ARTICLE IN PRESS
TOPOL:107783

Topology and its Applications ••• (••••) ••••••

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Convexity in topological betweenness structures

Daron Anderson [a], Paul Bankston [b,*], Aisling McCluskey [c]

[a] *School of Computer Science, Trinity College Dublin, Dublin, Ireland*
[b] *Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI 53201-1881, USA*
[c] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway, Ireland*

A R T I C L E   I N F O

Article history:
Received 22 July 2020
Received in revised form 22 November 2020
Accepted 30 November 2020
Available online xxxx

To Professor Mioduszewski

MSC:
primary 08A02
secondary 03B10, 06A06, 52A01, 52A05, 52A07, 54D05, 54E35, 54F15, 54G15, 54G20

Keywords:
Topological betweenness structures
Convexity
Fréchet systems
Roads
Intervals
Metric spaces
Normed vector spaces
Continua

A B S T R A C T

A **betweenness structure** on a set X is a ternary relation $[\cdot,\cdot,\cdot] \subseteq X^3$ that captures a rudimentary notion of one point of X lying *between* two others. The **interval** $[a,b]$ is the set of all points lying between a and b, and a subset C of X is **convex** if $[a,b] \subseteq C$ whenever $a,b \in C$. The **span** of a set A is the union of all intervals $[a,b]$, where $a,b \in A$; by iterating the span operator countably many times, we obtain the **convex hull** of A. The betweenness structure is **topological** if X carries a topology that satisfies certain compatibility conditions with respect to betweenness; in particular, intervals are closed subsets. We are guided by questions involving how the span and convex hull operators interact with the topological closure and interior operators, especially in the domains of metric spaces and of continua. With a metric space $\langle X,\varrho\rangle$, $[a,c,b]$ holds exactly when $\varrho(a,b) = \varrho(a,c) + \varrho(c,b)$; and one result about this betweenness structure is that the span of any compact subset is both closed and bounded. With a continuum X, $[a,c,b]$ holds exactly when c belongs to every subcontinuum of X that contains a and b; and one result about this betweenness structure is that when the continuum is either aposyndetic or hereditarily unicoherent, the closure of a convex subset is always convex.

1. Betweenness structures

A **betweenness structure**[1] is a pair $\langle X,[\cdot,\cdot,\cdot]\rangle$, where X is a set and $[\cdot,\cdot,\cdot] \subseteq X^3$ is a ternary relation on X, satisfying the following first-order axioms.

* Corresponding author.
E-mail addresses: Andersd3@tcd.ie (D. Anderson), paul.bankston@marquette.edu (P. Bankston), aisling.mccluskey@nuigalway.ie (A. McCluskey).

[1] Betweenness structures are also referred to as *basic ternary structures* in [3].

https://doi.org/10.1016/j.topol.2021.107783
0166-8641/

(B1) Inclusivity: $\forall xy\ ([x,y,y] \wedge [x,x,y])$
(B2) Symmetry: $\forall xzy\ ([x,z,y] \to [y,z,x])$
(B3) Uniqueness: $\forall xz\ ([x,z,x] \to x = z)$

In keeping with the betweenness theme, we use interval notation, defining $[a,b]$ to be $\{c \in X : [a,c,b]\}$. The points a,b are called **bracket points** of the interval.[2] In interval terms, inclusivity says $\{a,b\} \subseteq [a,b]$ always holds; symmetry says $[a,b] = [b,a]$; and uniqueness says $[a,a] = \{a\}$. An interval—or any set—is **nondegenerate** if it contains more than one point.

If $\langle X, [\cdot,\cdot,\cdot]\rangle$ is a betweenness structure, a subset $C \subseteq X$ is **interval-convex** if whenever $a,b \in C$, it follows that $[a,b] \subseteq C$. Since the default notion of convexity here is interval-convexity, we will drop the modifier and simply call a subset *convex* if it is interval-convex. Other kinds of convexity—e.g., linear convexity in vector spaces—will be clarified by context.

Clearly the family of all convex subsets of a betweenness structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ includes both $\emptyset$ and X, and is closed under arbitrary intersections and nested unions. This makes the family into a *convexity*, in the sense of [17]. Furthermore, by the uniqueness axiom (B3), all singletons are convex subsets.

1.1 Examples.

(i) Let X be a vector space (always assumed in this paper to have real scalars). The classical **linear interpretation of betweenness** is given by $[a,c,b]$ just in case $c = (1-t)a + tb$ for some $0 \le t \le 1$. The associated notions of *interval* and *convexity* are prefixed with the modifier *linear* in the sequel; we use the symbol $[\![a,b]\!]$ exclusively to denote the linear interval bracketed by a and b. What we call *linear convexity* is what is known simply as *convexity* in the functional analysis literature.

(ii) Each metric space $\langle X, \varrho\rangle$ admits an interpretation of betweenness, where $[a,c,b]$ means $\varrho(a,b) = \varrho(a,c) + \varrho(c,b)$. We call this M**-betweenness** after K. Menger [12], who introduced the notion.[3] Intervals and convexity in this interpretation are prefixed with the letter M; we follow a similar pattern when referring to other kinds of betweenness.

(iii) A combination of (i) and (ii) above has been studied by W. Takahashi [16]: A **convex structure** on a metric space $\langle X, \varrho\rangle$ is a continuous map $W : X^2 \times [0,1] \to X$ satisfying the condition that if $c,a,b \in X$ and $0 \le t \le 1$, then $\varrho(c, W(a,b,t)) \le (1-t)\varrho(c,a) + t\varrho(c,b)$. It is easy to show that $W(a,b,t)$ always lies on the M-interval $[a,b]_\varrho$, $W(a,b,0) = a$, $W(a,b,1) = b$, and $W(a,a,t) = a$ for all $0 \le t \le 1$. One can check that $W(a,b,\cdot) : [0,1] \to X$ is an embedding; so the set $L(a,b) := \{W(a,b,t) : 0 \le t \le 1\}$ is an arc with end points a,b. It is not generally the case that $L(a,b) = L(b,a)$, but defining $[a,c,b]$ to mean that $c \in L(a,b) \cup L(b,a)$ results in an interpretation of betweenness, which we call T**-betweenness**.

(iv) For any partially ordered set $\langle X, \le\rangle$ there is the **order interpretation of betweenness**, where $[a,c,b]$ means that either $a \le c \le b$ or $b \le c \le a$.

(v) For any connected topological space X we can write $[a,c,b]$ to mean that either $c \in \{a,b\}$ or a and b lie in different components of $X \setminus \{c\}$. Equivalently, each connected subset of X that contains $\{a,b\}$ also contains c. This is called C**-betweenness**.

(vi) In the previous example we can also define $[a,c,b]$ to mean that each subset of X that is both connected and closed contains c if it contains $\{a,b\}$. This is called K**-betweenness**, and has been of particular interest in the study of continua (i.e., connected compact Hausdorff spaces) [2–6].

If $[\cdot,\cdot,\cdot]_1 \subseteq [\cdot,\cdot,\cdot]_2$ are two betweenness relations on X, the first is said to **refine** the second. (So $[a,b]_1$ is always contained in $[a,b]_2$. For example, T-betweenness refines M-betweenness in a Takahashi-convex

[2] An interval may have many sets of bracket points.
[3] Menger excluded the bracket points from his intervals, but this is an inessential difference.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

metric space, and C-betweenness refines K-betweenness in a connected topological space.) A doubleton set $\{a, b\} \subseteq X$ (i.e., $a \neq b$) is called a **gap** if $[a, b] = \{a, b\}$. The betweenness relation is called **discrete** if each nondegenerate interval is a gap. The discrete betweenness relation on X is also called *minimal* in [6] because it refines all other betweenness relations on X.

If each nondegenerate interval contains at least three points, the betweenness relation is called **gap-free**. This condition is expressed as the following first-order axiom.

(B4) Gap-Freeness: $\forall xy\, \exists z\, (x \neq y \rightarrow (x \neq z \wedge z \neq y \wedge [x, z, y]))$

1.2 Remark. In the order interpretation of betweenness, being gap-free is synonymous with being a *dense* ordering. In the metric interpretation, this feature is what Menger [12] calls a *convex metric*. Notice that if we have a metric space that admits a Takahashi-convex structure, then the metric is automatically *convex* in Menger's sense. Even though Menger's terminology is time honored (see also [8,14]), we feel that it does not capture what "convexity"—as the subset property of being closed under betweenness—traditionally means.

The **span** of $C \subseteq X$ is defined to be $[C] := \bigcup\{[a, b] : a, b \in C\}$. Owing to inclusivity (B1), every subset is contained in its span, and a set equals its span if and only if it is convex. We say that C **spans** X (or X is **irreducible about** C) if $[C] = X$.

Spans of sets are not necessarily convex: even with the linear interpretation of betweenness in the Euclidean plane, the span of a set of three noncollinear points consists of the three legs of the corresponding triangle, and is not linearly convex. This limitation suggests iterating the span process.

For any subset C of a betweenness structure and $n \in \omega := \{0, 1, 2, \ldots\}$, let $[C]^0 := C$; for each n, let $[C]^{n+1} := [[C]^n]$. Finally let $[C]^\omega := \bigcup\{[C]^n : n \in \omega\}$.

1.3 Proposition. *If C is a subset of a betweenness structure, then $[C]^\omega$ is the smallest convex set containing C.*

Proof. By inclusivity, we have $[C]^0 \subseteq [C]^1 \subseteq \cdots \subseteq [C]^\omega$, so suppose $a, b \in [C]^\omega$. Then there is some $n < \omega$ such that $a, b \in [C]^n$. But then $[a, b] \subseteq [C]^{n+1} \subseteq [C]^\omega$, showing $[C]^\omega$ to be convex. Since the collection of all convex sets is closed under intersections, and X itself is convex, there is a smallest convex set C' containing C. Then $C' \subseteq [C]^\omega$. But clearly, by an easy induction, each $[C]^n$ is contained in C', simply because C' is convex. Hence $[C]^\omega \subseteq C'$ too, and we conclude that $[C]^\omega = C'$. □

So, in the parlance of convexity theory, $[C]^\omega$ is the *convex hull* of C (**hull**, for short). If it so happens that $[C]^{n+1} = [C]^n$ for some $n \in \omega$, then we have $[C]^\omega = [C]^n$, and we call C n**-convex**. (So 0-*convex* means *convex*.) In the context of linear convexity in Euclidean spaces, each n-point set, $n \geq 1$, is $(n-1)$-convex.[4]

We next introduce two axioms that are closely related to each other—with the second being formally stronger than the first—and which play an important role in our study.

