JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 97 (1994) 221–245

Topologies invariant under a group action

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Abstract

We study links between faithful group actions on a set and topologies on that set. In one direction, a group action has its invariant topologies (so we may regard members of the action to be homeomorphisms relative to those topologies); in the other direction, a topology has its preserving group actions (i.e., the subgroups of the homeomorphism group of the topology). This two-way passage allows us to discuss topological features of group actions as well as symmetry features of topologies.

0. Introduction

In this paper we consider group actions (permutation groups) and the topologies they leave invariant. One could think of this enterprise as a study of topological spaces from the perspective of symmetry; also as an investigation of topological features of group actions. For example, the usual topologies on the rational and real lines are "maximally symmetric" in a certain sense (see Theorems 4.3 and 5.5); also one can link transitivity/primitivity properties of a group action with the lower-level separation axioms satisfied by its invariant topologies (see Theorem 1.7). A principal device we employ is the passage from topology to group action via the homeomorphism group, as well as the return passage via the support topology (subbasically generated by supports of permutations). This topology is of key importance in our study because it is contained in every Hausdorff invariant topology, and provides the main vehicle for describing topologically how groups act on sets.

A large part of mathematics involves the study of symmetry and the ways in which one may describe symmetry precisely, in absolute as well as in relative terms. A "structured" set whose automorphism group is the full symmetric group on the set is symmetric in the extreme, while one with a "small" group of automorphisms has very little symmetry. Thus a good deal of effort has been spent developing vocabulary for describing the relative size of subgroups of permutation groups. (One way is via cardinality, another via index, a third via notions of transitivity/primitivity, while

SSDI 0022-4049(93)00004-3

a fourth is via the imposition of certain maximality conditions.) We make use of much of this important vocabulary in the sequel.

The overall layout of the paper is as follows: The first section introduces the support topology and other basic notions; the next section brings in the idea of when a group action preserves a smallest topology that satisfies specific conditions. For example, in Theorem 2.14, we characterize exactly when certain group actions, defined in terms of stabilizing a finite set, preserve a smallest Hausdorff topology. In the last three sections, we focus on groups acting on a field-ordered set. In this setting, the usual order topology serves as the support topology for most group actions of interest. This feature is exploited most fully in the last two sections, where group actions on the rational and real fields are investigated.

1. The support topology

The general setting is the following. Suppose $\mathscr{G}$ is a group acting faithfully on a set X. (We always assume X to be infinite, although several of our arguments do not require this assumption.) The *cardinality* of the action is just the usual cardinality $|\mathscr{G}|$ of $\mathscr{G}$; $|X|$ is called the *degree* of the action. Since the action is faithful (i.e., only the identity element of $\mathscr{G}$ fixes each element of X), we may, for concreteness, regard $\mathscr{G}$ as a subgroup of $\mathrm{Sym}(X)$, the full symmetric group on X. We adopt the convention that function application is made from the left. If $\mathscr{T}$ is a topology on X, we let $H(\mathscr{T})$ be the group of $\mathscr{T}$-homeomorphisms on X. We say $\mathscr{T}$ is *$\mathscr{G}$-invariant* (or $\mathscr{G}$ is *$\mathscr{T}$-compatible*) if $\mathscr{G} \leq H(\mathscr{T})$. (There are other phrases in common usage, and synonymous with the above; e.g., "$\mathscr{G}$ preserves (or stabilizes) $\mathscr{T}$.") Define the sets $\mathrm{Invar}(\mathscr{G}) := \{\mathscr{T}\colon \mathscr{T}$ is $\mathscr{G}$-invariant$\}$ and $\mathrm{Compat}(\mathscr{T}) := \{\mathscr{G}\colon \mathscr{G}$ is $\mathscr{T}$-compatible$\}$. Then of course $\mathrm{Invar}(\mathscr{G})$ is a meet-complete lattice under intersection, and $\mathrm{Compat}(\mathscr{T})$ is the subgroup lattice of $H(\mathscr{T})$.

As the operator $H(\ )$ is a means of proceeding from a topology to a group action, we introduce now a reverse operation, which we call the "support topology". Let $\mathscr{G}$ be a group action on a set X. For each $g \in \mathscr{G}$, the *support* $\mathrm{supp}(g)$ is the set of points moved by g, namely $\{x \in X\colon g(x) \neq x\}$; dually we define $\mathrm{fix}(g) := X \setminus \mathrm{supp}(g)$. We set $\mathrm{Supp}(\mathscr{G}) := \{\mathrm{supp}(g)\colon g \in \mathscr{G}\}$. Obviously, we have the relations $\mathrm{supp}(g^{-1}) = \mathrm{supp}(g)$ and $\mathrm{supp}(g) \cap \mathrm{fix}(h) \subseteq \mathrm{supp}(gh) \subseteq \mathrm{supp}(g) \cup \mathrm{supp}(h)$ (so, in particular, the elements of $\mathscr{G}$ whose supports are members of any given ideal of subsets of X, say the finite subsets, form a subgroup), but in general $\mathrm{Supp}(\mathscr{G})$ is not very well mannered. However, it does form a subbasis for a topology $S(\mathscr{G})$, the *support topology* of $\mathscr{G}$.

Proposition 1.1. *Let X be a set, $\mathscr{T}$ a topology on X, and $\mathscr{G}$ a group action on X. Then $H(\mathscr{T}) \in \mathrm{Compat}(\mathscr{T})$ and $S(\mathscr{G}) \in \mathrm{Invar}(\mathscr{G})$.*

Proof. The first statement is immediate from the definition; the second follows from the observation that if $g, h \in \mathrm{Sym}(X)$, then $g(\mathrm{supp}(h)) = \mathrm{supp}(ghg^{-1})$. Thus, $\mathrm{Supp}(\mathscr{G})$ itself is $\mathscr{G}$-invariant. $\square$

So $H(\ )$ and $S(\ )$ are operators connecting the topologies on a set X and the subgroups of Sym(X). As maps between lattices, clearly $S(\ )$ is order preserving; $H(\ )$ is not. (Neither preserves the lattice operations.) By Proposition 1.1, we always have $\mathcal{G} \leq HS(\mathcal{G})$, but there is generally no relationship between $\mathcal{T}$ and $SH(\mathcal{T})$. (If $\mathcal{T}$ is trivial, $SH(\mathcal{T})$ is discrete; on the other hand, there exist topologies $\mathcal{T}$ that are *rigid*, i.e., their homeomorphism groups are trivial, in which case $SH(\mathcal{T})$ is the trivial topology.) Call a group $\mathcal{G}$ (resp. a topology $\mathcal{T}$) *HS-fixed* (resp. *SH-fixed*) if $HS(\mathcal{G}) = \mathcal{G}$ (resp. $SH(\mathcal{T}) = \mathcal{T}$). The top elements of each of our lattices are "fixed" in their respective senses; the bottom elements are not. Quite straightforwardly, if $\mathcal{G}$ is HS-fixed, then $S(\mathcal{G})$ is SH-fixed; and if $\mathcal{T}$ is SH-fixed, then $H(\mathcal{T})$ is HS-fixed. The converses are easily seen to be false.

Before proceeding, we establish some (mostly) standard notions from the theory of group actions. Let $\mathcal{G}$ be a group action on a set X, with $A \subseteq X$. A is called *cofinite* if $X \setminus A$ is finite; A is a *moiety* if $|A| = |X \setminus A|$. We denote by $\mathcal{G}_A$ (resp. $\mathcal{G}_{(A)}$) the *setwise* (resp. *pointwise*) *stabilizer* of A, namely $\{g \in \mathcal{G}: g(A) = A\}$ (resp. $\{g \in \mathcal{G}: g(a) = a$ for all $a \in A\}$). For $a \in X$, we define $\mathcal{G}_a := \mathcal{G}_{\{a\}}$.

Let n be a natural number. (The set of natural numbers is denoted ω; this symbol also denotes the first infinite cardinal.) $\mathcal{G}$ is *n-transitive* if any bijection between n-element subsets of X can be extended to a member of $\mathcal{G}$. $\mathcal{G}$ is *n-homogeneous* if whenever A and B are an n-element subsets of X, there is a member of $\mathcal{G}$ taking A to B. When $n = 1$, we drop the numerical prefix and just write "transitive". A paraphrase of n-transitivity (resp. n-homogeneity) is that there is exactly one orbit on n-tuples of distinct points (resp. n-element sets). A group action is said to be *highly transitive* (resp. *highly homogeneous*) if it is n-transitive (resp. n-homogeneous) for all $n < \omega$. n-transitivity clearly becomes stronger with increasing n; the same is true for n-homogeneity, but much less trivial to show (see [7]). (Examples of highly transitive group actions include the full symmetric group, as well as the homeomorphism groups of the usual topologies on the rational line and euclidean n-spaces for $n \geq 2$. In the case $n = 1$, the homeomorphism group is 2-transitive and highly homogeneous, but not 3-transitive.) If each pair of elements of X can be interchanged by a member of $\mathcal{G}$, this is a weak form of 2-transitivity, and we say $\mathcal{G}$ is *flipping*. More strongly, if $\mathcal{G}_a$ is a flipping on $X \setminus \{a\}$ for each $a \in X$, then we say, after P. Neumann, that $\mathcal{G}$ is *generously 2-transitive*. 3-transitivity implies generous 2-transitivity, which in turn implies 2-transitivity; the homeomorphism group of the real line with its usual topology is a 2-transitive action that is not generously 2-transitive.

Along with transitivity notions, there are related notions of "primitivity". Let T be a first-order theory in the sense of [8]. A model of T with underlying set X is *obligatory* if Sym(X) is the automorphism group of that model. Define a group action $\mathcal{G}$ on X to be *T-primitive* if $\mathcal{G}$ fails to preserve the nonobligatory models of T with underlying set X. In the standard terminology [7], $\mathcal{G}$ is *primitive* if it is T-primitive, where T is the theory of equivalence (i.e., reflexive, symmetric, transitive binary) relations. (The obligatory equivalence relations are just the trivial one and the discrete one.) Also, when T is the theory of preorders (i.e., reflexive, transitive binary relations),

T-primitivity has been dubbed *strong primitivity* by H. Wielandt. There is, of course, an unlimited store of such first-order primitivity notions; some of which coincide with transitivity notions already defined. For example, when T is the theory of irreflexive binary relations, T-primitivity is 2-transitivity. (To see this, suppose $\mathscr{G}$ is 2-transitive on X, and let R be a nonobligatory irreflexive binary relation on X. Then there are $a \neq b$ and $c \neq d$ such that $\langle a, b\rangle \in R$ but $\langle c, d\rangle \notin R$. Since some element of $\mathscr{G}$ takes $\langle a, b\rangle$ to $\langle c, d\rangle$, $\mathscr{G}$ fails to preserve R. Conversely, suppose $\mathscr{G}$ is not 2-transitive, and $a \neq b$ and $c \neq d$ are such that no element of $\mathscr{G}$ takes $\langle a, b\rangle$ to $\langle c, d\rangle$. Let R be the orbit of $\langle a, b\rangle$ under $\mathscr{G}$. Then R is nonobligatory irreflexive binary relation that is $\mathscr{G}$-invariant.) If T is the theory of graphs (i.e., irreflexive, symmetric binary relations), then T-primitivity is 2-homogeneity. A much-cited fact is that a transitive group action is primitive if and only if its point stabilizes are all maximal proper subgroups (see [7]); a primitive flipping action is strongly primitive. Finally, being T-primitive for every first-order theory T is equivalent to being highly transitive. (Indeed, if $\mathscr{G}$ is not n-transitive on X ($n \geq 1$), let the theory T say of n-tuples that there are no repetitions. We can then argue that $\mathscr{G}$ is not T-primitive as we did above in the case $n = 2$. Conversely, if $\mathscr{G}$ is highly transitive and T is any first-order theory, let R be an n-ary relation on X witnessing a nonobligatory model of T with underlying set X. Then there is an n-tuple $\langle a_1, \ldots, a_n\rangle \in R$ and a permutation $x \mapsto \bar{x}$ such that $\langle \overline{a_1}, \ldots, \overline{a_n}\rangle \notin R$. Since some member of $\mathscr{G}$ takes $\langle a_1, \ldots, a_n\rangle$ to $\langle \overline{a_1}, \ldots, \overline{a_n}\rangle$, we see that $\mathscr{G}$ is T-primitive.)

