CHAINABILITY AND HEMMINGSEN'S THEOREM

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Abstract. On the surface, the definitions of chainability and Lebesgue covering dimension ≤ 1 are quite similar as covering properties. Using the ultracoproduct construction for compact Hausdorff spaces, we explore the assertion that the similarity is only skin deep. In the case of dimension, there is a theorem of E. Hemmingsen that gives us a first-order lattice-theoretic characterization. We show that no such characterization is possible for chainability, by proving that if κ is any infinite cardinal and $\mathcal{A}$ is a lattice base for a nondegenerate continuum, then $\mathcal{A}$ is elementarily equivalent to a lattice base for a continuum Y, of weight κ, such that Y has a 3-set open cover admitting no chain open refinement.

1. introduction

Throughout this paper, all topological spaces are assumed to be Hausdorff; a **compactum** is a compact Hausdorff space, and a **continuum** is a connected compactum. A finitely-indexed open cover $\{U_j\}_{j<n}$ of a compactum X is called a **chain open cover** of X if $U_i \cap U_j \neq \emptyset$ exactly when $|i-j| \leq 1$. A compactum X is **chainable** if every open cover of X refines to a chain open cover of X. This is an obvious generalization of the notion of chainability for metric compacta, defined by saying that $\langle X, d\rangle$ is chainable just in case, for every $\varepsilon > 0$, there is a chain open cover of X consisting of sets of d-diameter $\leq \varepsilon$.

Clearly every chainable compactum is a continuum. Chainability is a central notion in the theory of continua; not only is there a rich source of chainable metrizable continua through limits of inverse systems of arcs and continuous surjections, but important continua are characterizable in terms of chainability. For example, only the arc (resp., the pseudo-arc) is chainable among the nondegenerate locally connected (resp., hereditarily indecomposable) metrizable continua. (See, e.g., [16] for more background.)

Every nondegenerate chainable continuum also has (Lebesgue) covering dimension 1. Of particular interest to us is the fact that the definition of being of covering dimension ≤ 1 bears a strong resemblance to that of being chainable: A compactum X is of covering dimension ≤ 1 just in case every open cover refines to a finite open cover with the property that no three sets in the refinement contain a point in common. Here we show that this definitional resemblance is very superficial, in the sense that there can be no Hemmingsen-style theorem for chainability. Recall Hemmingsen's remarkable characterization of being of covering dimension $\leq n$ for

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normal spaces (see [9, 12, 13]); we quote the relevant case $n = 1$: *$dim(X) \leq 1$ if and only if, for every 3-set open cover $\{U_0, U_1, U_2\}$, there is a 3-set open cover $\{V_0, V_1, V_2\}$ with $V_j \subseteq U_j$, $j < 3$, and $V_0 \cap V_1 \cap V_2 = \emptyset$.*

In the context of compacta, the textbook definitions of chainability and of covering dimension ≤ 1 are clearly expressible in lattice-theoretic terms involving open sets (or, dually, closed sets). More precisely, let L be the alphabet $\{\sqcup, \sqcap, \bot, \top, =\}$ of bounded lattices. For purely technical reasons we focus primarily on closed sets, letting $F(X)$ denote the L-structure of all closed subsets of X. In more familiar terms, $F(X)$ is a bounded lattice, where $\sqcup$ is interpreted as set union, etc. A central theme of this paper is the consideration of whether, in formulating a topological property lattice-theoretically, one may restrict attention to a topological base. We thus define a **lattice base** for a compactum X to be a sublattice $\mathcal{A}$ of $F(X)$, satisfying the condition that every closed set is an intersection of members of $\mathcal{A}$. What makes the notion of lattice base of key importance in our study is a representation theorem, largely due to H. Wallman [17]: *There is a (universal-existential) sentence in the first-order language over L, whose models are precisely the lattices that are isomorphic to the lattice bases for compacta.*

Let us now call a property $\mathfrak{P}$ of compacta **finitely expressible** (resp., ω_1-**expressible**) if there is a sentence φ in the first-order language over L (resp., the infinitary language over L that allows countable conjuctions and disjunctions) such that, for any compactum X, X has property $\mathfrak{P}$ just in case $\mathcal{A} \models \varphi$ for every lattice base $\mathcal{A} \subseteq F(X)$. (I.e., the sentence φ is **base invariant**.) So, for example, if $\mathfrak{P}$ is finitely expressible, and if X and Y are two compacta such that some lattice base for X is elementarily equivalent to (i.e., satisfies the same first-order sentences as) some lattice base for Y, then X has property $\mathfrak{P}$ just in case Y does.

It is not difficult to show (see Lemma 5.1 below) that the textbook definitions of chainability and of covering dimension ≤ 1 may be straightforwardly phrased as base-invariant infinitary sentences over L, making both properties ω_1-expressible. The Hemmingsen reformulation of the latter property, though, gives it the added status of finite expressibility. We show in the sequel (see Theorem 3.1 below) that chainability differs quite markedly in this regard.

Remark 1.1. In addition to covering dimension ≤ 1 (or $\leq n$, for fixed $n < \omega$), there are several familiar properties known to be finitely expressible: being a continuum, being a decomposable (resp., indecomposable, hereditarily indecomposable) continuum, and being a continuum of multicoherence degree $\leq n$. (See [4].) Properties known *not* to be finitely expressible include: being locally connected, being a hereditarily decomposable continuum, and being of large inductive dimension $\leq n$ ($n > 0$). (The negative result for local connectedness may be found in [10] and in [2], the other two appear in [4]. It is not presently known whether these properties are ω_1-expressible.)