(B5) Transitivity: $\forall xwzy\, (([x, z, y] \wedge [x, w, z]) \rightarrow [x, w, y])$
(B6) Convexity: $\forall xuzvy\, (([x, u, y] \wedge [x, v, y] \wedge [u, z, v]) \rightarrow [x, z, y])$

A betweenness structure is called **transitive** (resp., **convex**) if it satisfies axiom (B5) (resp., (B6)).[5] In interval terms, transitivity says that $[a, c] \subseteq [a, b]$ whenever $c \in [a, b]$, while convexity is the stronger condition that $[c, d] \subseteq [a, b]$ for all $c, d \in [a, b]$. Put another way: transitivity says that intervals are *star-shaped* (see [17]) about each of their bracket points; convexity says the intervals themselves are convex.

[4] This is an easy induction using classical ideas (see [17], especially Theorem 4.11 and Proposition 4.14.1.)
[5] Transitive (resp., convex) betweenness structures are also referred to as τ-*basic* (resp., κ-*basic*) in [3].

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

If we fix a point a in a betweenness structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ and rewrite $[a,x,y]$ as $x \leq_a y$, the conjunction of axioms (B1), (B3), and (B5) says that each binary relation $\leq_a$ is a pre-ordering with least element a. (The second clause of (B1) gives us reflexivity, the first clause of (B1) says that a is a minimal element, (B3) then says no other element is minimal, and (B5) ensures usual binary transitivity.) What makes the pre-order into a partial ordering is antisymmetry; and the ternary form of this axiom is the following.

(B7) Antisymmetry: $\forall xyz\ (([x,z,y] \wedge [x,y,z]) \to y = z)$

1.4 Examples.

(i) Linear betweenness in a vector space is clearly convex, as well as antisymmetric.

(ii) It is shown in [12] that M-betweenness is transitive; and, by means of a finite example, that it is not necessarily convex. For a more geometric example of convexity's failure, consider the space $\Theta = C \cup H$, where C is the standard planar unit circle and H is the horizontal segment $[-1,1]\times\{0\}$. The distance between any two points of Θ is given as the shortest length of an arc in Θ containing them. For $a = \langle 0,1\rangle$ and $b = \langle 0,-1\rangle$, we have $[a,b] = C$, but $[C] = \Theta$. Hence we have an interval which is not M-convex.
The M-interpretation of betweenness is easily shown to be antisymmetric. To see this, suppose $[a,c,b]$ and $[a,b,c]$ hold; i.e., that $\varrho(a,c)+\varrho(c,b) = \varrho(a,b)$ and also $\varrho(a,b)+\varrho(b,c) = \varrho(a,c)$. Plug the second equation into the first, obtaining $(\varrho(a,b)+\varrho(b,c))+\varrho(c,b) = \varrho(a,b)$. Then we have $\varrho(b,c) = 0$, and hence $b = c$. (Since the M-interpretation satisfies antisymmetry, and T-betweenness refines M-betweenness, the T-interpretation is antisymmetric too.)

(iii) If X is a metric vector space—i.e., a topological vector space with distinguished topology-inducing metric—then there are two obvious interpretations of betweenness: the linear kind and the metric kind. When the metric arises from a norm, each linear interval is contained within the corresponding M-interval, but there may be a wide discrepancy. For example, suppose $X = \langle \mathbb{R}^n, \|\cdot\|_1\rangle$, Euclidean n-space with $\|a\|_1 := \sum_{i=1}^n |a(i)|$. Then the M-interval $[a,b]$ is the Cartesian product $\prod_{i=1}^n [a(i),b(i)]$ of the individual closed intervals in $\mathbb{R}$, substantially larger than $[\![a,b]\!]$ when $n \geq 2$. The two notions of betweenness coincide when the norm is *strictly convex*.[6] This says that the unit sphere contains no nondegenerate linear intervals, and is equivalent to saying that $\|a+b\| = \|a\|+\|b\|$ implies $a = tb$ for some $t \geq 0$ (see [6, Proposition 4.1]).

(iv) Even normed vector spaces are not necessarily M-convex: in [6, Example 4.6] it is shown that $\langle \mathbb{R}^3, \|\cdot\|_\infty\rangle$ fails in this regard. (Here $\|\cdot\|_\infty$ is the *supremum norm*, defined—in this case—by $\|a\|_\infty := \max\{|a(1)|,|a(2)|,|a(3)|\}$.) What *is* true about the convexity of M-intervals in normed vector spaces is that they are *linearly* convex [6, Proposition 4.3].

2. Related work

A betweenness structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ is **topological** if X carries a topology for which all intervals are closed. This of course implies that all singletons are closed, making the topology T_1. In Section 4 we begin in earnest our study of topological betweenness structures and will then insist that the topologies be regular as well.

Our topological betweenness structures can be compared to the *topological convexity structures* of M. van de Vel [17, Chapter III]. While his notion of *convex set* is primitive, our corresponding notion—which meets his defining criteria—is derived from the primitive notion of *interval*. For us, intervals, and hence spans of finite sets, are required to be closed; with topological convexity structures, all hulls of finite sets

[6] This is also known as *rotund*.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

are closed and there is no separate notion of *span*. Intervals are not necessarily convex, despite being spans of singleton or doubleton sets; so except for singletons, the empty set, and the space itself, none of our convex sets are required to be closed. The present work also has a stronger focus on specific examples and counterexamples; especially we are interested in what topological/metric properties force the topological and convexity-theoretic aspects of a structure to interact more cohesively. While *interval spaces*, something closely akin to betweenness structures, are studied in [17, Chapter I, §4], it is from the geometric and lattice-theoretic perspectives, and not the topological.

As outlined in Example 1.1 (iii), there is a conceptually similar strand of work on notions of *interval* and *convexity* due to W. Takahashi [16]. This approach achieves a significant generalization of the Banach space environment, especially in the study of convex functions and the fixed point theory of nonexpansive mappings.

Another approach to *interval* and *convexity* in metric spaces is via the notion of *geodesic*; see [9], for example. This was carried out to some extent in [6, Section 3].

3. Fréchet systems

In this section we induce betweenness structures on X via special families of subsets of X, called *Fréchet systems*. We characterize the betweenness structures so created in terms of the betweenness axioms already introduced.

If X is a set, denote its power set by $\wp(X)$. Any family $\mathcal{A} \subseteq \wp(X)$ induces a ternary relation $[\cdot,\cdot,\cdot]_{\mathcal{A}}$ as follows: First, given $a, b \in X$, define $\mathcal{A}(a,b) := \{A \in \mathcal{A} : a, b \in A\}$. Next, define $[a,c,b]_{\mathcal{A}}$ to hold just in case $c \in \bigcap \mathcal{A}(a,b)$. When viewed in this way, the members of a family $\mathcal{A}$ are called *roads*. The point c lies between a and b precisely when every road containing a and b "passes through" c.

3.1 Example. The C- and K-betweenness structures on a connected topological space X (see Examples 1.1 (v,vi)) are induced by the families $\mathcal{C}_X$ and $\mathcal{K}_X$ of connected and connected closed subsets, respectively.

Clearly $[\cdot,\cdot,\cdot]_{\mathcal{A}}$ satisfies inclusivity (B1) and symmetry (B2). Convexity (B6) is also easily seen to hold: Indeed, let $c, d \in [a,b]_{\mathcal{A}}$, with $x \in [c,d]_{\mathcal{A}}$. To prove $x \in [a,b]_{\mathcal{A}}$, let $A \in \mathcal{A}(a,b)$ be arbitrary. Then $c, d \in A$, by definition, hence $A \in \mathcal{A}(c,d)$. This gives $x \in A$, and we have $x \in [a,b]_{\mathcal{A}}$, as desired.

Arbitrary families $\mathcal{A} \subseteq \wp(X)$ induce convex ternary relations that fall short of being betweenness relations, in that the uniqueness axiom (B3) is generally violated. To remedy this, we define $\mathcal{F} \subseteq \wp(X)$ to satisfy the **Fréchet condition**−and the pair $\langle X, \mathcal{F}\rangle$ to be a **Fréchet system** (**F-system**, for short)−if, for any two points of X, each is contained in a member of $\mathcal{F}$ that excludes the other.[7]

It is straightforward to see that the ternary relation induced by an F-system satisfies uniqueness; hence it is a convex betweenness structure. Call a ternary structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ a **Fréchet structure** (or **F-structure**) if there is a Fréchet system $\mathcal{F} \subseteq \wp(X)$ such that $[\cdot,\cdot,\cdot] = [\cdot,\cdot,\cdot]_{\mathcal{F}}$.

For a ternary structure $\langle X, [\cdot,\cdot,\cdot]\rangle$, let $\mathcal{I}_X$ be the family $\{[a,b] : a, b \in X\}$ of intervals.

3.2 Proposition. *Suppose $\langle X, [\cdot,\cdot,\cdot]\rangle$ is a betweenness structure. Then $\mathcal{I}_X$ is an F-system whose associated ternary relation is an F-structure refining $[\cdot,\cdot,\cdot]$.*

Proof. By uniqueness (B3), $\mathcal{I} = \mathcal{I}_X$ contains the singletons and is hence Fréchet. Next, note that if $[a,c,b]_{\mathcal{I}}$ holds, then c is in every set in $\mathcal{I}$ that contains both a and b. This includes $[a,b]$ itself, by inclusivity (B1). Hence $[a,b]_{\mathcal{I}} \subseteq [a,b]$. □

[7] This is just M. Fréchet's T_1 axiom when $\mathcal{F}$ is a topology on X. In [3] we introduce *road systems* on X as subset families that include all singletons, as well as X itself. In light of Theorem 3.3, this is unnecessarily restrictive: indeed, given a Fréchet system $\mathcal{F}$, it is easy to see that both $\mathcal{F}$ and the road system $\mathcal{F} \cup \{X\} \cup \{\{a\} : a \in X\}$ induce the same betweenness relation.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

3.3 Theorem. *A ternary structure is an F-structure if and only if it satisfies inclusivity (B1), uniqueness (B3), and convexity (B6).*

Proof. As discussed above, F-structures clearly satisfy the betweenness axioms, as well as convexity. As for the converse, suppose $\langle X, [\cdot,\cdot,\cdot]\rangle$ satisfies the conditions of inclusivity, uniqueness, and convexity (B1, B3, B6). Then symmetry (B2) automatically follows, and we have a convex betweenness structure. Let $\mathcal{I} = \mathcal{I}_X$. Then, by Proposition 3.2, $\mathcal{I}$ is an F-system and $[\cdot,\cdot,\cdot]_{\mathcal{I}} \subseteq [\cdot,\cdot,\cdot]$. It suffices to prove the reverse inclusion; so given $a, b \in X$, suppose $A \in \mathcal{I}(a,b)$. Then $A = [u,v]$ for some $u, v \in X$. But then $a, b \in [u,v]$, and−by convexity (B6)–$[a,b] \subseteq A$. This shows $[a,b] \subseteq [a,b]_{\mathcal{I}}$. □

3.4 Examples.

(i) In a metric space M-betweenness is not an F-structure in general, as convexity may fail. Nevertheless, it is both transitive and antisymmetric (see [12] and Example 1.4 (ii)).