We next establish some topological notions. To begin with, our set-theoretic foundation is Zermelo–Fraenkel set theory with the Axiom of Choice. Infinite ordinal and cardinal numbers are denoted using lower-case Greek letters; the notation κ^λ can mean both the set of functions from λ to κ as well as the cardinality of that set. In particular, 2^ω is the cardinality of the continuum, commonly denoted $\mathbf{c}$. The *weight* of a topology $\mathscr{T}$, denoted $w(\mathscr{T})$, is the smallest cardinality of a possible open basis for $\mathscr{T}$, and is in many ways the most useful cardinal invariant in topology. The usual separation axioms are denoted T_n, where n is an integer between 0 and 4. Recall that T_0 (resp. T_1) says that, given two points, there is a neighborhood of one (resp. each) missing the other; T_2 says that each two points can be separated by disjoint neighborhoods; and T_3 (resp. T_4) is the conjunction of T_1 and the condition that a closed set and a point not contained in the set (resp. two disjoint closed sets) can be separated by disjoint neighborhoods. The adjectives Hausdorff, regular, and normal are often used in place of T_2, T_3, and T_4, respectively (see [27]). If κ is any infinite cardinal, then $\mathscr{C}_\kappa := \{A \subseteq X: |X \setminus A| < \kappa\} \cup \{\emptyset\}$. $\mathscr{C}_\kappa$ is a topology which is discrete if $\kappa > |X|$. Otherwise, $\mathscr{C}_\kappa$ is *perfect* (i.e., all nonempty open sets are infinite), T_1, and a *filterbase topology* (i.e., one in which two nonempty open sets always intersect). ($\mathscr{C}_\omega$ is also referred to as the *cofinite* topology).

The following assertion is easy to prove, and is well known in somewhat different contexts (see [15]).

Proposition 1.2. *The topologies $\mathscr{C}_\kappa$ are precisely those topologies on X that are $\mathscr{G}$-invariant for every group action $\mathscr{G}$ on X.*

Remark 1.3. Proposition 1.2 is interesting because it suggests a new notion of primitivity. Call a group action $\mathcal{G}$ on X *topologically primitive* if the only $\mathcal{G}$-invariant topologies are the obligatory ones, namely the topologies $\mathcal{C}_\kappa$. Then $\mathrm{Sym}(X)$ is topologically primitive. P. Cameron, J. McDermott, and others have studied this property, but the work so far has not been published. McDermott has found examples of topologically primitive group actions of countable degree (e.g., actions that are transitive on moieties), and it follows from sophisticated work of Macpherson and Praeger [18] that all such actions are highly transitive. In addition Cameron has developed a simple proof of the result that a primitive group action of countable degree preserves a nonobligatory topology if and only if it preserves a nonobligatory filter (a result which also follows from the methods in [18]).

Given a topological property π, define $\mathcal{G}$ to be *support-π* if $S(\mathcal{G})$ has property π.

Remark 1.4. Checking directly to see whether the support topology has a particular property can often be cumbersome, since $\mathrm{Supp}(\mathcal{G})$ is generally not a topological basis. In important special cases, however, it is indeed a basis, as we see in Section 3. Under such circumstances, our task is made considerably easier.

We first consider the situation when π is the property "trivial". Define a group action $\mathcal{G}$ on X to be *sharp* if whenever $x, y \in X$ are in the same $\mathcal{G}$-orbit, then there is a unique $g \in \mathcal{G}$ taking x to y. An action is *sharply transitive* [7] if it is both sharp and transitive; thus a sharp group action is sharply transitive on each of its orbits. If $Y \subseteq X$ is a $\mathcal{G}$-orbit, then there is a bijection between $\mathcal{G}$ and Y, namely fix $y_0 \in Y$, and assign $g(y_0)$ in Y to each g in $\mathcal{G}$. So in an obvious way one may regard a sharp group action as an abstract group $\mathcal{G}$ acting on $\mathcal{G} \times I$, where I is an arbitrary nonempty set, via left (or right) multiplication on the first coördinate (i.e., $g(h, i) = \langle gh, i\rangle$). (A subgroup of a group, acting on the larger group via left (or right) multiplication, is a case in point; the orbits are just the left (or right) cosets of the subgroup.) The following characterization of support-triviality is an easy consequence of the definition and the remarks above.

Proposition 1.5. *Let $\mathcal{G}$ be a group action on X. Then $\mathcal{G}$ is support-trivial if and only if $\mathcal{G}$ is sharp. In particular, if $\mathcal{G}$ is support-trivial, then $|\mathcal{G}| \leq |X|$, and $\mathcal{G}$ is neither HS-fixed nor primitive.*

Proof. We show that a primitive group action cannot be sharp. Indeed, let $\mathcal{G}$ be primitive. Then the point stabilizes are maximal proper subgroups. If $\mathcal{G}$ were also sharp, hence sharply transitive, then $\mathcal{G}$ would be infinite with trivial point stabilizers. This would say that $\mathcal{G}$ had no proper nontrivial subgroups. But the only groups with this property are cyclic of finite (prime) cardinality. □

One conclusion of Proposition 1.5 concerns the cardinality/degree relationship in a group action. Clearly all actions satisfy $|\mathcal{G}| \leq 2^{|X|}$, and this gives us one notion of how large $\mathscr{G}$ is in Sym(X). Proposition 1.5 then says that support-trivial group actions are not "large" in this sense. The following gives us more information in this vein.

Proposition 1.6. *Suppose $\mathscr{G}$ is a group action on a set X.*

(i) *If $\mathscr{G}$ is support-T_0 then $|X| \leq 2^{|\mathscr{G}|}$. (One can obtain a support-metrizable action for which equality holds).*

(ii) *If $\mathscr{G}$ is either transitive or support-discrete, then $|X| \leq |\mathscr{G}|$. (One can obtain a support-discrete highly transitive action for which equality holds.)*

(iii) *If $\mathscr{G}$ has at most $|X|$ orbits on moieties, then $|\mathscr{G}| = 2^{|X|}$.*

Proof. (i) Assume $\mathscr{G}$ is support-T_0, and let $\mathscr{B}$ be any open basis for $S(\mathscr{G})$. For each $x \in X$, let $\mathscr{B}(x)$ be all members of $\mathscr{B}$ containing x. If $x \neq y$ in X, then $\mathscr{B}(x) \neq \mathscr{B}(y)$. This tells us that $|X| \leq 2^{w(S(\mathscr{G}))}$. Thus $S(\mathscr{G})$, hence $\mathscr{G}$, is infinite; so $w(S(\mathscr{G})) \leq |\mathscr{G}|$.

To prove the parenthetical assertion, let X be the real line, and let $\mathscr{G}$ be all those increasing bijections on X whose graphs are broken lines satisfying: (1) the number of breaks is finite; and (2) the coordinates of each break are rational. Then the desired equality holds; also every bounded open interval with rational endpoints is the support of some member of $\mathscr{G}$ (see Proposition 3.3). This says that $S(\mathscr{G})$ contains the usual real topology $\mathscr{U}$. But $\mathscr{U}$ is clearly $\mathscr{G}$-invariant, so the two topologies are equal (see Proposition 2.1). In particular, $\mathscr{G}$ is support-metrizable.

(ii) Suppose first $\mathscr{G}$ is transitive, and let $a \in X$ be fixed. Define $F: \mathscr{G} \to X$ by $F(g) = g(a)$. Then F is surjective.

Next assume $S(\mathscr{G})$ is discrete. Then its weight is at least $|X|$ and at most $|\mathscr{G}|$.

To prove the parenthetical assertion, assign to each $n < \omega$ and to each bijection f between two n-element subsets of X, a permutation g_f on X extending f. Let $\mathscr{G}$ be the subgroup of Sym(X) generated by the permutations g_f, plus all the transpositions (i.e., permutations with doubleton support). Then $|\mathscr{G}| = |X|$; moreover $\mathscr{G}$ is support-discrete and highly transitive.

(iii) If $\mathscr{G}$ has $\leq |X|$ orbits on moieties, an easy application of the Jourdain–König inequality (that for every infinite cardinal α, $\alpha < \alpha^{cf(\alpha)}$, see A.29 in [8]) ensures that some orbit $\mathscr{M}$ on moieties has cardinality $2^{|X|}$. Then we quickly obtain a surjection from $\mathscr{G}$ to $\mathscr{M}$, as in (ii) above. □

Primitivity properties of a group action affect its invariant topologies vis à vis connectedness and the lower-level separtion axioms (i.e., T_n, $n \leq 2$). We first define a topology to be *completely Hausdorff* (resp. *ultra-Hausdorff*) if each pair of points can be separated by a continuous real-valued function (resp. a clopen set). (NB: The ultra-Hausdorff condition should not be confused with the strictly weaker condition of being totally disconnected. The latter merely says that no two points lie in a connected subset.)

