posed by high-dimensional data and unearthed strategies to tackle them. The extension of forecasting techniques to the multivariate landscape illuminated avenues for capturing intricate dependencies across multiple functional variables. The MFDLM provided us a structured approach to incorporate dynamic relationships into our analysis.

As we concluded our survey, we turned our attention to the array of software packages developed to facilitate FTS analysis, providing practitioners with essential tools for implementing these techniques. While reflecting on the accomplishments in this field, we acknowledged the challenges and future directions that lie ahead. The evolving nature of FTS analysis beckons further research to address emerging complexities and refine existing methodologies.

In this review, we have journeyed through a landscape of FTS methodologies, uncovering a rich tapestry of techniques, models, and tools. Our exploration has demonstrated the interdisciplinary nature of this field, bridging the realms of statistics, functional analysis, and time series forecasting. As researchers continue to push the boundaries of knowledge, we anticipate that the insights gained from FTS analysis will not only enrich our understanding of data but also contribute significantly to informed decision-making in various domains.

CONFLICT OF INTEREST

The authors have declared no conflicts of interest for this article.

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