(ii) In a vector space linear betweenness is an F-structure, as witnessed by the family of all linear intervals. As noted earlier, antisymmetry is easily seen to hold in this setting as well.

(iii) In a connected topological space X, C-betweenness is an F-structure induced by the F-system $\mathcal{C} = \mathcal{C}_X$ of connected subsets (see Example 1.1 (v)). What is not completely obvious is that $[\cdot,\cdot,\cdot]_{\mathcal{C}}$ satisfies antisymmetry. To see this, suppose $a, b, c \in X$, with $b \neq c$. Let C be the connected component of $X \setminus \{b\}$ that contains c. If $a \in C$, then C is a road containing a and c, but not b; hence $[a,b,c]_{\mathcal{C}}$ fails. On the other hand, if $a \notin C$, then both a and b are in $X \setminus C$. By [11, Theorem §46.III.5], $X \setminus C$ is connected; i.e., in $\mathcal{C}$. Hence $[a,c,b]_{\mathcal{C}}$ fails.

(iv) In a connected T_1 space X, K-betweenness is an F-structure induced by the F-system $\mathcal{K} = \mathcal{K}_X$ of connected closed subsets (see Example 1.1 (vi)). Antisymmetry may fail, even when X is a metrizable continuum. For example, take X to be the $\sin\frac{1}{x}$-curve, defined to be the union (in $\mathbb{R}^2$) of the function graph $G = \{\langle t, \sin\frac{1}{t}\rangle : 0 < t \leq 1\}$ and the arc $I = \{0\} \times [-1,1]$. If $a \in G$ and $b, c \in I$ are arbitrary, then it is easy to check that $[a,b,c]_{\mathcal{K}}$ and $[a,c,b]_{\mathcal{K}}$ both hold.

The following condition on a family of sets, introduced in [3], does for the antisymmetry axiom (B7) what the Fréchet condition does for the uniqueness axiom (B3) in a betweenness relation. Define $\mathcal{S} \subseteq \wp(X)$ to be **separative** (and $\langle X, \mathcal{S}\rangle$ an **S-system**) if whenever $a, b, c \in X$ are such that $b \neq c$, then some member of $\mathcal{S}$ contains a and exactly one of b and c.

It is easy to see that every S-system is an F-system. Also, at the axiom level, inclusivity and antisymmetry together imply uniqueness. The following is an analogue of Theorem 3.3, and is proved in much the same way.

3.5 Theorem. *A ternary structure is an S-structure if and only if it satisfies inclusivity (B1), antisymmetry (B7), and convexity (B6).*

3.6 Remarks.

(i) From Example 3.4 (ii), the family of all linear intervals of a vector space is actually an S-system inducing linear betweenness.

(ii) From Examples 3.4 (iii,iv), we see that when X is a continuum, $\mathcal{C}_X$ is separative, but $\mathcal{K}_X$ need not be.

(iii) The class of continua X for which $\mathcal{K}_X$ is separative has been referred to as the *antisymmetric* continua in [3–6]. (This is consistent with our terminology, in light of Theorem 3.5.) Antisymmetric metrizable continua were studied earlier by B. E. Wilder [18], who called them *C-continua*; more recently, K. Królicki and P. Krupski [10] have called them−metrizable or not− *Wilder continua*. While we appre-

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

ciate this terminology, we prefer sticking with *antisymmetric* because of its close link with the theory of partial orderings.

A family $\mathcal{A}$ of subsets of X is **additive** if the union of two overlapping roads in $\mathcal{A}$ is also a road in $\mathcal{A}$.

For example, if $\mathcal{A}$ is either the connected subsets or the connected closed subsets of a topological space, then $\mathcal{A}$ is additive. A straightforward betweenness consequence of additivity in an inducing F-system is the following.

(B8) Disjunctivity: $\forall xuzy\ ([x,u,y] \to ([x,u,z] \vee [z,u,y]))$

Disjunctivity says of a betweenness structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ that for any $a,b,c \in X$, $[a,b] \subseteq [a,c] \cup [c,b]$. To see that additive families induce disjunctive betweenness structures, suppose $\mathcal{A}$ is an additive F-system on X. Given $a,b,c \in X$, choose $d \in X$ such that $[a,d,b]_{\mathcal{A}}$ holds. Assuming $[a,d,c]_{\mathcal{A}}$ fails, we wish to show $[c,d,b]_{\mathcal{A}}$ holds. Fix $A \in \mathcal{A}(a,c)$ such that $d \notin A$, and let $B \in \mathcal{A}(c,b)$ be arbitrary. Then, since $c \in A \cap B$, additivity tells us that $A \cup B \in \mathcal{A}(a,b)$. But now $d \in A \cup B$. Since $d \notin A$, we have $d \in B$; hence $[c,d,b]_{\mathcal{A}}$ holds, as desired.

3.7 Remark. In [3, Theorem 4.0.5] it is proved that an F-structure satisfies disjunctivity (B8) if and only if each of its inducing F-systems is contained within an inducing F-system that is additive.

We call a betweenness structure **disjunctive** if its betweenness relation satisfies the disjunctivity axiom. The next result shows that transitivity (B5) and disjunctivity (B8) imply something much stronger than convexity (B6): not only is the span of every doubleton subset convex, the span of *every* subset is convex.

3.8 Theorem. *Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be both transitive and disjunctive. Then every subset of X is 1-convex; i.e., the span of any subset coincides with its hull.*

Proof. Let $A \subseteq X$, with $a \in A$ fixed. Then $[A]^\omega \supseteq [A] \supseteq A' := \bigcup\{[a,b] : b \in A\} \supseteq A$. So, in light of Proposition 1.3, it suffices to show A' is convex. Let $u,v \in A'$. Then there are $b,c \in A$ such that $u \in [a,b]$ and $v \in [a,c]$. By transitivity, both $[a,u]$ and $[a,v]$ are subsets of A'. By disjunctivity, $[u,v] \subseteq [a,u] \cup [a,v]$; hence $[u,v] \subseteq A'$. $\square$

3.9 Corollary. *Suppose X is a connected T_1 space, and $\mathcal{A}$ is either $\mathcal{C}_X$ or $\mathcal{K}_X$. Then, with respect to $[\cdot,\cdot,\cdot]_{\mathcal{A}}$, the span of each subset of X is convex.*

In the next section we properly begin our study of betweenness structures that have topological structure as well.

4. Local convexity

As stated in Section 2, our topological betweenness structures (TBS, for short) carry regular topologies for which all intervals—and hence singletons—are closed subsets. In the sequel we write A^- and A° for the closure and interior, respectively, of a subset A of a topological space. If $\langle X, \varrho\rangle$ is a metric space, $c \in X$, and $r > 0$, the *open ball with center c and radius r* is denoted $B(c;r) := \{x \in X : \varrho(x,c) < r\}$. The family of open balls is the *standard base* for the induced metric topology.

4.1 Examples.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

(i) The K-interpretation of betweenness in a continuum X clearly gives rise to a TBS because continua have regular topologies, and intervals—being intersections of subcontinua—are closed.
(ii) The same cannot be said for the C-interpretation. For example, let X be the $\sin\frac{1}{x}$-curve from Example 3.4 (iv). If a is the end point of the graph G and b is any point in the arc I, then $[a,b]_{\mathcal{C}} = G \cup \{b\}$, which is not closed.
(iii) The M-betweenness structure of a metric space $\langle X, \varrho\rangle$ gives rise to a TBS because metric topologies are regular, and each interval $[a,b]_\varrho$ is the zero set corresponding to the continuous function $x \mapsto \varrho(a,x) + \varrho(x,b) - \varrho(a,b)$.
If the metric space is equipped with a Takahashi-convex structure, see Example 1.1 (iii)), then we have a TBS because each T-interval—being the union of two arcs joined at the end points—is a subcontinuum of the space.
(iv) For X a topological vector space, linear betweenness defines a TBS, provided the topology is T_1. For then the topology is (completely) regular (see, for example, [13, Chapter 1]); and, given $a, b \in X$, $[\![a,b]\!]$ is the image of the compact set $[0,1] \subseteq \mathbb{R}$ under the continuous map $t \mapsto (1-t)a + tb$. Since X is Hausdorff, $[\![a,b]\!]$ is closed in X (indeed, it is topologically an arc).

4.2 Remark. Referring to Example 4.1 (iii), M-intervals are not only closed in the metric space, they are bounded [6, Proposition 2.1]. (Indeed, the diameter of $[a,b]_\varrho$ is $\varrho(a,b)$.) They are not necessarily compact, however (see Remark 4.17 below). A metric space for which all closed bounded subsets are compact is said to satisfy the **Heine-Borel property**.[8]

The K-interpretation of betweenness is "determined by topology," in the sense that a homeomorphism of topological spaces is an isomorphism of the associated betweenness structures. The M-interpretation, however, is "determined by geometry," not topology, as the following two examples illustrate.