Theorem 1.7. *Suppose $\mathscr{G}$ is a group action on a set X.*

(i) *$\mathscr{G}$ is primitive if and only if every $\mathscr{G}$-invariant nontrivial topology on X is T_0, if and only if every $\mathscr{G}$-invariant nonconnected topology on X is ultra-Hausdorff. (In particular, a primitive group action on an infinite set is support-T_0.)*

(ii) *If $\mathscr{G}$ is primitive of degree $< \mathbf{c}$, then every completely Hausdorff $\mathscr{G}$-invariant topology on X is ultra-Hausdorff.*

(iii) *$\mathscr{G}$ is strongly primitive if and only if every $\mathscr{G}$-invariant nontrivial topology on X is T_1. (In particular, a strongly primitive group action on an infinite set is support-T_1.)*

(iv) *If $\mathscr{G}$ is 2-homogeneous, then every $\mathscr{G}$-invariant nonfilterbase topology on X is T_2. (In particular, a 2-homogeneous group action is support-Hausdorff if it contains finitely many nonidentity elements whose supports have empty intersection.) As a weak converse: if every $\mathscr{G}$-invariant nonfilterbase topology on X is T_2, then $\mathscr{G}$ is primitive.*

Proof. (i), (iii) Given a topology $\mathscr{T}$ on a set X, define the binary relations L, C, and Q on X as follows: xLy (i.e., $\langle x, y\rangle \in L$) if x lies in the $\mathscr{T}$-closure of y; xCy if xLy and yLx; and xQy if there is no $\mathscr{T}$-clopen separation of x and y (i.e., x and y lie in the same $\mathscr{T}$-quasicomponent). Then L is a preorder, and both C and Q are equivalence relations on X. $\mathscr{T}$ is trivial if and only if L is trivial, if and only if C is trivial; $\mathscr{T}$ is connected (resp. T_0, T_1, ultra-Hausdorff) if and only if Q is trivial (resp. C is discrete, L is discrete, Q is discrete). If $\mathscr{G}$ is a group acting on X and $\mathscr{T}$ is $\mathscr{G}$-invariant, then so are these relations. So if $\mathscr{T}$ is nontrivial and fails to be T_0, then C witnesses the imprimitivity of $\mathscr{G}$. The other analogous assertions follow as easily, establishing the left-to-right direction of both (i) and (iii).

Conversely, given a preorder L on X, define the topology $\mathscr{T}$ using the sets $[x] := \{y \in X: xLy\}$, $x \in X$, for an open subbasis. Clearly L is trivial (resp. discrete) if and only if $\mathscr{T}$ is trivial (resp. discrete). Moreover, if L is nondiscrete, then there exist distinct $x, y \in X$ with $y \in [x]$. If $x \in U = [x_1] \cap \cdots \cap [x_n]$, then, by transitivity of L, we have $y \in U$ also, hence $\mathscr{T}$ is not T_1. If L is now an equivalence relation, then the sets $[x]$ (the L-equivalence classes) constitute a clopen basis for $\mathscr{T}$. Thus if L is nontrivial and nondiscrete, then $\mathscr{T}$ is nonconnected and not T_0. Finally, if L is $\mathscr{G}$-invariant, so is $\mathscr{T}$. This establishes the other direction of (i) and (iii).

The parenthetical statements follow immediately from Proposition 1.5.

(ii) This follows immediately from (i), plus the observation that a connected completely Hausdorff space with more than one point surjects continuously onto a nondegenerate real interval, and must therefore have cardinality $\geq \mathbf{c}$.

(iv) Assume $\mathscr{G}$ is 2-homogeneous, and let $\mathscr{T}$ be a $\mathscr{G}$-invariant topology on X. Define the binary relation E on X by saying xEy if $x \neq y$ and every $\mathscr{T}$-neighborhood of x intersects every $\mathscr{T}$-neighborhood of y. Then E is the adjacency relation for a $\mathscr{G}$-invariant graph on X, and $\mathscr{T}$ is a filterbase (resp. Hausdorff) topology if and only if E is complete (resp. discrete). Since $\mathscr{G}$ is 2-homogeneous, these are the only choices for E.

The parenthetical assertion then follows from Proposition 1.1; as for the weak converse, the same argument used in the converse of (i) will do. □

Remark 1.8. McDermott has also independently observed the connection between primitivity (resp. strong primitivity) and the T_0 (resp. T_1) axiom in

Theorem 1.7(i), (iii). As for Theorem 1.7(iv), the original hypothesis was for $\mathscr{G}$ to be generously 2-transitive. This assumption is certainly too strong, as is shown in Theorem 3.10. We do not know whether the weaker hypothesis of 2-homogeneity is necessary; it is definitely the case that the approach used in parts (i) and (ii) does not work: One can easily devise nonobligatory adjacency relations that do not give rise to nonfilterbase non-Hausdorff topologies (e.g., the "random graph" of Erdős–Rado, which gives rise, according to our recipe, to a filterbase topology).

2. T_n-complete group actions

We begin with an easy, but important, observation.

Proposition 2.1. *If $\mathscr{T}$ is Hausdorff and $\mathscr{G}$-invariant, then $S(\mathscr{G}) \subseteq \mathscr{T}$.*

Proof. This is a special case of the well-known fact that if f and g are two continuous functions with the same domain and the same Hausdorff range, then the set $\{x: f(x) \neq g(x)\}$ is an open set in the domain. □

Proposition 2.2. (i) *If $\mathscr{T}$ is a Hausdorff topology on X, then $SH(\mathscr{T}) \subseteq \mathscr{T}$. (In particular, if $\mathscr{T}$ is Hausdorff and $H(\mathscr{T}) = \mathrm{Sym}(X)$, then $\mathscr{T}$ is discrete.)*

(ii) *If $\mathscr{G}$ is a support-Hausdorff group action, then $S(\mathscr{G})$ is SH-fixed; the converse is false.*

Proof. (i) This is an immediate restatement of Proposition 2.1; the parenthetical assertion easily follows.

(ii) We have in general that $\mathscr{G} \leq HS(\mathscr{G})$. Since $S(\ )$ is order-preserving, we then get $S(\mathscr{G}) \subseteq SH(S(\mathscr{G}))$. Now assume $S(\mathscr{G})$ is Hausdorff. Then, by (i), we have $SH(S(\mathscr{G})) \subseteq S(\mathscr{G})$, hence $S(\mathscr{G})$ is SH-fixed.

For the failure of the converse, let a be a fixed element of X, an infinite set, and let $\mathscr{G} = \mathrm{Sym}(X)_a$. Then points $x \in X\setminus\{a\}$ are $S(\mathscr{G})$-isolated, since every doubleton set in $X\setminus\{a\}$ is the support of some member of $\mathscr{G}$. But the only $S(\mathscr{G})$-neighborhood of a is X, so $S(\mathscr{G})$ is T_0 but not T_1. Clearly, however, $H(S(\mathscr{G})) = \mathscr{G}$, so $S(\mathscr{G})$ is SH-fixed. □

The support topology is of use in the study of various minimal members of $\mathrm{Invar}(\mathscr{G})$. Let π be an arbitrary topological property. An action $\mathscr{G}$ on X is *π-complete* if there is a smallest $\mathscr{G}$-invariant topology on X having property π. Denote this topology, when it exists, by $\min(\mathscr{G}, \pi)$. Of course $\mathscr{G}$ is always T_1-complete, and $\min(\mathscr{G}, T_1)$ is just the cofinite topology $\mathscr{C}_\omega$.

Example 2.3. The trivial action on an infinite set is not T_n-complete for any $n \neq 1$. To see this, let $a \in X$ be given; define the topology $\mathscr{E}_a$ by declaring all points $x \neq a$ to be isolated, and by having X as the only neighborhood of a. (As we saw in the proof of

Proposition 2.2(ii), $\mathscr{E}_a = S(\mathrm{Sym}(X)_a)$.) If $A \subseteq X$ is proper nonempty and $a \in A$, then $\mathscr{E}_a$ is a trivial-invariant T_0 topology for which A is not an open set. This says that the intersection of all T_0 trivial-invariant topologies on X is the trivial topology. Now for $a \in X$ define the topology $\mathscr{D}_a$ as above, except that open neighborhoods of a are sets of the form $X \backslash F$, where F is a finite subset of $X \backslash \{a\}$. ($\mathscr{D}_a$ is the classic one-point compactification of the discrete space $X \backslash \{a\}$, with a as point-at-infinity.) This topology is compact Hausdorff, hence T_n for all n. If $A \subseteq X$ is nonempty and not cofinite, and $a \in A$, then A is not $\mathscr{D}_a$-open. This implies that the intersection of all T_n trivial-invariant topologies on X, $n \geq 1$, is the cofinite topology.

Proposition 2.4. *For $n \geq 1$, every support-T_n group action is T_n-complete; in fact, for $n \geq 2$, $\min(\mathscr{G}, T_n) = S(\mathscr{G})$.*

Proof. This is immediate from Proposition 2.1, plus the fact that every group action is T_1-complete. □

Example 2.5. For $n \geq 1$, T_n-complete group actions need not be support-T_1. A support-trivial action will do for the case $n = 1$; for the other cases, suppose $a \in X$ are given, and let $\mathscr{G} = \mathrm{Sym}(X)_a$. As we saw in the proof of Proposition 2.2(ii), $S(\mathscr{G}) = \mathscr{E}_a$, a T_0 topology that is not T_1. On the other hand, $\mathscr{D}_a$ is $\mathscr{G}$-invariant, as well as T_n for all n. If $\mathscr{T}$ is any $\mathscr{G}$-invariant Hausdorff topology, then $\mathscr{E}_a \subseteq \mathscr{T}$, hence $\mathscr{D}_a \subseteq \mathscr{T}$. Thus, for $n \geq 2$, $\min(\mathscr{G}, T_n) = \mathscr{D}_a$.

In Example 2.5, $\min(\mathscr{G}, T_2)$ is compact Hausdorff, hence normal. A necessary condition for T_n-completeness, $n \geq 2$, can be phrased in terms of compactness because compact Hausdorff topologies are minimal Hausdorff.

Proposition 2.6. *Suppose $n \geq 2$ and $\mathscr{G}$ is a T_n-complete group action on X. If $\mathscr{G}$ preserves a compact Hausdorff topology $\mathscr{T}$, then $\min(\mathscr{G}, T_n) = \mathscr{T}$. In particular, $\mathscr{G}$ cannot preserve two compact Hausdorff topologies.*

Example 2.7. Let $\mathscr{G}$ be the group $H(\mathscr{U})$ of homeomorphisms of the usual topology $\mathscr{U}$ on the closed unit interval $X = [0, 1]$ in the real line $\mathbf{R}$. Then there are two $\mathscr{G}$-invariant compact metrizable topologies on X, and $\mathscr{G}$ is therefore not T_n-complete for $n \geq 2$, by Proposition 2.6. Moreover, $\mathscr{G}$ is not support-T_0. To see this, note first that, because of the intermediate value theorem of elementary calculus, $\mathscr{G}$ consists of all the bijections on X that are either increasing or decreasing. Thus every member of $\mathscr{G}$ either fixes the endpoints or interchanges them. Since each decreasing $g \in \mathscr{G}$ has exactly one fixed point, the $S(\mathscr{G})$-neighborhoods of the endpoints are of the form $X \backslash F$, where F is a finite subset of the open interval $(0, 1)$. Thus $S(\mathscr{G})$ is not T_0. On the other hand, each open subinterval of $(0, 1)$ is the support of an increasing member of $\mathscr{G}$. Thus the topology on $(0, 1)$ inherited from $S(\mathscr{G})$ is the usual one. (See the proof of Proposition 3.3 for a more detailed explanation.) Define the topology $\mathscr{T}$ on X as follows: Basic neighborhoods of points in $(0, 1)$ are open subintervals as usual; basic neighborhoods of 0 (resp. 1) are of the form $\{0\} \cup (t, 1)$ (resp. $(0, t) \cup \{1\}$), $t \in (0, 1)$. Then it is easy to show that $\mathcal{T}$ is a $\mathcal{G}$-invariant compact metrizable topology that is distinct from $\mathcal{U}$.