4.3 Examples.

(i) Metrics with nondiscrete betweenness structures are commonplace; however, for any given metric, there is a topologically equivalent metric whose associated betweenness structure *is* discrete. To see this, let $\langle X, \varrho\rangle$ be given, and define $\varrho'(a,b) := \sqrt{\varrho(a,b)}$. Then, because $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$ fixes zero, is strictly increasing, and is subadditive (i.e., $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$), ϱ' is a metric on X. And because both $\sqrt{\cdot}$ and its inverse are continuous at zero, ϱ' generates the same topology as ϱ. Finally, because of *strict* subadditivity (i.e., $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$ for any $x, y > 0$), it follows that if $c \notin \{a,b\}$, then $\varrho'(a,c) + \varrho'(c,b) > \sqrt{\varrho(a,c) + \varrho(c,b)} \geq \sqrt{\varrho(a,b)} = \varrho'(a,b)$. Hence $c \notin [a,b]_{\varrho'}$; i.e., $[a,b]_{\varrho'} = \{a,b\}$.
(ii) Define a metric to be *Peano* if its induced topology is that of a locally connected continuum. By independent results of Bing and Moise (see [12, Page 98], [8, Theorem 8], [14, Theorem 4]), each Peano metric generates the same topology as a metric whose associated betweenness structure is gap-free (B4). (Menger [12] had already proved that any gap-free metric on a continuum is Peano.)

Henceforth, when we refer to *betweenness*—and its attendant notions—in the metric (resp., continuum) context, we implicitly mean the M-interpretation (resp., K-interpretation). However, when more than one betweenness interpretation is under consideration, we retain disambiguating modifiers.

For $n \in \omega$, a TBS $\langle X, [\cdot,\cdot,\cdot]\rangle$ is called **locally n-convex** at $a \in X$ if for any open neighborhood U of a, there is an n-convex set C, with $a \in C^\circ \subseteq C \subseteq U$. Being **locally n-convex** means being locally n-convex at each point; the term *locally convex* is short for *locally 0-convex*.

[8] Such metrics are also called *proper*, but that word is overused.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

4.4 Examples.

(i) In the topological vector space literature, being "locally convex" means what we refer to as being locally convex with respect to *linear betweenness*. If the topology on our space is induced by a norm, then open balls are easily seen to be linearly convex. And if the norm is strictly convex (see [6, Proposition 4.1]), then M-betweenness and linear betweenness coincide; hence strictly convex normed vector spaces are locally M-convex.

(ii) Here is an easy example of where open balls are not M-convex, but where local M-convexity still holds. Let $X = \langle \mathbb{R}^2, \|\cdot\|_\infty \rangle$. Then the open unit ball $B = B(\langle 0,0 \rangle; 1)$ is the open square region $(-1,1) \times (-1,1)$. If we let $a = \langle -\frac{1}{2}, \frac{1}{2} \rangle$ and $b = \langle \frac{1}{2}, \frac{1}{2} \rangle$, then the M-interval $[a,b]$ is the closed square region with corner points a, b, $\langle 0,0 \rangle$, and $\langle 0,1 \rangle$. The bracket points are in B, but the corner point $\langle 0,1 \rangle$ is not. Local convexity does hold nevertheless because this metric space has an open base consisting of open squares with sides of slope ± 1. Such sets are clearly M-convex.

In Section 1 we pointed out the difference between the span and the hull for general betweenness structures. With this in mind one can ask about the difference between local convexity and local n-convexity for $n > 0$. In Corollary 4.6 (i) below we show the two notions to be equivalent for metric spaces. In order to prove this, as well as other results involving interactions between topology and convexity theory, we recall some terminology from [6].

A TBS $\langle X, [\cdot,\cdot,\cdot] \rangle$ is defined to be **upper semicontinuous** (**USC**) (resp., **lower semicontinuous** (**LSC**)) at a pair $\langle a,b \rangle \in X^2$ if for each open set $U \subseteq X$ containing (resp., intersecting) $[a,b]$, there is an open neighborhood $V_a \times V_b$ of $\langle a,b \rangle$ such that U contains (resp., intersects) $[a',b']$ for all $\langle a',b' \rangle \in V_a \times V_b$. The relation is **USC/LSC** if it is USC/LSC at every pair; it is **USC/LSC at** a if it is USC/LSC at the pair $\langle a,a \rangle$. If the relation is USC/LSC at each point, we say it is **USC/LSC at singletons**. (While being USC at singletons is an important special case of upper semicontinuity, being LSC at singletons is a trivial consequence of the inclusivity (B1) and uniqueness (B3) axioms.)

4.5 Theorem. *If a TBS is USC at a point and also locally n-convex at the point, for some $n \in \omega$, then it is locally convex at that point.*

Proof. Let $\langle X, [\cdot,\cdot,\cdot] \rangle$ be a TBS which is USC at $a \in X$ and also locally n-convex at a for some $n > 0$. Let U an open neighborhood of a. Then U contains $[a,a] = \{a\}$, so—by the definition of being USC at a point—there is an open neighborhood V of a with $[V] \subseteq U$. By local n-convexity at a, there is an n-convex set C with $a \in C^\circ \subseteq C \subseteq V$. Then $[C]$ is $(n-1)$-convex and $a \in [C]^\circ$. Also, since the span operator is clearly monotone, we have $[C] \subseteq [V] \subseteq U$. This proves that our TBS is locally $(n-1)$-convex at a; and, after n repetitions of the argument, that it is locally 0-convex at a. □

By [6, Proposition 2.10], every metric space is USC at singletons; and by Corollary 3.9, each continuum is locally 1-convex. Hence we have the following.

4.6 Corollary.

(i) *A metric space is locally convex if it is locally n-convex for some $n \in \omega$.*
(ii) *A continuum is locally convex if it is USC at singletons.*

We are interested in identifying some familiar metric/topological properties which force a TBS to be locally convex, but first we address the issue of when local convexity fails. We start with two examples of Banach spaces that are not locally M-convex. The first example is finite-dimensional, but does not have convex betweenness; the second has convex betweenness, but does not have finite dimension. In light of Corollary 4.6 (i), these spaces are not locally M-n-convex for any $n \in \omega$.

4.7 Examples.

(i) Let our Banach space be $X = \langle \mathbb{R}^3, \|\cdot\|_\infty \rangle$, and fix $\epsilon > 0$. For $n = 1, 2, \ldots$, let $a_n = \langle -\epsilon, 0, (2n-2)\epsilon \rangle$, $b_n = \langle \epsilon, 0, (2n-2)\epsilon \rangle$, $c_n = \langle 0, -\epsilon, (2n-1)\epsilon \rangle$, $d_n = \langle 0, \epsilon, (2n-1)\epsilon \rangle$, with $A = A(\epsilon) = \{a_1, b_1\}$. Then c_1 and d_1 are in $[a_1, b_1] = [A]$, a_2 and b_2 are in $[c_1, d_1] \subseteq [A]^2$, etc. In general, $n \geq 2$, we have $c_n, d_n \in [a_n, b_n] \subseteq [A]^{2n-1}$ and $a_n, b_n \in [c_{n-1}, d_{n-1}] \subseteq [A]^{2n-2}$. Since the convex hull $[A]^\omega$ is $\bigcup_{n=1}^\infty [A]^n$, we have an inductive proof that all the points a_n, b_n, c_n, d_n lie in $[A]^\omega$. But, for $n \geq 2$, $\|a_n\|_\infty = (2n-2)\epsilon$; hence $[A]^\omega$ is unbounded. Consequently, if C is any convex neighborhood of the origin, then C is unbounded because it contains $A(\epsilon)$ for suitably small ϵ. So X cannot be locally M-convex. (To see that X does not have convex M-betweenness, notice that $a_2 \in [c_1, d_1]$, $c_1, d_1 \in [a_1, b_1]$, but $a_2 \notin [a_1, b_1]$.)

(ii) Consider the Banach space $X = \ell^1$ of all real sequences with absolutely convergent series, and norm given by $\|a\|_1 := \sum_{i=1}^\infty |a(i)|$. Given $a, b \in X$, it is routine to show that $[a, b]$ is the Cartesian product $\prod_{i=1}^\infty [a(i), b(i)]$. As a result, all M-intervals are M-convex (i.e., X has convex M-betweenness). To see this space is not locally convex, it suffices to show that every open ball $B(0; \epsilon)$ has unbounded convex hull. So, given $\epsilon > 0$, $n \geq 1$, let $e_n(i)$ be $\frac{\epsilon}{2}$ if $i = n$, and 0 otherwise. Let $B' = [B(0; \epsilon)]^\omega$. Then B' contains $[e_1, e_2] = [0, \frac{\epsilon}{2}]^2 \times \prod_{i=3}^\infty \{0\}$, and hence it contains the point $a_2 = e_1 + e_2$. B' also contains $[a_2, e_3] = [0, \frac{\epsilon}{2}]^3 \times \prod_{i=4}^\infty \{0\}$, and hence the point $a_3 = a_2 + e_3$. Proceeding by induction, where $a_{n+1} = a_n + e_{n+1}$, we see that B' contains each a_n–whose norm is precisely $\frac{n\epsilon}{2}$–and is therefore unbounded.

The "failings" of the previous two examples suggest the following.

4.8 Question. If a normed vector space has convex M-betweenness and is finite-dimensional, is it locally M-convex?

We can answer this question positivitely−even without the dimension hypothesis−if we assume something stronger than convex M-betweenness, namely the following softening of the disjunctivity axiom (B8).

(B9) Weak Disjunctivity: $\forall xuzy\ (([x, u, y] \wedge [x, z, y]) \rightarrow ([x, u, z] \vee [z, u, y]))$

In the language of intervals, a betweenness structure $\langle X, [\cdot, \cdot, \cdot] \rangle$ satisfies weak disjunctivity if and only if $[a, b] \subseteq [a, c] \cup [c, b]$ whenever $a, b \in X$ and $c \in [a, b]$. Since we do not require $c \in [a, b]$ in the statement of disjunctivity, (B9) is formally weaker than (B8).

Transitivity (B5) and weak disjunctivity (B9) formally imply convexity (B6): For if $c, d \in [a, b]$ and $e \in [c, d]$, (B9) implies that $d \in [a, c]$ or $d \in [c, b]$. In the first instance, we have $e \in [a, c]$, by transitivity; another application of transitivity gets us $e \in [a, b]$. The story is similar in the case $d \in [c, b]$. Since M-betweenness always satisfies transitivity, the presence of weak disjunctivity implies that convexity (B6) holds too.