Example 2.8. Let $\mathcal{G}$ be the group $H(\mathcal{U})$ of homeomorphisms of the usual topology $\mathcal{U}$ on the standard unit circle $X = S^1$ in the Euclidean plane $\mathbf{R} \times \mathbf{R}$. Then there is exactly one $\mathcal{G}$-invariant compact Hausdorff topology on X, namely $\mathcal{U}$ itself. Moreover, $\mathcal{U} = S(\mathcal{G})$. To see this, let U be a typical basic $\mathcal{U}$-open set, i.e., a proper open subarc of the circle, and fix $p \in X \setminus U$. We identify $X \setminus \{p\}$ in a standard way with the usual real line; hence there is a $\mathcal{U}$-homeomorphism whose support is U. This tells us that $\mathcal{U} = S(\mathcal{G})$ (again see the proof of Proposition 3.3), and implies that $\mathcal{U}$ is the only $\mathcal{G}$-invariant compact Hausdorff topology on X.

A point $a \in X$ is a *fixed point* of $\mathcal{G}$ if $g(a) = a$ for all $g \in \mathcal{G}$; i.e., if $\mathcal{G} \le \mathrm{Sym}(X)_a$.

The compact Hausdorff topologies $\mathcal{D}_a$, defined in Example 2.3, can help us elaborate on Proposition 2.6.

Proposition 2.9. *Suppose $\mathcal{G}$ is a group action on a set X.*

(i) *If $\mathcal{G}$ is T_n-complete, $n \ge 2$, and $a \in X$ is a fixed point of $\mathcal{G}$, then* $\min(\mathcal{G}, T_n) = \mathcal{D}_a$.

(ii) *If $n \ge 2$, $\mathcal{G}$ has a fixed point, and preserves a T_n topology with more than one nonisolated point, then $\mathcal{G}$ is not T_k-complete for $2 \le k \le n$.*

(iii) *If $\mathcal{G}$ has two fixed points, then $\mathcal{G}$ is not T_n-complete for any $n \ge 2$.*

Proof. (i) If a is a fixed point of $\mathcal{G}$, then clearly $\mathcal{D}_a$ is a $\mathcal{G}$-invariant topology that is compact Hausdorff. By Proposition 2.6, $\min(\mathcal{G}, T_n) = \mathcal{D}_a$.

(ii) Let $n \ge 2$. If $2 \le k \le n$ and $\mathcal{G}$ is T_k-complete, then, by (i), every $\mathcal{G}$-invariant T_n topology must contain $\mathcal{D}_a$ for some $a \in X$, and hence can have at most one nonisolated point.

(iii) If $\mathcal{G}$ has two fixed points a and b, then $\mathcal{D}_a$ and $\mathcal{D}_b$ are two distinct $\mathcal{G}$-invariant compact Hausdorff topologies; hence $\mathcal{G}$ is not T_n-complete for any $n \ge 2$, by Proposition 2.6. □

Proposition 2.10. *If $\mathcal{G}$ preserves a compact Hausdorff topology, then $\mathcal{G}$ is support-compact; the converse fails. Indeed, if $|X| < \mathbf{c}$, and if $\mathcal{G}$ is transitive, then no $\mathcal{G}$-invariant topology is compact Hausdorff.*

Proof. The first assertion follows from Proposition 2.1, plus the fact that any topology coarser than a compact topology is also compact. For the failure of the converse, suppose $\mathcal{G}$ acts on the infinite set X, and let $\mathcal{T}$ be any compact Hausdorff topology on X. If $\mathcal{T}$ is a perfect topology (which means having no isolated points, in the presence of the T_1 axiom), then, by a binary tree argument, it is possible to construct $\mathbf{c}$ infinite descending chains of nonempty $\mathcal{T}$-closed sets such that any two of these chains eventually give rise to disjoint sets. By compactness, each chain has nonempty intersection; hence $|X| \ge \mathbf{c}$. (See, e.g., [27]. This is the standard way one shows that any compact Hausdorff space with $< \mathbf{c}$ points is scattered.) So if $|X| < \mathbf{c}$, then there must be (infinitely many) $\mathscr{T}$-isolated points. But there must also be $\mathscr{T}$-nonisolated points. So if $\mathscr{G}$ is transitive as well, then $\mathscr{T}$ cannot be $\mathscr{G}$-invariant. To get an example of the failure of the converse, then, let $\mathscr{G}$ be a sharply transitive action on an infinite set X, where $|X| < \mathbf{c}$. Then $\mathscr{G}$ is support-trivial (hence support-compact) by Proposition 1.5; however, no $\mathscr{G}$-invariant topology is compact Hausdorff. □

Example 2.11. A group action may preserve exactly one compact Hausdorff topology, and still fail to be T_n-complete for any $n \geq 2$. Let $\mathscr{U}$ be the usual topology on $\mathbf{Q}$, $a \in \mathbf{Q}$, and $\mathscr{G} = H(\mathscr{U})_a$. Then $\mathscr{D}_a$ is a $\mathscr{G}$-invariant compact Hausdorff topology. Because $\mathscr{U}$ is a normal perfect $\mathscr{G}$-invariant topology, Proposition 2.9(ii) says that $\mathscr{G}$ fails to be T_n-complete for any $n \geq 2$. If $\mathscr{T}$ is any $\mathscr{G}$-invariant compact Hausdorff topology, then there must be $\mathscr{T}$-nonisolated points as well as infinitely many $\mathscr{T}$-isolated points, by the proof of Proposition 2.10. But $\mathscr{G}$ is transitive on $\mathbf{Q}\backslash\{a\}$, hence all points $x \neq a$ are $\mathscr{T}$-isolated. This forces $\mathscr{D}_a \subseteq \mathscr{T}$, hence $\mathscr{D}_a = \mathscr{T}$.

Example 2.5 suggests a line of questioning that concerns certain "large" subgroups of particularly rich groups. Let $\mathscr{G}$ act on an infinite set X. Define a subgroup $\mathscr{H} \leq \mathscr{G}$ to be *finitely restricted* if there is a finite $A \subseteq X$ such that $\mathscr{G}_{(A)} \leq \mathscr{H} \leq \mathscr{G}_A$. It is easy to show that the index of any finitely restricted subgroup of a group acting on X is at most $|X|$, so such a subgroup can be said to be relatively large under many circumstances of interest; e.g., when the big group is the full symmetric group. Define a subgroup of a group $\mathscr{G}$ to have *small index* in $\mathscr{G}$ if the index of the subgroup is less than $|\mathscr{G}|$. Thus every finitely restricted subgroup of $\mathrm{Sym}(X)$ has small index, and a remarkable result of Dixon–Neumann–Thomas [11] shows the converse in the countable degree case.

Theorem 2.12 (Theorem 1 in [11]). *Let X be a countably infinite set. Then every small index subgroup of* $\mathrm{Sym}(X)$ *is finitely restricted.* □

Remark 2.13. Theorem 2.12 is interesting partly because it equates a condition on subgroup actions with a condition on abstract subgroups. Any group action on a countable set is said [7] to satisfy the *strong small index property* if it can be substituted for $\mathrm{Sym}(X)$ in Theorem 2.12. We return to this theme in the next section.

The following result specifies when a finitely restricted subgroup of $\mathrm{Sym}(X)$ is T_n-complete.

Theorem 2.14. *Let X be an infinite set, $n \geq 2$, and suppose $\mathscr{H}$ is a finitely restricted subgroup of* $\mathrm{Sym}(X)$. *Then $\mathscr{H}$ is T_n-complete if and only if $\mathscr{H}$ has at most one fixed point. If $\mathscr{H}$ has no fixed points, then* $\min(\mathscr{H}, T_n)$ *is the discrete topology on X; if $\mathscr{H}$ has exactly one fixed point $a \in X$, then* $\min(\mathscr{H}, T_n) = \mathscr{D}_a$.

Proof. If $\mathscr{H}$ is T_n-complete, the conclusion follows from Proposition 2.9(iii), so we show the converse.

Let A be a finite subset of X such that $\mathrm{Sym}(X)_{(A)} \leq \mathscr{H} \leq \mathrm{Sym}(X)_A$. $\mathscr{H}$ then moves all elements of $X \setminus A$, so any fixed point of $\mathscr{H}$ must lie in A. $|X \setminus A| > 2$, and any two points of $X \setminus A$ constitute the support of some element of $\mathscr{H}$, so all points of $X \setminus A$ are $S(\mathscr{H})$-isolated. Let $\mathscr{T}$ be a given $\mathscr{H}$-invariant T_n topology. Then every point of $X \setminus A$ is $\mathscr{T}$-isolated, by Proposition 2.1. Suppose $a \in A$ is not fixed by $\mathscr{H}$, say $a \in \mathrm{supp}(g)$ for some $g \in \mathscr{H}$. Since $g(A) = A$, we may define $h \in \mathrm{Sym}(X)$ to be the identity on A and to be the inverse of g on $X \setminus A$. Then $h \in \mathscr{H}$, and $a \in \mathrm{supp}(gh) \subseteq A$. This says that a is contained in a finite $S(\mathscr{H})$-open set, so must be $\mathscr{T}$-isolated (again by Proposition 2.1). Thus, if $\mathscr{H}$ has no fixed points, then $\min(\mathscr{H}, T_n)$ exists and is discrete. If $\mathscr{H}$ has just one fixed point a, then the only $S(\mathscr{H})$-neighborhood of a is X, and all other points are $S(\mathscr{H})$-isolated. This forces $\mathscr{T}$ to contain $\mathscr{D}_a$; whence $\min(\mathscr{H}, T_n) = \mathscr{D}_a$. □

3. Group actions on field-ordered sets

We now turn our attention to groups acting on linearly ordered sets X in which the ordering is compatible with a field structure on X. The most well-known examples are the real field **R** and the rational field **Q**, but there are many others. What is important to us is the order structure and that there is some field (one of many) for which that order is compatible. However, we will usually treat X as an ordered field, rather than as a field-ordered set, bearing in mind that the field structure is in some sense secondary to the order structure. (See [13, 14] for background on ordered fields.)

We denote the extra structure of an ordered field X generically, writing just X when we mean $\langle X, +, -, \cdot, 0, 1, < \rangle$. As an ordering, X is dense without endpoints. As a field, X is of characteristic 0, and its minimal subfield is isomorphic to **Q**. The usual order topology on X, basically generated by bounded open (i.e., endpoint-free) intervals of X, is generically denoted $\mathscr{U}$. This topology is well known to be normal. When $\mathscr{U}$ is understood as the topology under consideration, we often write $H(X)$ instead of $H(\mathscr{U})$.