The following establishes the equivalence of weak disjunctivity (a betweenness-theoretic property) and strict convexity (a geometric property) in normed vector spaces.

4.9 Theorem. *Let X be a normed vector space. Then X is strictly convex if and only if its M-betweenness satisfies weak disjunctivity (B9). Hence normed vector spaces with weakly disjunctive M-betweenness are locally M-convex.*

Proof. If X has a strictly convex norm, then its M-betweenness and linear betweenness agree. Linear betweenness trivially satisfies weak disjunctivity.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

For the converse, suppose X has weakly disjunctive M-betweenness. For any $a, b \in X$—because linear intervals are contained in corresponding M-intervals—it suffices to show $[a, b] \subseteq [\![a, b]\!]$. If $a = b$ there is nothing to prove. So assume $a \neq b$ and fix $c \in [a, b]$. Then $\|c - a\| \leq \|b - a\|$; so

$$c' = \left(1 - \frac{\|c-a\|}{\|b-a\|}\right)a + \frac{\|c-a\|}{\|b-a\|}b \in [\![a, b]\!] \subseteq [a, b].$$

Now, by rewriting

$$c' = a + \frac{\|c-a\|}{\|b-a\|}(b - a),$$

we have immediately that $\|c' - a\| = \|c - a\|$. Since $c' \in [a, b]$, we also have $\|b - c'\| = \|b - c\|$. If it happens that $c \in [a, c']$, then we quickly conclude that $c = c'$. By weak disjunctivity (B9), the alternative is that $c \in [c', b]$, from which we obtain the same conclusion. Hence $c \in [\![a, b]\!]$. □

4.10 Remark. The authors originally used [6, Theorem 3.8 (ii)] to prove the finite-dimensional version of Theorem 4.9. We are grateful to the anonymous referee for suggesting the simple direct approach above to obtain a stronger result.

Weak disjunctivity also plays a role in the study of betweenness structures *à la* [17]. Paraphrasing slightly, a transitive betweenness structure is **geometric** if it satisfies the following first-order axiom.

(B10) Inversion: $\forall xuzy\ (([x, u, y] \wedge [x, z, y] \wedge [x, u, z]) \rightarrow [u, z, y])$

In interval terms, inversion[9] says that if $c, d \in [a, b]$ and $c \in [a, d]$, then $d \in [c, b]$. It is shown in [12] that M-betweenness structures satisfy this axiom, and [17, Chapter I, §4] offers an extensive study of geometric betweenness structures—*geometric interval spaces*, in the parlance of [17]—from the perspective of classical geometry. The following is an immediate consequence of [3, Theorem 5.0.6].

4.11 Proposition. *A transitive, weakly disjunctive betweenness structure is geometric if and only if it is antisymmetric.*

4.12 Corollary. *The* K*-betweenness structure of a continuum satisfies inversion (B10) if and only if it satisfies antisymmetry (B7).*

So, among the continua, all (and only) the antisymmetric ones are *geometric*, in the sense of [17].

Examples 4.7 (i,ii) fail to be locally M-convex, but are of course locally *linearly* convex as normed vector spaces. On the other hand there exist metric vector spaces that are locally M-convex—for trivial reasons in fact—but which fail dramatically to be locally linearly convex.

4.13 Example. Let $X = L^p([0, 1])$, where $0 < p < 1$. Elements of X are continuous $f : [0, 1] \rightarrow \mathbb{R}$, and $\varrho(f, g) := \int_0^1 |f(t) - g(t)|^p dt$. Because of the subadditivity condition (i.e., $(a + b)^p \leq a^p + b^p$, for $a, b \geq 0$), this defines a metric on X. (Indeed, it is a metric which is both complete and translation-invariant.) Because of strict subadditivity (see Example 4.3 (i)) it is easy to show that all M-intervals are gaps; hence *every* subset is M-convex. However (see [13, Paragraph 1.47]), every linearly convex proper subset of X has empty interior.

[9] This is called *reciprocity* in [3].

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

The next two examples illustrate how local convexity—and, by implication (Corollary 4.6 (ii)), USC at singletons—can fail in continua.

Nondegenerate metrizable continua which are antisymmetric are well known to be **decomposable**; i.e., they can be written as the union of two proper subcontinua. Continua that are not decomposable are called **indecomposable**.

4.14 Examples.

(i) Let X be the *harmonic fan*; namely the planar continuum $H \cup \bigcup_{n=1}^{\infty} D_n$, where $H = [0,1] \times \{0\}$ and each D_n is $\{\langle t, \frac{t}{n}\rangle : t \in [0,1]\}$. This continuum is easily seen to be antisymmetric. Let $a = \langle 0,0\rangle$; $b = \langle 1,0\rangle$; and, for $n \geq 1$, $b_n = \langle 1, \frac{1}{n}\rangle$. Then, since $\langle b_n\rangle \to b$, any convex set containing b in its interior must contain some b_n; and hence it must contain a. If U is an open subset of $X \setminus \{a\}$ containing b, then no convex subset contained in U contains b in its interior. Hence X is not locally convex at b.

(ii) Recall that a continuum X is *irreducible* if there are two points $a, b \in X$ such that $[a, b] = X$ (i.e., if some doubleton set spans X). An irreducible indecomposable continuum is not locally convex at any of its points. Indeed, recall (see, for example, [15]) that each such continuum is partitioned into at least two dense connected subsets, called *composants*,[10] such that $[a, b] = X$ whenever a and b lie in distinct composants. Every nonempty open subset of X meets every composant; hence every proper convex subset of X has empty interior.

4.15 Question. If a nondegenerate continuum is locally convex at some point, is the continuum decomposable? (By Example 4.14 (ii), a counterexample would have to be an indecomposable continuum with exactly one composant. Such continua exist, with weights as little as $\aleph_1$ [7], but none of the examples we know of (see [2]) seem to be locally convex.)

A continuum X is **aposyndetic** if for any two distinct points, each has a subcontinuum neighborhood that misses the other; i.e., if $a, b \in X$ and $a \neq b$, then there is a subcontinuum $K \in \mathcal{K}_X$ such that $a \in K^\circ \subseteq K \subseteq X \setminus \{b\}$. Aposyndesis is considerably weaker than local connectedness, and implies antisymmetry [5, Theorem 3.2]. In the remainder of this section we show that while antisymmetry is not enough to ensure local convexity in a continuum—see the harmonic fan above—aposyndesis implies local convexity very decisively: while being USC at singletons is sufficient, aposyndetic continua are USC at *all* pairs.

We introduce the following notion in order to deal with upper semicontinuity by means of (net) convergence. Define a TBS $\langle X, [\cdot,\cdot,\cdot]\rangle$ to satisfy the **interval convergence property** (**ICP**) if the following condition holds: Given convergent nets $\langle a_\lambda\rangle \to a$, $\langle b_\lambda\rangle \to b$, and $\langle c_\lambda\rangle \to c$ (where the variable λ ranges over the elements of a directed set $\Lambda = \langle \Lambda, \leq\rangle$), if $c_\lambda \in [a_\lambda, b_\lambda]$ for each $\lambda \in \Lambda$, then $c \in [a, b]$.

4.16 Theorem. *A TBS that is USC also satisfies the ICP. A compact TBS satisfying the ICP is USC.*

Proof. Assume $\langle X, [\cdot,\cdot,\cdot]\rangle$ is USC, and suppose the convergent nets $\langle a_\lambda\rangle \to a$, $\langle b_\lambda\rangle \to b$, and $\langle c_\lambda\rangle \to c$ are given, where $c_\lambda \in [a_\lambda, b_\lambda]$, for each $\lambda \in \Lambda$. Suppose $c \notin [a, b]$. Then, by regularity and the fact that $[a, b]$ is closed, there is an open set $U \supseteq [a, b]$ such that $c \in X \setminus U^-$. By USC, there are open neighborhoods V_a of a and V_b of b such that $[a', b'] \subseteq U$ for any $\langle a', b'\rangle \in V_a \times V_b$. By net convergence, there is a $\lambda_0 \in \Lambda$ such that $a_\lambda \in V_a$ and $b_\lambda \in V_b$ for all $\lambda \geq \lambda_0$. Thus $c_\lambda \in U$ for all $\lambda \geq \lambda_0$, implying that $c \in U^-$, a contradiction.

Now suppose X is compact and satisfies the ICP. If USC fails at $\langle a, b\rangle$, we have an open $U \supseteq [a, b]$, convergent nets $\langle a_\lambda\rangle \to a$, $\langle b_\lambda\rangle \to b$, and a net $\langle c_\lambda\rangle$ such that, for each $\lambda \in \Lambda$: (1) $c_\lambda \in [a_\lambda, b_\lambda]$; and (2) $c_\lambda \in X \setminus U$. By compactness, there is a subnet of $\langle c_\lambda\rangle$ converging to some $c \in X$. That is, there is monotone

[10] The number of composants in a nondegenerate indecomposable metrizable continuum is uncountable [15].

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

cofinal mapping $f : \Delta \to \Lambda$ of directed sets such that $\langle c'_\delta \rangle \to c$, where $c'_\delta := c_{f(\delta)}$. Since subnets of convergent nets converge to the same limit, the corresponding nets $\langle a'_\delta \rangle$ and $\langle b'_\delta \rangle$ converge to a and b, respectively. But then, by the ICP, we have $c \in [a, b] \subseteq U$, a contradiction. □

4.17 Remark. By continuity of the metric function, it follows that every metric space satisfies the ICP; and so, by Theorem 4.16, each compact metric space is USC (see also [6, Theorem 2.5]). Compactness is crucial here, for consider the Banach space c_0 consisting of all real sequences converging to zero. Equipped with the supremum norm, the resulting metric space is USC at a pair $\langle a, b \rangle$ if and only if $[a, b]$ is compact, if and only if $a = b$ (see [6, Example 4.19]).

The following is a significant improvement on [6, Theorem 5.2].