Because X has such rich structure, it has a vast assortment of naturally definable group actions. The ones of most relevance to us are:

(i) $M(X)$, the monotone (increasing or decreasing) bijections;
(ii) $I(X) \leq M(X)$, the increasing bijections;
(iii) $PLM(X) \leq M(X)$, the piecewise linear monotone bijections definable in finitely many pieces;
(iv) $PLI(X) := PLM(X) \cap I(X)$;
(v) $LM(X)$, the linear bijections; and
(vi) $LI(X) := LM(X) \cap I(X)$.

Remark 3.1. It is easy to verify that all the inclusions above are proper, with the possible exception of $M(X) \leq H(X)$. Because of the intermediate value theorem equality holds when $X = \mathbf{R}$. (This situation is unique, however; see Theorem 3.2.) We reiterate that it is the ordering on X that is the most relevant for our purposes. Thus, of the several group actions defined above, the ones defined in terms of linearity have a secondary (though important) status. All group actions $LI(X) \leq \mathcal{G} \leq I(X)$ are 2-homogeneous without being strongly primitive (viz. the field ordering on X); all actions containing $LM(X)$ are 2-transitive. If $\mathcal{G}$ contains $PLI(X)$, then $\mathcal{G}$ is highly homogeneous. $M(X)$ is never 3-transitive (or even generously 2-transitive); $H(X)$ is highly transitive when X is not the real field (again, refer to Theorem 3.2).

The special position of **R** among ordered fields is indicated in the following (mostly) well-known result. Since we know of no single source where a proof is presented, we outline one here. (Recall that a space is *zero-dimensional* if it has a basis of clopen sets; clearly zero-dimensional T_0 spaces are ultra-Hausdorff.)

Theorem 3.2. *Let X be an ordered field. The following are equivalent:*
(a) *X is order isomorphic to the real field.*
(b) *X is a (Dedekind-)complete ordering.*
(c) *$\mathcal{U}$ is connected.*
(d) *$\mathcal{U}$ is not zero-dimensional.*
(e) *$H(X) = M(X)$.*
(f) *$H(X)$ is not generously 2-transitive.*
(g) *$H(X)$ is not 3-transitive.*
(h) *$H(X)$ is not highly transitive.*

Proof. ((a) iff (b)) **R** is well known to be complete as an ordered set; so let X be an ordered field whose ordering is complete. Then X is easily seen to be archimedean, and hence is order isomorphic to a subfield of **R** (see, e.g., [14]). By completeness, this subfield must be all of **R**.

((b) iff (c)) A complete ordering gives rise to a connected order topology; the proof is much the same as for **R** itself (see e.g., [27]). Conversely, a Dedekind cut in X that is not an interval is a proper nonempty $\mathcal{U}$-clopen set.

((c) iff (d)) Only one direction is nontrivial, so assume X is an ordered field whose usual topology is nonconnected. By the above, we then have clopen cuts, which we can shift using the field structure of X. The result is that, given any two points of X, there is a clopen cut containing one and missing the other. Now a cut that is unbounded to the left is the complement of a cut that is unbounded on the right. Thus, by intersecting clopen cuts of "opposite parity", we can quickly establish zero-dimensionality.

((d) iff (e)) If $\mathcal{U}$ is not zero-dimensional, then X is order isomorphic to the real field, hence $H(X) = M(X)$ as indicated in Remark 3.1. If $\mathcal{U}$ is zero-dimensional, then we can easily find a proper nonempty $\mathcal{U}$-clopen set U that is "symmetric about the origin", i.e., $x \in U$ if and only if $-x \in U$. We then define $g \in H(X)\backslash M(X)$ fixing $x \in U$ and taing x to $-x$ otherwise.

((e) only if (f)) $M(X)$ is never generously 2-transitive.

((f) only if (g)) Trivial.

((g) only if (h)) Trivial.

((h) only if (e)) Assume $H(X) \neq M(X)$. Then $\mathscr{U}$ is zero-dimensional. Suppose $A \subseteq X$ is finite, with $s \in \mathrm{Sym}(A)$. Finite symmetric groups are well known to be generated by transpositions, so we may assume s interchanges $a, b \in A$, fixing everything else. Let I_a and I_b be disjoint bounded open intervals containing a and b, respectively; using zero-dimensionality, let U be a $\mathscr{U}$-clopen neighborhood of a contained in I_a. Let $h \in PLI(X)$ take I_a onto I_b so that $h(a) = b$. Then $g \in \mathrm{Sym}(X)$, defined by taking $x \in U$ to $h(x)$, $x \in h(U)$ to $h^{-1}(x)$, and fixing x otherwise, is a $\mathscr{U}$-homeomorphism extending s.

Now suppose $f: B \to A$ is a bijection between finite subsets of X. Then $f = st$, where $t: B \to A$ preserves the order inherited from X, and $s \in \mathrm{Sym}(A)$. We are done since t extends to a member of $PLI(X)$, and s extends to a member of $H(X)$ as shown above. □

Proposition 3.3. *Let $PLI(X) \leq \mathscr{G} \leq H(X)$. Then $S(\mathscr{G}) = \mathscr{U}$; consequently $\mathscr{U}$ is SH-fixed (hence $H(X)$ is HS-fixed).*

Proof. Let I be any bounded open interval in X, say $I = (a, b)$. We set $c = (a + b)/2$ and $d = (c + b)/2$, and define $g: X \to X$ to be the identity on $X \backslash I$, to take $x \in (a, c]$ to $a + (d - a)(x - a)/(c - a)$, and to take $x \in (c, b]$ to $d + (b - d)(x - c)/(b - c)$. Then $g \in PLI(X)$ and $\mathrm{supp}(g) = I$, so $\mathscr{U} \subseteq S(PLI(X)) \subseteq S(\mathscr{G}) \subseteq S(H(X))$. Since $\mathscr{U}$ is $\mathscr{G}$-invariant and Hausdorff, equality holds by Proposition 2.1. (In fact, $\mathrm{Supp}(PLI(X))$ basically generates $\mathscr{U}$.) The rest of the statement of the proposition follows easily. □

Remark 3.4. If $PLI(X) \leq \mathscr{G} \leq \mathrm{Sym}(X)$, then $\mathscr{U} \subseteq S(\mathscr{G})$ by the argument above, hence $\mathscr{G}$ is support-Hausdorff. On the other hand, if $\mathscr{G}$ is either $LM(X)$ or $LI(X)$, then $S(\mathscr{G})$ is the cofinite topology on X. Thus, if $LI(X) \leq \mathscr{G} \leq \mathrm{Sym}(X)$, then $\mathscr{C}_\omega \subseteq S(\mathscr{G})$, so $\mathscr{G}$ is support-T_1. (Note that Theorem 1.7 guarantees only that $LI(X)$ is support-T_0).

An easy consequence of Proposition 3.3 and Remark 3.4 is the following.

Corollary 3.5. *Every $PLI(X)$-invariant Hausdorff topology on X is an enrichment of $\mathscr{U}$.*

Example 3.6. Let $X = \mathbf{R}$, and let $\mathscr{T}$ be the density topology (see [9, 17, 24]). (A Lebesgue-measurable set E, of measure $m(E)$, is $\mathscr{T}$-open just in case $m((x - \varepsilon, x + \varepsilon) \cap E)/2\varepsilon$ tends to 1 as positive ε tends to 0, for each $x \in E$.) $\mathscr{T}$ is an enrichment of $\mathscr{U}$ that is completely regular (i.e., a point and a nonempty closed set not containing the point can be separated by a continuous real-valued function); moreover, the $\mathscr{T}$-connected subsets of X are precisely the intervals. This latter feature ensures that $H(\mathscr{T}) \leq H(X)$ (the intermediate value theorem again). Equality does not hold; $H(\mathscr{T})$ does not even contain $I(X)$. (To see this, there are two classic results [21]: one that says every set of first category (= meager) can be mapped, via a member of $I(X)$, to a nullset (= set of measure 0); the other that says every subset of X is the union of a set of first category and a nullset. Now nullsets are $\mathscr{T}$-closed; hence, if

$\mathscr{T}$ were $I(X)$-invariant, it would be discrete.) However, $PLM(X) \leq H(\mathscr{T})$, so $\mathscr{U} = S(H(\mathscr{T}))$, by Proposition 3.3.

We wish to investigate the behavior of $\mathscr{G}$-invariant topologies, where $\mathscr{G}$ is a group acting on an ordered field X, and $\mathscr{G}$ arises naturally from the structure on X. As an illustration of what we mean, we first define a topology to have *maximal dispersion character* if all nonempty open sets are equinumerous.

Proposition 3.7. *Every nondiscrete $LI(X)$-invariant topology has maximal dispersion character.*

Proof. Suppose $\mathscr{T}$ is a $LI(X)$-invariant topology, with $U \in \mathscr{T}$ nonempty of cardinality $< |X|$. Because $LI(X)$ is transitive, we may assume, for convenience, that $0 \in U$. For each positive $b \in X$, define $g_b \in LI(X)$ by $g_b(x) = bx$, and define $G_b \subseteq X \times X$ to be the graph of g_b. Then for $b > c > 0$, $G_b \cap G_c = \{\langle 0,0 \rangle\}$. Since there are $|X|$ pairwise disjoint sets $G_b \backslash \{\langle 0,0 \rangle\}$, $b > 0$, and since $|U \times U| < |X|$, we infer that there is some $b > 0$ such that $G_b \cap (U \times U) = \{\langle 0,0 \rangle\}$. This says that $U \cap g_b(U) = \{0\}$, so 0 is $\mathscr{T}$-isolated. By transitivity of $LI(X)$, $\mathscr{T}$ is discrete. □

Let $\mathscr{S}$ and $\mathscr{T}$ be two topologies on a set X (forgetting any added structure on X for the moment). $\mathscr{T}$ is an *H-enrichment* of $\mathscr{S}$ if $\mathscr{S} \subseteq \mathscr{T}$ and $\mathscr{T}$ is $H(\mathscr{S})$-invariant. H-enrichments were introduced in [6], and studied further in [3,4], with special emphasis on H-enrichments of the usual topologies on the rational line and on the euclidean spaces. We continue that study here, focusing on H-enrichments of the usual topology $\mathscr{U}$ on a field-ordered set X. For simplicity, we use the term "H-enrichment," unmodified, to mean "H-enrichment of the usual topology" in the context of ordered fields. H-enrichments are obviously Hausdorff, and are, unless discrete, of maximal dispersion character by Proposition 3.7.

Remark 3.8. In [6] a version of Proposition 3.7 is proved for H-enrichments of euclidean topologies (Theorem 2.21); the proof idea if similar to the above. Another property of proper H-enrichments of euclidean topologies, as well as of the usual rational topology, is that the only convergent sequences are those that are eventually constant (Theorem 2.19 in [6]). (A topological space with this property is often called *contrasequential.*) One of many consequences of this is that the usual topology, in the euclidean and rational cases, admits a unique smallest proper H-enrichment (Proposition 3.7 in [4]).

Remark 3.9. The density topology on **R** (see Example 3.6) is not an H-enrichment; in fact the only H-enrichment that contains the density topology is discrete. This topology does, however, share some of the properties of proper nondiscrete H-enrichments, including being contrasequential and of maximal dispersion character.