4.18 Theorem. *Every aposyndetic continuum is USC, and hence locally convex.*

Proof. Let X be an aposyndetic continuum. Since X is compact, Theorem 4.16 tells us that all we need to show is that the ICP holds. Assume we have three convergent nets $\langle a_\lambda \rangle \to a$, $\langle b_\lambda \rangle \to b$, and $\langle c_\lambda \rangle \to c$, such that $c_\lambda \in [a_\lambda, b_\lambda]$ for all $\lambda \in \Lambda$. We need to show that $c \in K$ for every subcontinuum of X containing $\{a, b\}$. Assuming $c \in X \setminus \{a, b\}$, we use aposyndesis to find subcontinua K_a and K_b such that $a \in K_a^\circ$ and $b \in K_b^\circ$, but $c \in X \setminus (K_a \cup K_b)$. Let $K \in \mathcal{K}_X(a, b)$. By net convergence there is some $\lambda_0 \in \Lambda$ such that $a_\lambda \in K_a^\circ$ and $b_\lambda \in K_b^\circ$ for all $\lambda \geq \lambda_0$. Since $K_a \cup K \cup K_b \in \mathcal{K}_X(a_\lambda, b_\lambda)$ for $\lambda \geq \lambda_0$, we know that $[a_\lambda, b_\lambda] \subseteq K_a \cup K \cup K_b$ on a tail of Λ. Hence, on this tail, we know c_λ belongs to the closed set $K_a \cup K \cup K_b$; so $c \in K_a \cup K \cup K_b$. But $c \notin K_a \cup K_b$; so $c \in K$. We conclude that the ICP holds. Theorem 4.16 says X is USC, and Corollary 4.6 (ii) says X is locally convex. □

4.19 Question. Call a TBS **openly locally n-convex** at a point if the point has a neighborhood base consisting of open n-convex sets. Does being locally n-convex at a point imply being openly so? (We do not know how to obtain the "open" version of Theorem 4.5; nor can we improve Theorem 4.18 to conclude that aposyndetic continua are openly locally convex. However, it is worth noting that all strictly convex normed vector spaces are openly locally M-convex and all Takahashi-convex metric spaces are openly locally T-convex.)

5. Spans and hulls of subsets

This section is about when topological properties of subsets are preserved by taking the span/hull.

Let P be a property of subsets of a set. The TBS $\langle X, [\cdot,\cdot,\cdot] \rangle$ is called **span-P** (resp., **hull-P** if the span (resp., hull) of each subset of X with property P also has property P.

5.1 Examples.

(i) A TBS with discrete betweenness structure is clearly span-P for any property P.

(ii) Span-open implies hull-open: If $U \subseteq X$ is open, then each $[U]^n$ is open because of span-openness and induction. Thus the hull $[U]^\omega = \bigcup_{n=1}^{\infty} [U]^n$ is open.

(iii) For continua, span-P is equivalent to hull-P: This follows from Corollary 3.9, which says that spans and hulls coincide.

(iv) Each topological vector space X is span-open (hence hull-open) with respect to linear betweenness: For each open U we can write the span as $[U] = \bigcup\{(1-t)U + tU : 0 \leq t \leq 1\}$. By continuity of the vector space functions, the maps $x \mapsto x + a$ and $x \mapsto sx$ are homeomorphisms on X for any vector a and nonzero scalar s. This quickly implies that each $(1-t)U + tU$ is open; hence so is $[U]$. Moreover, if the topology is generated by a strictly convex norm, then linear betweenness coincides with M-betweenness and X is span-open with respect to the latter.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

(v) Each topological vector space X is span-compact with respect to linear betweenness: For let $K \subseteq X$ be compact. Then $[K]$ is the image of the compact set $K^2 \times Y$, where $Y = \{\langle t_1, t_2\rangle \in [0,1]^2 : t_1 + t_2 = 1\}$, under the continuous map $\langle a_1, a_2, t_1, t_2\rangle \mapsto t_1a_1 + t_2a_2$, and is hence compact. If the space is finite-dimensional, then it is hull-compact too: Recall Carathéodory's theorem in this regard (see [1, Theorem 5.32]), which says that if a is in the convex hull of $E \subseteq \mathbb{R}^n$, then a is in the convex hull of a subset of E of cardinality $\leq n + 1$. So if $K \subseteq \mathbb{R}^n$ is compact, then $[K]^\omega$ is the image of the compact set $K^{n+1} \times Y$, where $Y = \{\langle t_1, \ldots, t_{n+1}\rangle \in [0,1]^{n+1} : \sum_{i=1}^{n+1} t_i = 1\}$, under the continuous map $\langle a_1, \ldots, a_{n+1}, t_1, \ldots, t_{n+1}\rangle \mapsto \sum_{i=1}^{n+1} t_ia_i$. On the other hand, it is known [1, Example 5.34] that convex hulls of compact subsets of a Hilbert space need not even be closed.

(vi) Not every finite-dimensional topological vector space is span-closed (or hull-closed) with respect to linear betweenness: Let X be $\mathbb{R}^2$, with the Euclidean norm. If A is the closed set $(\mathbb{R} \times \{0\}) \cup \{\langle 0, 1\rangle\}$, then $[A] = [A]^\omega = (\mathbb{R} \times [0, 1)) \cup \{\langle 0, 1\rangle\}$, which is not closed.

(vii) Let X be the standard unit circle in the plane, equipped with its intrinsic (i.e., "shortest arc") metric. Then X is span-open with respect to M-betweenness: To see this, note that the nondegenerate convex proper subsets of X are precisely those circular arcs−with or without end points−that do not contain a semicircle. So let U be any nonempty open set. If U is contained in one of these arcs, then $[U]$ is a circular arc without end points; otherwise $[U] = X$. Either way, $[U]$ is open in X.

(viii) Let X be the triod in the plane, given by $X = A \cup B$, where $A = [-1, 1] \times \{0\}$ and $B = \{0\} \times [0, 1]$. With the shortest arc metric on X, the M-interpretation and the K-interpretation of betweenness are identical. Let $c = \langle 0, 0\rangle$, with $U = A \setminus \{c\}$. Then U is open in X; however $[U] = [U]^\omega = A$ is not open in X. This is simultaneously a metric and a continuum example of how hull-openness and span-openness can fail.

(ix) Let X be any irreducible indecomposable continuum (see Example 4.14 (ii)). Then we have $[U] = X$ for each nonempty open $U \subseteq X$. Thus such continua are span-open for trivial reasons.

(x) Let $X = G \cup I$ be the $\sin \frac{1}{x}$-curve from Example 3.4 (iv). Then X is both span-closed and span-open. To see this, let $\pi : X \to [0, 1]$ be projection onto the x-axis. It is straightforward to see that the span/hull of any $A \subseteq X$ is $\pi^{-1}([\pi(A)])$. The projection is a closed map, and arcs are easily seen to be span-closed; hence X is span-closed. To see that X is span-open, let $U \subseteq X$ be a proper open set. If U intersects I, then there is a unique point $t_U \in (0, 1]$ such that $[U] = I \cup \{\langle t, \sin \frac{1}{t}\rangle : 0 < t < t_U\}$. If U does not intersect I, then $[U]$ is the image, under the sine function, of an open interval in $(0, 1]$. In either case, $[U]$ is open in X.

Our first result resembles the well-known characterization of closed maps as: "the closure of the image is contained in the image of the closure." Recall that a subset is *relatively compact* if its closure is compact.[11]

5.2 Theorem. *Suppose a TBS $\langle X, [\cdot, \cdot, \cdot]\rangle$ satisfies the ICP. If $A \subseteq X$ is relatively compact, then $[A]^- \subseteq [A^-]$. In particular, spans of compact subsets are closed.*

Proof. Let $\langle X, [\cdot, \cdot, \cdot]\rangle$ be a TBS which satisfies the ICP, and fix $A \subseteq X$ a relatively compact subset. Suppose $c \in [A]^-$; we want to show $c \in [A^-]$. Let $\mathcal{U}$ be the family of open neighborhoods of c; and for each $U \in \mathcal{U}$, pick $c_U \in [A] \cap U$. Then, for each such U, we have $a_U, b_U \in A$ such that $c_U \in [a_U, b_U]$. By the relative compactness of A, the nets $\langle a_U\rangle$ and $\langle b_U\rangle$ have subnets converging in A^-. And, since $\langle c_U\rangle \to c$, and subnets of convergent nets converge to the same limit, we lose no generality in assuming the existence of $a, b \in A^-$ with $\langle a_U\rangle \to a$ and $\langle b_U\rangle \to b$. By the ICP, we have $c \in [a, b]$; i.e., $c \in [A^-]$. For the last assertion, suppose A is compact. Then A is closed, since X is Hausdorff, and we have $[A]^- \subseteq [A^-] = [A]$. Thus $[A]^- = [A]$ as desired. □

[11] Many authors also use the term, *precompact*.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

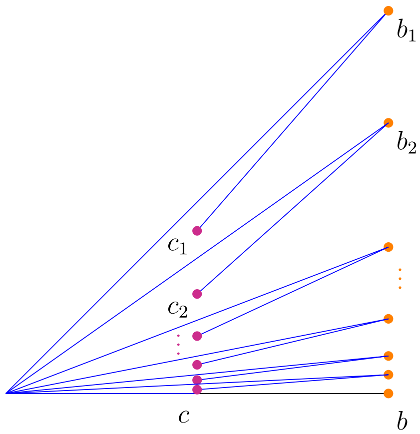

Fig. 1. The *Harmonic rake* from Example 5.6 (ii). The points $c_1, c_2, \ldots$ are shown in pink and $[C]$ is shown in blue. The span is not closed since it does not contain the black segment $[c, b]$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The following is an immediate consequence of Theorem 5.2.

5.3 Corollary. *Every compact TBS satisfying the ICP is span-closed.*

When we combine Corollaries 3.9 and 5.3 with Theorems 4.16 and 4.18, we have the following.

5.4 Corollary. *Every aposyndetic continuum is span-closed, as well as hull-closed.*

5.5 Remark. Corollary 5.4 specifies a large class of well-behaved continua which are span/hull-closed. On the other hand it is easy (see Example 5.1 (viii)) to find continua in this class which are not span-open. Span-openness seems to be an elusive property: aside from a few *ad hoc* examples, the only general result along these lines makes the unremarkable assertion that irreducible indecomposable continua are span-open.