Theorem 3.10. *Any $PLM(X)$-invariant nontrivial topology on X is either a T_1 filterbase topology, a connected enrichment of $\mathscr{U}$ (a possibility just in case X is the real field), or an ultra-Hausdorff enrichment of $\mathscr{U}$.*

Proof. Suppose $\mathscr{T}$ is a $PLI(X)$-invariant nontrivial topology. Because $PLM(X)$ is 2-transitive, we may invoke Theorem 1.7 directly to infer that $\mathscr{T}$ is: either T_1 filterbase or T_1 nonfilterbase; if T_1 nonfilterbase, then either T_2 connected or T_2 non-connected; if T_2 nonconnected, then ultra-Hausdorff. If $\mathscr{T}$ is Hausdorff, then it is an enrichment of $\mathscr{U}$ by Corollary 3.5. The fact that $\mathscr{T}$ can be a connected enrichment of $\mathscr{U}$ just in case X is the real field follows from Theorem 3.2. □

Define an $H(X)$-invariant topology to be *sharply* $H(X)$-invariant if its homeomorphism group is precisely $H(X)$. Sharply $H(X)$-invariant topologies are nonobligatory since $H(X) \neq \mathrm{Sym}(X)$ (Proposition 1.2). They are therefore T_1 (Theorem 3.10), and of maximal dispersion character (Proposition 3.7). By Theorem 3.10, such topologies are either filterbase, connected Hausdorff, or ultra-Hausdorff (enriching $\mathscr{U}$ in the last two cases). Here the situation splits. We do not know anything further in general when X is incomplete, except that the second case is vacuous. When $X = \mathbf{R}$, we have the following result from [3].

Lemma 3.11. (Theorem 0.7 and Proposition 4.2 in [3]). *An H-enrichment $\mathscr{T}$ of $\mathbf{R}$ is sharpy $H(\mathbf{R})$-invariant if and only if it is connected, if and only if the $\mathscr{T}$-connected subsets of $\mathbf{R}$ are precisely the intervals.*

From Theorem 3.10 and Lemma 3.11, we immediately infer the following.

Theorem 3.12. *Any sharply $H(\mathbf{R})$-invariant topology is either a T_1 filterbase topology or a connected H-enrichment.*

The question naturally arises as to what filterbase topologies can be sharply $H(X)$-invariant. To begin to answer this, let $\mathscr{T}$ be a topology on X, and let $DO(\mathscr{T}) := \{\emptyset\} \cup \{U \in \mathscr{T}: U \text{ is } \mathscr{T}\text{-dense}\}$. Then $DO(\mathscr{T})$ is a filterbase topology, which is T_1 just in case $\mathscr{T}$ is T_1 and perfect. In general $H(\mathscr{T}) \leq H(DO(\mathscr{T}))$; the reverse inequality need not hold. (An easy example: Let X be infinite, $a \in X$: $\mathscr{T} = \{\emptyset, X, \{a\}, X \backslash \{a\}\}$.) We do not know whether equality holds for the usual topology in an arbitrary ordered field, but it does hold when the usual topology is metrizable (e.g., in the real and rational cases). In a slightly more general setting, we have the following.

Lemma 3.13. *Let $\mathscr{T}$ be a perfect metrizable topology on a set X. Then $H(\mathscr{T}) = H(DO(\mathscr{T}))$.*

Proof. Suppose $g \in H(DO(\mathscr{T})) \backslash H(\mathscr{T})$. Then we may assume that g is not $\mathscr{T}$-continuous at some $c \in X$. This says that there is a sequence (c_n) of distinct points of X such that (c_n) converges to c, but $(g(c_n))$ fails to converge to $g(c)$. We can arrange for c to be distinct from all the points c_n, so that the set $C = \{c\} \cup \{c_1, c_2, \ldots\}$ has c as its only limit point. Then C, as well as each $C \backslash \{c_n\}$, is closed nowhere dense, but $C \backslash \{c\}$ is nowhere dense without being closed. (Nonempty interiors would give rise to isolated points.) Since $g \in H(DO(\mathscr{T}))$, the same can be said, respectively, for $g(C)$, each $g(C) \backslash \{g(c_n)\}$, and $g(C) \backslash \{g(c)\}$. Thus $g(c)$ is the only limit point of $g(C)$. This implies that the sequence $(g(c_n))$, which does not converge to $g(c)$, has the property that any convergent subsequence must converge to $g(c)$; moreover there is a neighborhood of $g(c)$ that fails to contain infinitely many points of $g(C) \backslash \{g(c)\}$. From this we infer that there is a subsequence $(g(c_{n_i}))$ which itself has no convergent subsequence; hence the set $\{g(c_{n_1}), g(c_{n_2}), \ldots\}$ is closed nowhere dense. But then so is the set $\{c_{n_1}, c_{n_2}, \ldots\}$, contradicting the fact that (c_n) converge to c. □

The following result is then an immediate consequence.

Theorem 3.14. *Let X be an ordered field whose usual topology is metrizable (e.g., if X is archimedean, or, more generally, if X has a countable order-dense subset). Then there exists a sharply $H(X)$-invariant filterbase topology, namely $DO(\mathscr{U})$.* □

We next look at the issue of detecting T_n-completeness in certain subgroups of $H(X)$, in analogy with Theorem 2.14. First we mention an analogue of Theorem 2.12, due to Truss [27].

Theorem 3.15 (Theorems 2.12 and 3.5 in [27]). *Both $I(\mathbf{Q})$ and $H(\mathbf{Q})$ have the strong small index property.*

Remark 3.16. We note in passing that the proof Truss gives for establishing the strong small index property in $H(\mathbf{Q})$ relies on the much earlier result of Anderson [1], saying that $H(\mathbf{Q})$ is a simple group.

While the strong small index property is of great interest in itself, our results actually pertain to subgroups that stabilize a finite set (e.g., finitely restricted subgroups), rather than subgroups of small index. One analogue of Theorem 2.14 is the following.

Theorem 3.17. *Let $\mathscr{G}$ be either $I(X)$ or $M(X)$, $n \geq 2$, with $\mathscr{H} \leq \mathscr{G}$ a subgroup that stabilizes a finite set. The following are equivalent:*

(a) *$\mathscr{H} = \mathscr{G}$.*

(b) *$S(\mathscr{H}) = \mathscr{U}$.*

(c) *$\mathscr{H}$ is support-T_n.*

(d) *$\mathscr{H}$ is T_n-complete.*

Proof. The implications (a) only if (b) only if (c) only if (d) follow immediately from Proposition 3.3, properties of the usual topology, and Proposition 2.4, respectively.

((d) only if (a)) Let $\mathscr{H} \leq \mathscr{G}$ stabilize a finite set, say $A \subseteq X$ is finite and $\mathscr{H} \leq \mathscr{G}_A$. In the case $\mathscr{G} = I(X)$, it is clear that $\mathscr{H} \leq \mathscr{G}_{(A)}$. If (a) does not hold, then $A \neq \emptyset$; hence, $\mathscr{H}$ has a fixed point and preserves a T_n topology with more than one nonisolated point (i.e., $\mathscr{U}$). By Proposition 2.9(ii), $\mathscr{H}$ is not T_n-complete.

So assume now that $\mathscr{G} = M(X)$, and that (a) fails. Then A is nonempty. If $\mathscr{H}$ has a fixed point, we are done; so we may assume otherwise (hence $|A| \geq 2$). Let $\mathscr{H}|A := \{h|A: h \in \mathscr{H}\}$ be the restriction of $\mathscr{H}$ to A. Then $\mathscr{H}|A$ has two elements. If $s \in \mathrm{Sym}(X)$ is the permutation that reverses the order of the elements of A and fixes everything else, then $s|A$ is the nonidentity element of $\mathscr{H}|A$. Let $\mathscr{T} = s(\mathscr{U})$. Then $s: \langle X, \mathscr{U}\rangle \to \langle X, \mathscr{T}\rangle$ is a homeomorphism, hence $\mathscr{T}$ is a normal topology. It remains to show that $\mathscr{T}$ is $\mathscr{H}$-invariant and that $\mathscr{T} \cap \mathscr{U}$ is non-Hausdorff. Let $h \in \mathscr{H}$. Then $h|A \in \{s|A, id_A\}$ and $s|(X\backslash A) = id_{X\backslash A}$, hence $hs = sh$. Thus $h = shs^{-1} \in H(\mathscr{T})$, so $\mathscr{T}$ is indeed $\mathscr{H}$-invariant. Finally let a and b be, respectively, the least and greatest elements of A in the field ordering on X, and let U and V be $(\mathscr{T} \cap \mathscr{U})$-neighborhoods of a and b, respectively. By hypothesis, $a \neq b$, and s interchanges a and b. Since U is a $\mathscr{U}$-neighborhood of a, there is a bounded open interval I containing a and missing $A\backslash\{a\}$, such that $I \subseteq U$. Since V is a $\mathscr{T}$-neighborhood of b, there is a bounded open interval J containing a and missing $A\backslash\{a\}$, such that $s(J) \subseteq V$. But $s(J) = \{b\} \cup (J\backslash\{a\})$. Thus $\emptyset \neq I \cap s(J) \subseteq U \cap V$, hence $\mathscr{T} \cap \mathscr{U}$ is non-Hausdorff. □

At present we do not know of a complete analogue of Theorems 2.14 and 3.17 involving $H(X)$ (different from $M(X)$ exactly when X is not the real field, by Theorem 3.2). As a partial analogue, we have the following.

Theorem 3.18. *Let $\mathscr{G}$ be $H(X)$, $H(X) \neq M(X)$, $n \geq 2$, with $\mathscr{H} \leq \mathscr{G}$ finitely restricted.*

(i) *If $\mathscr{H}$ lies in the centralizer of a nonidentity permutation on X, then there is a normal $\mathscr{H}$-invariant topology whose intersection with $\mathscr{U}$ is non-Hausdorff; hence, $\mathscr{H}$ is not T_n-complete.*

(ii) *If $\mathscr{H}$ is support-T_1, then $S(\mathscr{H}) = \mathscr{U}$; hence, $\mathscr{H}$ is T_n-complete.*

Proof. Since $\mathscr{H}$ is finitely restricted in $\mathscr{G} = H(X)$, we find a finite $A \subseteq X$ such that $\mathscr{G}_{(A)} \leq \mathscr{H} \leq \mathscr{G}_A$. As in the proof of Theorem 3.17, $\mathscr{H}|A \leq \mathrm{Sym}(A)$ is the restriction of $\mathscr{H}$ to A. Observe that $\mathscr{H}|A$ "determines" $\mathscr{H}$ in the sense that if $\mathscr{G}_{(A)} \leq \mathscr{H}' \leq \mathscr{G}_A$ and $\mathscr{H}|A = \mathscr{H}'|A$, then $\mathscr{H} = \mathscr{H}'$. (Indeed, if $g \in \mathscr{H}$, then $g|A = h|A$ for some $h \in \mathscr{H}'$, so $(gh^{-1})|A = id_A$. Thus $gh^{-1} \in \mathscr{H}'$, hence $g \in \mathscr{H}'$.) This tells us that if $g \in \mathscr{G}_A$ and $g|A \in \mathscr{H}|A$, then $g \in \mathscr{H}$.

(i) Let $s \in \mathrm{Sym}(X)$ be a nonidentity permutation that commutes with every element of $\mathscr{H}$. Then we claim $\mathrm{supp}(s) \subseteq A$. For let $b \in \mathrm{supp}(s)\backslash A$. If $s(b) \in A$, we can find $g \in \mathscr{G}_{(A)}$ that moves b; if $s(b) \notin A$, we can find $g \in \mathscr{G}_{(A)}$ that fixes b and moves $s(b)$. In either case, gs and sg disagree at b. This contradiction proves the claim.

As in the proof of Theorem 3.17, let $\mathscr{T} = s(\mathscr{U})$. Then we know $s: \langle X, \mathscr{U}\rangle \to \langle X, \mathscr{T}\rangle$ is a homeomorphism, so $\mathscr{T}$ is a normal topology. It remains to show that $\mathscr{T}$ is

$\mathscr{H}$-invariant and that $\mathscr{T} \cap \mathscr{U}$ is non-Hausdorff. Let $h \in \mathscr{H}$. Then $hs = sh$ by hypothesis, hence $\mathscr{H}$-invariance is assured. To see that $\mathscr{T} \cap \mathscr{U}$ is non-Hausdorff, we recall that $\mathrm{supp}(s) \subseteq A$. If $a \in \mathrm{supp}(s)$ and $b = s(a)$, then a and b are distinct points of A, and we may argue exactly as in the proof of Theorem 3.17.

(ii) Now assume $\mathscr{H}$ is support-T_1, and let $I \subseteq X$ be a bounded open interval, with $x \in I$. It suffices to find $g_1, \dots, g_k \in \mathscr{H}$ such that $x \in \mathrm{supp}(g_1) \cap \cdots \cap \mathrm{supp}(g_k) \subseteq I$. For then we have $\mathscr{U} \subseteq S(\mathscr{H})$; hence, $\mathscr{U} = S(\mathscr{H})$ by Proposition 2.1 (since $\mathscr{U}$ is Hausdorff and $\mathscr{H}$-invariant). If our finite set A is empty, Proposition 3.3 applies. If $|A| \in \{1, 2\}$, then $\mathscr{H}$ is not support-T_0; so we assume $|A| \geq 3$.

If $x \notin A$, then pick a bounded open interval J missing A such that $x \in J \subseteq I$. By Proposition 3.3, there is some $g \in \mathscr{G}$ such that $\mathrm{supp}(g) = J$. Since $g \in \mathscr{G}_{(A)}$, g is also in $\mathscr{H}$.

If $x \in A$, we use the assumption that $\mathscr{G} \neq M(X)$. By Theorem 3.2, $\mathscr{U}$ has a clopen basis consisting of bounded convex sets. So pick a $\mathscr{U}$-clopen convex set J containing x such that: (1) $J \subseteq I$; (2) J is "symmetric about x" (i.e., $u \in J$ if and only if $2x - u \in J$ for all $u \in X$); and (3) the translates $J_a := \{u + a - x\colon u \in J\}$ (also $\mathscr{U}$-clopen and convex) are pairwise disjoint for $a \in A$. For any $s \in \mathscr{H} \mid A$ containing x in its support, we define $g_s\colon X \to X$ to be the function taking $u \in J_a$ to $u + s(a) - a$, for $a \in \mathrm{supp}(s)$, and fixing u otherwise. Then clearly $g_s \in \mathscr{G}_A$ extends s; therefore, $g_s \in \mathscr{H}$. Moreover, $\mathrm{supp}(g_s) = \bigcup \{J_a\colon a \in \mathrm{supp}(s)\}$.

By assumption, $|A| \geq 3$. In addition $\mathscr{H} \mid A$ is support-T_1, hence support-discrete, so there exist $s_1, \dots, s_k \in \mathscr{H} \mid A$ such that $\{x\} = \mathrm{supp}(s_1) \cap \cdots \cap \mathrm{supp}(s_k)$. Thus $J = \mathrm{supp}(g_{s_1}) \cap \cdots \cap \mathrm{supp}(g_{s_k}) \subseteq I$, as desired. □

An easy corollary of the proof of Theorem 3.18 is the following.

Corollary 3.19. *Let $A \subseteq X$ be finite, $H(X) \neq M(X)$, $n \geq 2$. Then $S(H(X)_A) = \mathscr{U}$ if and only if $H(X)_A$ is T_n-complete, if and only if $|A| \notin \{1, 2\}$.*

Proof. Let $\mathscr{H} = H(X)_A$. If $S(\mathscr{H}) = \mathscr{U}$, then $\mathscr{H}$ is T_n-complete, by Proposition 2.4. If $|A| \in \{1, 2\}$, then $\mathscr{H} \mid A$ is abelian; so we may apply the proof of Theorem 3.18(i). If A is empty, then $\mathscr{H} = H(X)$, and we may invoke Proposition 3.3. If $|A| \geq 3$, then $\mathscr{H} \mid A = \mathrm{Sym}(A)$ is support-T_1, and we may use the proof of Theorem 3.18(ii). □

4. Group actions on the rational line

The usual topological space of rational numbers is characterized, thanks to a celebrated theorem of Sierpiński [23], by the conjunction of properties: countable, second countable (i.e., having a countable open basis), regular, and perfect. In the parlance of model theory (see, e.g., [8]), this is a weak sort of ω-categoricity, when one chooses one's language appropriately, and has proved itself very useful in recent times (see, e.g., [2, 20]). In a more familiar context, the linearly ordered set of rational numbers is characterized, by Hausdorff, who pioneered the so-called "back-and-forth" method, by saying that the ordering is dense without endpoints. This translates into model-theoretic terms as the statement that the first-order theory of dense linear orderings without endpoints is ω-categorical, and has seen much application recently. In particular, the following result from [6] appeals strongly to this result.

Lemma 4.1 (Theorem 2.12 in [6]). *The usual topology and the discrete topology are the only regular H-enrichments of* $\mathbf{Q}$.

When one combines Lemma 4.1 with Corollary 3.5 and Proposition 2.2(i), the following characterization obtains.

Theorem 4.2. *The only regular* $H(\mathbf{Q})$*-invariant topologies are the usual topology and the discrete topology; the only regular sharply* $H(\mathbf{Q})$*-invariant topology is the usual topology.*

The next result can best be stated if we introduce a new notion. If π is a topological property, define a topology $\mathscr{T}$ on a set X to be *symmetry-maximal* (π) if whenever $\mathscr{S}$ is any topology on X such that $\mathscr{S}$ satisfies π and $H(\mathscr{T})$ is a proper subgroup of $H(\mathscr{S})$, then $H(\mathscr{S}) = \mathrm{Sym}(X)$.

Theorem 4.3. *The usual topology on the rational line is symmetry-maximal (nonfilterbase), but not symmetry-maximal* (T_1 *filterbase*). *(In fact, if* $\langle X, \mathscr{T}\rangle$ *is any perfect* T_1 *space possessing a nondiscrete closed nowhere dense subset of cardinality* $|X|$*, then* $\mathscr{T}$ *is not symmetry-maximal* (T_1 *filterbase*).)

Proof. Let $\mathscr{T}$ be an $H(\mathbf{Q})$-invariant nonfilterbase topology. By Theorem 3.10, $\mathscr{T}$ is an H-enrichment. Assume $\mathscr{T}$ is a proper H-enrichment. If $H(\mathscr{T}) = \mathrm{Sym}(\mathbf{Q})$, we are done; otherwise $\mathscr{T}$ is nondiscrete, hence nonregular by Lemma 4.1. Suppose $A \subseteq \mathbf{Q}$ is not $\mathscr{U}$-clopen, and define $\mathscr{S}$ to be the smallest H-enrichment containing both A and $\mathbf{Q}\backslash A$. Then an open basis for $\mathscr{S}$ consists of sets of the form $U \cap B$, where U is $\mathscr{U}$-clopen, and B is an intersection of finitely many $H(\mathscr{U})$-homeomorphs of A and of $\mathbf{Q}\backslash A$. Since B is $\mathscr{S}$-clopen, $\mathscr{S}$ is zero-dimensional. Since $\mathscr{S}$ is a proper H-enrichment, we infer that $\mathscr{S}$ is the discrete topology, again by Lemma 4.1.

Thus $\mathscr{T}$ and $\mathscr{U}$ share the same clopen sets. So if $g \in H(\mathscr{T})$, then g preserves $\mathscr{T}$-clopen sets; hence g preserves $\mathscr{U}$-clopen sets. Since $\mathscr{U}$ is zero-dimensional, we conclude that $g \in H(\mathscr{U})$. Thus $\mathscr{U}$ is symmetry-maximal (nonfilterbase).

To show $\mathscr{U}$ is not symmetry-maximal (T_1 filterbase), we prove the decidedly stronger assertion in parentheses. Let $\langle X, \mathscr{T}\rangle$ satisfy the hypotheses, with $C \subseteq X$ a nondiscrete closed nowhere dense set of cardinality $|X|$. Let $\mathscr{F}$ be the T_1 filterbase topology obtained by adding in all supersets of nonempty members of $DO(\mathscr{T})$. Since C if $\mathscr{F}$-closed and of cardinality $|X|$, we know $\mathscr{F}$ is not an obligatory topology; hence

$H(\mathscr{F}) \neq \mathrm{Sym}(X)$. Clearly $H(\mathscr{T}) \leq H(\mathscr{F})$, so it remains to show the inequality is proper. Indeed, pick $a \in X \setminus C$ (possible because C is nowhere dense), pick $c \in C$ a limit point of C (possible because C is nondiscrete and closed), and let $t \in \mathrm{Sym}(X)$ transpose a and c. If $A \subseteq X$ is nowhere dense, then so is $t(A)$, since singletons are closed but not open, hence nowhere dense. Thus $t \in H(\mathscr{F})$. On the other hand, C is closed but $t(C) = (C \cup \{a\}) \setminus \{c\}$ is not. Thus $t \notin H(\mathscr{T})$. □

Remark 4.4. With regard to Lemma 4.1, there are plenty of H-enrichments of **Q** (see [4]). With regard to Theorem 4.3, interesting (nonobligatory) symmetry-maximal topologies seem to be rare; one source of examples comes from the result of Richman [22] that the stabilizer of an ultrafilter on an infinite set is a maximal proper subgroup of the full symmetric group. This clearly implies that if X is an infinite set, with $\mathscr{F}$ an ultrafilter on X, then the (filterbase) topology $\mathscr{F} \cup \{\emptyset\}$ is symmetry-maximal (i.e., symmetry-maximal (π), where π is the class of all spaces). We also have the compact Hausdorff topologies $\mathscr{D}_a$ (see Example 2.3). They are always symmetry-maximal since $\mathrm{Sym}(X)$ is primitive and $H(\mathscr{D}_a) = \mathrm{Sym}(X)_a$. With regard to spaces satisfying the hypothesis of the parenthetical assertion of Theorem 4.3, one may take countable perfect Hausdorff spaces where some point has a countable neighborhood basis (nowhere dense copies of the ordinal space $\omega + 1$ exist); also one may take perfect completely metrizable spaces of cardinality **c** (nowhere dense Cantor sets exist).