5.6 Examples.

(i) A continuum can be span-closed while not satisfying the ICP, so the converse of Corollary 5.3 is false. To see this, let $X = I \cup G$ be the $\sin \frac{1}{x}$-curve. Then X is span-closed, by Example 5.1 (x). For $n = 1, 2, \ldots$, let $a_n = \langle \frac{2}{(4n+1)\pi}, 1 \rangle$. Then $\langle a_n \rangle$ is a sequence in G converging to $a = \langle 0, 1 \rangle$. Let $\langle b_n \rangle$ and $\langle c_n \rangle$ be constant sequences, where $b_n = b = a$ and $c_n = c = \langle 0, -1 \rangle$. Then each $[a_n, b_n]$ contains the limiting arc I, and so $[a_n, c_n, b_n]$ always holds. But $[a, b] = \{a\}$ and $c \neq a$. Hence $[a, c, b]$ does not hold.

(ii) A continuum X is **unicoherent** if whenever X is the union of two subcontinua K, M, then $K \cap M$ is connected. We say that a continuum satisfies a property *hereditarily* if each nondegenerate subcontinuum satisfies the property. (For example, the continuum is hereditarily unicoherent if no two of its subcontinua have disconnected intersection; equivalently [4, Proposition 2.1], if each of its intervals is connected.) The continuum in (i) above is hereditarily unicoherent, as well as hereditarily decomposable, but it is not antisymmetric. Here is an example of a continuum (see Fig. 1) that is both hereditarily unicoherent and hereditarily antisymmetric,[12]
but which fails to be span-closed: Start with X the harmonic fan from Example 4.14 (i), so we have $X = H \cup \bigcup_{n=1}^{\infty} D_n$. For each $n \geq 1$, let $b_n = \langle 1, \frac{1}{n} \rangle$ (the end point of D_n other than the origin a), with

[12] Hereditary antisymmetry implies being hereditarily decomposable [5, Theorem 3.6]. Wilder [18] first showed that nondegenerate antisymmetric continua are decomposable if they are metrizable, but it is unknown whether nonmetrizable antisymmetric continua are always decomposable.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

$c_n = \langle \frac{1}{2}, \frac{1}{2}(\frac{1}{2n} + \frac{1}{2n+2}) \rangle$. Let C_n be the linear interval joining b_n and c_n, and now define the *harmonic rake* Y to be $X \cup \bigcup_{n=1}^{\infty} C_n$. With $c = \lim_{n\to\infty} c_n = \langle \frac{1}{2}, 0 \rangle$ and $C = \{c\} \cup \{c_n : n = 1, 2, \dots\}$, we have that C is a closed set in Y. However, $[C] = [a, c] \cup \bigcup_{n=1}^{\infty}(D_n \cup C_n)$. This set is not closed because it contains the sequence $\langle b_n \rangle$, which converges to $b \notin [C]$.

By Remark 4.17, every metric space satisfies the ICP. Hence Theorem 5.2 gives the following facts about M-betweenness.

5.7 Corollary.

(i) *In a metric space, spans of compact subsets are closed.*
(ii) *Every compact metric space is span-closed.*

5.8 Theorem.

(i) *Every metric space is span-bounded.*
(ii) *In any metric space, the span of a compact subset is both closed and bounded; hence a metric space with the Heine-Borel property is span-compact.*

Proof. To prove Item (i), let A be a bounded subset of the metric space $\langle X, \varrho \rangle$. Fix $M \geq \sup\{\varrho(x, y) : x, y \in A\}$, and let $u, v \in [A]$. Then we have $a, b, c, d \in A$ with $u \in [a, b]$ and $v \in [c, d]$. Then $\varrho(u, v) \leq \varrho(u, b) + \varrho(b, d) + \varrho(d, v)$, by two applications of the triangle inequality. Since $\varrho(a, u) + \varrho(u, b) = \varrho(a, b)$, we have $\varrho(u, b) \leq \varrho(a, b)$. Similarly, $\varrho(v, d) \leq \varrho(c, d)$; hence $\varrho(u, v) \leq \varrho(a, b) + \varrho(b, d) + \varrho(c, d) \leq 3M$. Item (ii) now follows from Item (i), Corollary 5.7 (i), and the fact that compact subsets of metric spaces are closed and bounded. □

5.9 Examples.

(i) Examples 4.7 (i,ii) are Banach spaces that are not hull-bounded; hence we cannot generally get the diameter of $[A]$ to equal that of A in the proof of Theorem 5.8 (i).
(ii) Consider again the Banach space c_0 from Remark 4.17. The supremum norm metric fails quite strongly to have the Heine-Borel property: nondegenerate M-intervals, known [6, Proposition 2.1] to be both closed and bounded, are never compact ([6, Example 4.19]); hence we cannot conclude in Theorem 5.8 (ii) that spans of compact−even finite−subsets are compact.

6. Closures and interiors of convex subsets

The last section was about how taking the span/hull affects topological properties; in this section we shift perspective and focus on how topological operators affect convexity. A TBS is said to be **closure-stable** if closures of convex subsets are convex; similarly we define a TBS to be **interior-stable**. In the classical theory of convexity, Euclidean spaces are well known to be stable in both senses.

6.1 Proposition. *Every hull-open TBS is interior-stable.*

Proof. Let $\langle X, [\cdot, \cdot, \cdot] \rangle$ be a hull-open TBS, with $C \subseteq X$ convex. Then $C^\circ \subseteq [C^\circ]^\omega \subseteq C$. Since $[C^\circ]^\omega$ is open, we have $C^\circ = [C^\circ]^\omega$; hence C° is convex. □

6.2 Remarks. Here are some easy applications of Proposition 6.1.

(i) With reference to Examples 5.1 (iv,ix): Strictly convex normed vector spaces and irreducible indecomposable continua are interior-stable.
(ii) With reference to Examples 5.1 (vii,x): The standard unit circle, with the intrinsic metric, and the $\sin \frac{1}{x}$-curve are interior-stable. (These facts can also be easily verified directly.)

6.3 Example. With reference to Example 5.1 (viii): The triod $X = A \cup B$ has a convex subset, namely A, whose interior, namely $A \setminus \{c\}$, is not convex. This serves as an example−for both the M- and the K-interpretations of betweenness−to show interiors of convex sets need not be convex.

6.4 Theorem. *If a TBS is LSC, then it is closure-stable.*

Proof. Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be a TBS that is LSC, with $C \subseteq X$ convex. Let $a, b \in C^-$, with $c \in [a, b]$. If U is any open neighborhood of c, then lower semicontinuity provides us with a neighborhood $V_a \times V_b$ of $\langle a, b\rangle \in X^2$ so that $[a', b'] \cap U \neq \emptyset$ for all $\langle a', b'\rangle \in V_a \times V_b$. Because both a and b are in C^-, both V_a and V_b intersect C. So there exist $a', b' \in C$ such that $[a', b']$ intersects U. Since C is convex, $U \cap C \neq \emptyset$. Hence $c \in C^-$, and we conclude that $[a, b] \subseteq C^-$. Therefore C^- is convex. □

6.5 Remark. Lower semicontinuity holds in the following circumstances; and hence closure stability holds, by Theorem 6.4:

(i) Unique geodesic metric spaces with the Heine-Borel property [6, Theorem 3.10].
(ii) Strictly convex normed vector spaces [6, Theorem 4.2].
(iii) Normed vector spaces of dimension ≤ 2 [6, Proposition 4.13].
(iv) Hereditarily unicoherent continua [6, Theorem 5.9].

Closure-stability in a hereditarily unicoherent continuum is much easier to prove than LSC: intervals are connected, and hence so are convex subsets. Thus the closure of a convex subset is a subcontinuum, and therefore convex.

Hereditary unicoherence is an important continuum-theoretic property that implies LSC; we claim another such property is aposyndesis. We know that aposyndetic continua are USC (Theorem 4.18); now we show they are LSC as well.

6.6 Theorem. *Every aposyndetic continuum is LSC, and hence closure-stable.*

Proof. Assume X is aposyndetic, with $a, b \in X$ and U an open set intersecting $[a, b]$. We need to find an open neighborhood $V_a \times V_b$ of $\langle a, b\rangle$ such that $[a', b']$ intersects U for all $\langle a', b'\rangle \in V_a \times V_b$. Fix $c \in [a, b] \cap U$. If $c \in \{a, b\}$, say $c = a$, then we may let V_a equal U, with V_b arbitrary. So assume $c \notin \{a, b\}$. By aposyndesis there are subcontinua K_a and K_b such that $a \in K_a^\circ$, $b \in K_b^\circ$, and $c \in X \setminus (K_a \cup K_b)$. Let $V_a = K_a^\circ$ and $V_b = K_b^\circ$. For any $\langle a', b'\rangle \in V_a \times V_b$, let K be a subcontinuum containing $\{a', b'\}$. Then $K_a \cup K \cup K_b$ is a subcontinuum containing $\{a, b\}$; hence $c \in [a, b] \subseteq K_a \cup K \cup K_b$. But $c \notin K_a \cup K_b$, so $c \in K$. Thus $c \in [a', b']$, and the proof of lower semicontinuity is complete. Closure-stability follows from Theorem 6.4. □

Aposyndetic continua satisfy a property somewhat stronger than closure-stability, as stated in Theorem 6.9 below. To formulate this, first define a subset C of a TBS $\langle X, [\cdot,\cdot,\cdot]\rangle$ to be **strictly convex** if for any $a, b \in C^-$, we have $[a, b] \setminus \{a, b\} \subseteq C$. Every strictly convex set is convex; also the closure of a strictly convex set is clearly strictly convex. Closed convex sets are strictly convex, but the open upper half-plane in $\mathbb{R}^2$ (with the Euclidean norm) is clearly M-convex without being strictly M-convex.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

6.7 Remark. Our terminology is justified because a normed vector space is strictly convex in the traditional sense if and only if its open unit ball—hence any open ball—is strictly linearly convex in the sense above. This further implies that every open ball is strictly M-convex because M-intervals and linear intervals then agree. Because M-intervals contain the corresponding linear ones, we conclude that a normed space is strictly convex if and only if each of its open balls is strictly M-convex.

6.8 Question. What should it mean for a TBS to be *strictly convex*? From the remark above, one could define a metric space to be strictly convex if each open ball is strictly M-convex, but it is unclear how to define strict convexity in the absence of a "standard open base," or whether this would be a useful notion outside of the context of normed vector spaces.

We now define a TBS to be **strictly closure-stable** if each of its convex subsets is strictly convex. It is easy to see that a strictly convex normed vector space is strictly closure-stable if and only if its dimension is at most one. In the case of continua, we have the following.

6.9 Theorem. *Every aposyndetic continuum is strictly closure-stable.*

Proof. Let X be an aposyndetic continuum, with $C \subseteq X$ a convex subset. Let $a, b \in C^-$; we wish to show that $[a, b] \setminus \{a, b\} \subseteq C$. So arbitrarily pick $c \in [a, b] \setminus \{a, b\}$, and use aposyndesis to find subcontinua K_a and K_b as in the proof of Theorem 6.6. Then there exist $a' \in C \cap K_a^\circ$ and $b' \in C \cap K_b^\circ$. If K is any subcontinuum containing $\{a', b'\}$, then we argue as before to infer that $c \in K$. Since C is convex, we have $c \in C$, as desired. □

6.10 Examples.

(i) The harmonic fan $X = H \cup \bigcup_{n=1}^{\infty} D_n$ from Example 4.14 (i) is antisymmetric and hereditarily unicoherent; so by Theorem 6.4 and Remark 6.5, it is LSC and therefore closure-stable. However, $\bigcup_{n=1}^{\infty} D_n$ is a convex subset which is not strictly convex; hence X is not strictly closure-stable. This continuum also demonstrates that we cannot replace *aposyndetic* with *antisymmetric* in Theorem 6.9, but one can do better: antisymmetric continua need not even be closure-stable, let alone strictly so (see the *harmonic needle cushion*, Example 6.13 (iii) below).

(ii) If X is an irreducible indecomposable continuum, then each of its composants is convex without being strictly convex. Indeed, pick composant A and let B be a composant different from A. If $a \in A$ and $b \in B$, then (because composants are dense) we have $a, b \in A^-$. However, $[a, b] \setminus \{a, b\} = X \setminus \{a, b\}$, a set with finite complement and therefore not contained in A. Irreducible indecomposable continua are often hereditarily unicoherent, and are hence closure-stable without being strictly closure-stable.

The converses of Theorems 4.18, 6.6, and 6.9 are all false, as witnessed by the following example.

6.11 Example. Let X be the *thick Warsaw circle* $Y \cup A \cup B$ (see Fig. 2), where $Y = G \cup I$ is the sin $\frac{1}{x}$-curve from Example 3.4 (iv), A is the rectangle $[-1, 0] \times I$, and B is an arc joining the end point of G to $\langle -\frac{1}{2}, -1 \rangle$ on the boundary of A and missing all other points of $Y \cup A$. It is straightforward to see that the K-betweenness structure of X is discrete, and hence X is trivially USC, LSC, and strictly closure-stable. On the other hand, no two points of I are aposyndetic relative to one another.

6.12 Question. What "reasonable" continuum-theoretic properties could one add to either USC, LSC, or strict closure-stability to infer aposyndesis?

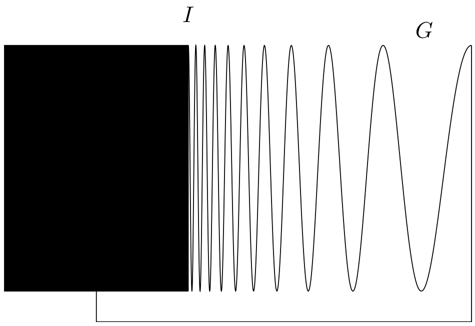

Fig. 2. The *thick Warsaw circle* from Example 6.11.

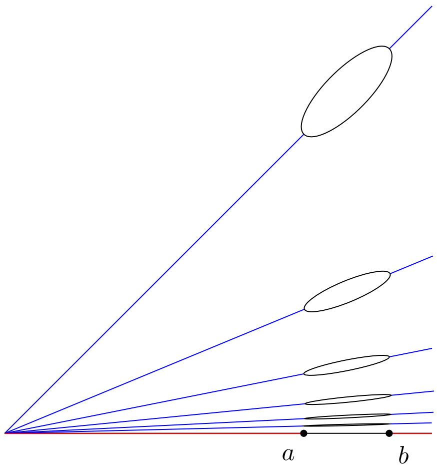

Fig. 3. The *Harmonic needle cushion* from Example 6.13 (iii). The blue region is convex. Its closure is the union of the blue and red regions. The closure is not convex as it does not contain the segment $[a, b]$.

We end with three examples in which closure-stability fails. The first two occur in the metric setting, the third in the setting of continua.

6.13 Examples.

(i) Let X be the unit circle from Example 5.1 (vii), with C a semicircle, minus an end point. Then C is convex, but C^- includes a pair of antipodal points without being all of X. Hence C^- is not convex.

(ii) Note that, as mentioned above in Remark 6.5, every normed vector space of dimension ≤ 2 is closure-stable. The dimension restriction is important because there is a norm $\|\cdot\|$ on $\mathbb{R}^3$ resulting in a space that fails to be closure-stable. To see this, let $P = \mathbb{R}^2 \times \{0\}$ be the xy-plane. Then—as explained in [6, Example 4.12]—the norm, when restricted to P, coincides with the supremum norm (i.e., $\|\langle x, y, 0\rangle\| = \max\{|x|, |y|\}$), and hence M-intervals with bracket points in P are almost always quadrilaterals. In all other situations, however, M-intervals are linear. So let H be the *open half-plane* $\{\langle x, 0, z\rangle : z > 0\}$. Then H is M-convex, while H^- is not: If $a = \langle 0, 0, 0\rangle$ and $b = \langle 1, 0, 0\rangle$ in H^-, then $[a, b]$ is the square in P with diagonal whose end points are a and b. So $[a, b] \not\subseteq H^-$.

(iii) For our continuum example, we construct the *harmonic needle cushion* (see Fig. 3) as a continuum that, while being neither aposyndetic nor unicoherent, is still antisymmetric and hereditarily decomposable. Start with the harmonic fan $X = H \cup \bigcup_{n=1}^{\infty} D_n$ (Example 4.14 (i)), and then replace a section of each arm D_n with an *eye*, in such a way that the eyes narrow with increasing n and converge to an arc. Formally we take the following steps: (1) write $H = A \cup B \cup C$, where $A = [0, \frac{1}{3}] \times \{0\}$, $B = [\frac{1}{3}, \frac{2}{3}] \times \{0\}$,

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783

and $C = [\frac{2}{3}, 1] \times \{0\}$; (2) for $n = 1, 2, \dots$, write $D_n = A_n \cup B_n \cup C_n$, where $A_n = \{\langle t, \frac{t}{n}\rangle : 0 \le t \le \frac{1}{3}\}$, $B_n = \{\langle t, \frac{t}{n}\rangle : \frac{1}{3} \le t \le \frac{2}{3}\}$, and $C_n = \{\langle t, \frac{t}{n}\rangle : \frac{2}{3} \le t \le 1\}$; (3) for $n = 1, 2, \dots$, set $V_n = \{\frac{1}{3}\} \times [\frac{1}{3n+3}, \frac{1}{3n}]$ and $W_n = \{\frac{2}{3}\} \times [\frac{2}{3n+3}, \frac{2}{3n}]$; for $n = 1, 2, \dots$, let $E_n = B_n \cup B_{n+1} \cup V_n \cup W_n$ (the nth *eye*); for $n = 1, 2, \dots$, set $F_n = A_n \cup E_n \cup C_n$ (the nth *needle*). Finally, we set $Y = H \cup \bigcup_{k=0}^{\infty} F_{2k+1}$. (Note: We are using only the odd-numbered needles to make our continuum Y, in order to avoid successive needles intersecting where they should not. In Fig. 3, the upper part of the−slightly rounded−eye E_n is B_n, and the lower part−also rounded−is $V_n \cup B_{n+1} \cup W_n$.)

The set $G = \bigcup_{k=0}^{\infty} A_{2k+1} \cup C_{2k+1}$ is convex because the eyes introduce convenient gaps. However, $G^- = G \cup (A \cup C)$ is not convex, because the "eye" belonging to H has degenerated into the arc B. Hence any interval $[a, c]$, with $a \in A$ and $c \in C$, has to include $B \nsubseteq G^-$.

References

[1] D. Aliprantis, K.C. Border, Infinite-Dimensional Analysis: A Hitchhiker's Guide, third ed., Springr-Verlag, Berlin, Heidelberg, 2006.
[2] D. Anderson, P. Bankston, Extreme points of a continuum, Topol. Appl. 257 (2019) 106–121.
[3] P. Bankston, Road systems and betweenness, Bull. Math. Sci. 3 (3) (2013) 389–408.
[4] P. Bankston, When Hausdorff continua have no gaps, Topol. Proc. 44 (2014) 177–188.
[5] P. Bankston, The antisymmetry betweenness axiom and Hausdorff continua, Topol. Proc. 45 (2015) 189–215.
[6] P. Bankston, A. McCluskey, R.J. Smith, Semicontinuity of betweenness functions, Topol. Appl. 246 (2018) 22–47.
[7] D. Bellamy, Indecomposable continua with one and two composants, Fundam. Math. 101 (1978) 129–134.
[8] R.H. Bing, Partitioning a set, Bull. Am. Math. Soc. 55 (1949) 1101–1110.
[9] M. Bridson, A. Haelfliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, New York-Berlin-Heidelberg, 1999.
[10] K. Królicki, P. Krupski, Wilder continua and their subfamilies as coanalytic absorbers, Topol. Appl. 220 (2017) 146–157.
[11] K. Kuratowski, Topology, Volume II, Academic Press, New York, 1968.
[12] K. Menger, Untersuchungen über allgemeine Metrik, Math. Ann. 100 (1928) 75–163.
[13] W. Rudin, Functional Analysis, second ed., McGraw-Hill, New York, 1991.
[14] E.E. Moise, Grille decomposition and convexification theorems for compact metric locally connected continua, Bull. Am. Math. Soc. 55 (1949) 1111–1121.
[15] S.B. Nadler Jr., Continuum Theory, an Introduction, Marcel Dekker, New York, 1992.
[16] W. Takahashi, A convexity in metric space and nonexpansive mappings, I, Kōdai Math. Semin. Rep. 22 (1970) 143–149.
[17] M.L.J. van de Vel, Theory of Convex Structures, North Holland, 1993.
[18] B.E. Wilder, Between aposyndetic and indecomposable continua, Topol. Proc. 17 (1992) 325–331.

Please cite this article in press as: D. Anderson et al., Convexity in topological betweenness structures, Topol. Appl. (2021), https://doi.org/10.1016/j.topol.2021.107783