A recurrent theme in the present paper is the issue of what kinds of topologies are invariant under a given group action. Trivial actions and topologically primitive actions (see Remark 1.3) sit at opposite extremes in this connection, and results like Proposition 2.10 and Theorem 3.10 give a small indication of what can happen in the middle. A natural question one can ask of a group action is whether it preserves, say, a perfect Hausdorff topology. There are many results (see [7]) that place certain transitivity/homogeneity assumptions on the group and imply the existence of invariant relational structures on the set. Often these structures give rise to interesting invariant topologies. For example, a result of McDermott ((3.11) in [7]) says that a 3-homogeneous but not 2-transitive group on a countable set preserves a linear ordering that is isomorphic to $\langle \mathbf{Q}, < \rangle$. The main part of the proof, which is not difficult, actually applies to arbitrary infinite sets, and asserts the existence of an invariant linear ordering that is dense without endpoints. Since such order topologies are perfect and Hausdorff (indeed hereditarily normal), we have an affirmative answer to our question in the case of 3-homogeneous but not 2-transitive group actions.

By Proposition 2.1, $\mathscr{G}$ preserves a perfect Hausdorff topology only if $\mathscr{G}$ is support-perfect; i.e., finite intersections of supports are either empty or infinite. Mekler [19] defines $\mathscr{G}$ to satisfy the *mimicking property* if whenever $x \in X$ and $g_1, \ldots, g_m \in \mathscr{G}$, there are infinitely many $y \in X$ such that for each $1 \leq i, j \leq m$, if $g_i(y) = g_j(y)$, then $g_i(x) = g_j(x)$. It is quite straightforward to show that satisfying the mimicking property is equivalent to being support-perfect (a fact first noted by Neumann [25]), and a natural question is: when does being support-perfect imply preserving a perfect Hausdorff topology. We do not know the answer in general, but Mekler has given us a good partial solution. The main result of [19] (Theorem 1.5) is that a countable group of countable degree embeds, as a group action, in $H(\mathbf{Q})$, if and only if that group action satisfies the mimicking property. Thanks to Sierpiński's theorem [23], Mekler's theorem translates into the statement that every countable support-perfect group of countable degree preserves a perfect metrizable topology.

Results that show when group actions preserve perfect T_n topologies can be useful in detecting the lack of T_n-completeness because of the following.

Theorem 4.5. *Let the group $\mathscr{G}$ preserve a perfect T_n topology on X, $n \geq 2$. If $\mathscr{H} \leq \mathscr{G}_A$ for some nonempty finite $A \subseteq X$, and if $\mathscr{H} \mid A$ is abelian, then $\mathscr{H}$ is not T_n-complete.*

Proof. Let $\mathscr{T}$ be a perfect T_n $\mathscr{G}$-invariant topology, and let A be a nonempty finite subset of X such that $\mathscr{H} \leq \mathscr{G}_A$ and $\mathscr{H} \mid A$ is abelian. If $\mathscr{H}$ has a fixed point, we can appeal to Proposition 2.9(ii). So assume $\mathscr{H}$ has no fixed points, hence $|A| \geq 2$. Let $s \in \mathrm{Sym}(X)$ satisfy $\emptyset \neq \mathrm{supp}(s) \subseteq A$, and $s \mid A \in \mathscr{H} \mid A$. Then, arguing as in Theorems 3.17 and 3.18(i), we show that $s(\mathscr{T})$ is a perfect T_n $\mathscr{H}$-invariant topology such that $\mathscr{T} \cap s(\mathscr{T})$ is non-Hausdorff. (Invariance comes from the fact that $\mathscr{H} \mid A$ is abelian; non-Hausdorffness comes from the facts that $2 \leq |A| < \omega$, and $\mathscr{T}$ is a perfect T_1 topology.) □

5. Group actions on the real line

The ordered field $\mathbf{R}$ of real numbers is unique, by Theorem 3.2, in having a complete ordering, a connected usual topology, and a homeomorphism group that does not extend past the group of monotonic bijections. This makes the theory of $H(\mathbf{R})$-invariant topologies dramatically different from that of any other field-ordered set.

Let $\mathscr{T}$ be a topology on $\mathbf{R}$. We say $\mathscr{T}$ is a *Darboux* topology if the $\mathscr{T}$-connected subsets of $\mathbf{R}$ are precisely the intervals of $\mathbf{R}$.

Remark 5.1. The density topology (see Example 3.6) is a Darboux topology, as are all sharply H-invariant Hausdorff topologies (see Theorem 3.12). In the theory of real functions, $f: \mathbf{R} \to \mathbf{R}$ *satisfies the Darboux property* if it satisfies the conclusion of the intermediate value theorem (see [16]). This property does not imply usual continuity; the function that fixes 0 and takes $x \neq 0$ to $\sin(1/x)$ is a well-known counterexample. Also, if $\mathscr{S}$ and $\mathscr{T}$ are any two Darboux topologies, then any $(\mathscr{S}, \mathscr{T})$-continuous function (i.e., one that pulls $\mathscr{T}$-open sets back to $\mathscr{S}$-open sets) satisfies the Darboux property.

Lemma 5.2. *Suppose X is a set, and $\mathscr{S}$ and $\mathscr{T}$ are two topologies on X such that the $\mathscr{S}$-connected subsets of X are precisely the $\mathscr{T}$-connected subsets. If $\mathscr{S}$ is regular, then every $\mathscr{S}$-connected $\mathscr{S}$-closed set is also $\mathscr{T}$-closed.*

Proof. Let $C \subseteq X$ be $\mathcal{S}$-connected and $\mathcal{S}$-closed, and let D be the $\mathcal{T}$-closure of C. If $B \subseteq X$ lies between C and D, then B must be $\mathcal{T}$-connected. Suppose $x \notin C$. By regularity of $\mathcal{S}$, $C \cup \{x\}$ is not $\mathcal{S}$-connected, and hence not $\mathcal{T}$-connected. This implies that $C = D$, so C is $\mathcal{T}$-closed. □

Theorem 5.3. *Let $\mathcal{T}$ be any topology on* **R**. *Each of the following statements implies the next:*

(a) *$\mathcal{T}$ is a nonfilterbase topology and $PLI(\mathbf{R}) \leq H(\mathcal{T}) \leq H(\mathbf{R})$.*
(b) *$\mathcal{U} \subseteq \mathcal{T}$ and $\mathcal{T}$ is connected.*
(c) *$\mathcal{T}$ is a Darboux topology.*
(d) *$\mathcal{U} \subseteq \mathcal{T}$ and $H(\mathcal{T}) \leq H(\mathbf{R})$.*

Proof. ((a) only if (b)) Suppose $\mathcal{T}$ satisfies (a). By Theorem 1.7(iv) and Corollary 3.5, $\mathcal{U} \subseteq \mathcal{T}$, so it remains to show that $\mathcal{T}$ is connected. Suppose otherwise, and U is a proper nonempty $\mathcal{T}$-clopen set. Because $LI(\mathbf{R}) \leq H(\mathcal{T})$, Proposition 3.7 tells us that $|U| = |\mathbf{R} \setminus U|$; moreover, we may assume $0 \notin U$. Let $V = U \cup \{-x : x \in U\}$. Then V is $\mathcal{T}$-clopen, $0 \notin V$, and $|V| = |\mathbf{R} \setminus V|$, so the map that fixes x in V and takes x to $-x$ otherwise is a $\mathcal{T}$-homeomorphism that is not in $H(\mathbf{R})$. Thus we conclude that $\mathcal{T}$ is connected.

((b) only if (c)) Assume $\mathcal{T}$ is a connected enrichment of $\mathcal{U}$, with $[a,b]$ any closed bounded interval of **R**. Then $[a,b]$ is $\mathcal{T}$-closed. If $\{U, V\}$ is a $\mathcal{T}$-disconnection of $[a,b]$, then both U and V are $\mathcal{T}$-closed sets. Suppose $a \in U$ and $b \in V$. Then $\{(-\infty, a] \cup U, V \cup [b, \infty)\}$ is a $\mathcal{T}$-disconnection of **R**, a contradiction. If both a and b are in, say, U, then $\{(-\infty, a] \cup [b, \infty) \cup U, V\}$ is a $\mathcal{T}$-disconnection of **R**, another contradiction. Thus all closed bounded intervals of **R** are $\mathcal{T}$-connected. Since every interval of **R** is a chain union of closed bounded intervals, we conclude that $\mathcal{T}$ is a Darboux topology.

((c) only if (d)) Assume $\mathcal{T}$ is a Darboux topology. By Lemma 5.2, every closed interval is a $\mathcal{T}$-closed set. Consequently, all the usual basic open sets are $\mathcal{T}$-open; hence $\mathcal{U} \subseteq \mathcal{T}$. Then $H(\mathcal{T}) \leq H(\mathbf{R})$ by the intermediate value theorem. □

Remark 5.4. The density topology and analogous topologies designed around Baire category instead of Lebesgue measure (see [9]) are intriguing examples of Darboux topologies that share many of the features of connected H-enrichments without being H-enrichments themselves. Of these features, the most notable is contrasequentiality, which implies, among other things, failing to be metrizable or locally compact. (The density topology is contrasequential because countable sets are density-closed; proper H-enrichments are contrasequential even when countable sets are not closed.) Significantly, most of the density-type topologies are regular; indeed, completely regular, even realcompact. (See [12, Theorem 3.6]; realcompactness becomes an issue when studying rings of continuous functions (see also [5, 10, 13, 28]).) We have been able to construct vast numbers of proper nondiscrete completely regular H-enrichments (see [6, 3, 4]), but at the expense of connectedness. At every turn, the price of regularity has been zero-dimensionality; so the question of the consistency of connectedness and regularity in H-enrichments of $\mathbf{R}$ has gained in stature over time. We would very much like to see an analogue of Theorem 4.2 for $\mathbf{R}$. We know that the only metrizable (locally compact Hausdorff) sharply $H(\mathbf{R})$-invariant topology is the usual one, and wonder whether regularity is enough to characterize $\mathscr{U}$ in these terms.

An easy consequence of Theorem 5.3 and Remark 5.4 is the following analogue of Theorem 4.3.

Theorem 5.5. *The usual topology on the real line is symmetry-maximal (connected nonfilterbase), but not symmetry-maximal (completely regular) (or even symmetry-maximal (zero-dimensional)).*

Acknowledgements

I would like to take the opportunity to thank Dugald Macpherson and Alex Ryba for many stimulating conversations regarding permutation groups. To a large extent, I credit my understanding and appreciation of the subject to them. (Any misunderstandings I can safely credit to myself.)

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