2. CHARACTERIZING A WEAK FORM OF CHAINABILITY

In this paper we enlarge the list of properties that are not finitely expressible by adding chainability; as well as a host of related properties, almost for free.

Following [16], we define a continuous map $f : X \to Y$ between topological spaces to be **essential** if f is not homotopic to any constant map from X to Y

(and **inessential** otherwise); following [7], we define a topological space X to be **acyclic** if every map from X to the unit circle S^1 in the plane is inessential. Acyclic continua clearly may have high covering dimension, but we are interested only in those that are **curves**; i.e., continua of covering dimension ≤ 1. The class of acyclic curves contains all tree-like continua and thus is much larger than the class of chainable (i.e., arc-like) continua (see [7]). Furthermore, since there are acyclic curves that are not even tree-like (see [5]), the gap between being chainable and being an acyclic curve is quite wide indeed. In spite of this, there is a sense in which these two properties are very close, and we may explain this as follows.

For any $n < \omega$, we define a compactum X to be n**-chainable** if each n-set open cover of X admits a chain open refinement. It is clear that a compactum is chainable if and only if it is n-chainable for every n, and that the properties n-chainable, $n < \omega$, become more restrictive with increasing n. What is less obvious is that the gap between chainability and being an acyclic curve is precisely the gap between 4-chainability and 3-chainability. The proof that 4-chainability is equivalent to chainability will appear in a later paper; the other assertion is much more germane to the present topic, and we prove it below.

Theorem 2.1. *A compactum X is 3-chainable if and only if it is an acyclic curve.*

Proof. Assume first that X is an acyclic curve, and that $\mathcal{U} = \{U_0, U_1, U_2\}$ is a 3-set open cover of X for which we wish to obtain a chain open refinement. Since X is a curve, we may assume from Hemmingsen's theorem that $U_0 \cap U_1 \cap U_2 = \emptyset$.

Since X is, in particular, a normal Hausdorff space, we may find (see, e.g., Theorem 4.2 in [8]) a *partition of unity subordinated to* $\mathcal{U}$; i.e., a family $\{\lambda_j : X \to [0,1]\}_{j<3}$ of continuous maps such that:

(i) $\lambda_j(x) = 0$ for $x \in X \setminus U_j$; and
(ii) $\sum_{j<3} \lambda_j(x) = 1$ for all $x \in X$.

Let $[0,1]$ be the closed unit interval in the real line $\mathbb{R}$, and define the map $\lambda : X \to [0,1]^3$, into the unit cube, by the assignment $x \mapsto \langle \lambda_0(x), \lambda_1(x), \lambda_2(x)\rangle$. Then λ maps X into the triangle $T = \{\langle x_0, x_1, x_2\rangle \in [0,1]^3 : x_0 + x_1 + x_2 = 1, x_0x_1x_2 = 0\}$. Moreover, for each $j < 3$, if $W_j := \{\langle x_0, x_1, x_2\rangle \in T : x_j > 0\}$, we have $\lambda^{-1}[W_j] \subseteq U_j$.

Let $p : \mathbb{R} \to T$ be a universal covering map. Since X is acyclic, the map $\lambda : X \to T$ is inessential; consequently there is a "lifting" map $\overline{\lambda} : X \to \mathbb{R}$ such that $p \circ \overline{\lambda} = \lambda$. (See, e.g., [7].) Using the chainability of the image $\overline{\lambda}[X] \subseteq \mathbb{R}$, find a chain open cover $\{V_i\}_{i<m}$ of $\overline{\lambda}[X]$ that refines $\{p^{-1}[W_j]\}_{j<3}$. Then $\{\overline{\lambda}^{-1}[V_i]\}_{i<m}$ is a chain open cover of X that refines $\mathcal{U}$, witnessing the fact that X is 3-chainable.

For the converse, assume that X is 3-chainable. Then Hemmingsen's theorem implies that X has covering dimension ≤ 1, so it remains to prove that X is acyclic. Let $f : X \to S^1$ be a map that we wish to show is inessential, and let $p : \mathbb{R} \to S^1$ be a fixed universal covering map (e.g., $p(t) := \langle \cos t, \sin t\rangle$). Since the circle S^1 can be covered by three open arcs of length $< \pi$, we may use the 3-chainability of X to find a chain open cover $\{U_i\}_{i<m}$ of X such that all the images $f[U_i]$, $i < m$, lie in arcs of length $< \pi$. This allows us to lift the map f to a map $\overline{f} : X \to \mathbb{R}$ such that $p \circ \overline{f} = f$. Since the map $\overline{f}$ is inessential, so too is the composition map f. □

3. STATEMENT OF THE MAIN RESULT

Our ultimate goal, showing that (3-)chainability fails dramatically to be finitely expressible, is a proof of the following theorem.

Theorem 3.1. *If $\mathcal{A}$ is a lattice base for a nondegenerate continuum X, and if κ is any infinite cardinal, then there is a non-3-chainable continuum Y that has weight κ, and which has a lattice base elementarily equivalent to $\mathcal{A}$.*

Remark 3.2. Theorem 3.1 implies that any lattice base for the interval $[0, 1]$ is elementarily equivalent to a lattice base for a metrizable continuum X that is not 3-chainable. Being of covering dimension 1 as well, this continuum fails to be acyclic, and hence is not tree-like. This says that any topological property of the interval (such as chainability, tree-likeness, etc.) is not finitely expressible if it implies acyclicity for metrizable compacta. Since unicoherence is finitely expressible, the continuum X is unicoherent. Now unicoherence is equivalent to acyclicity in the class of locally connected continua [7]. Hence the continuum X fails to be locally connected, but has a lattice base elementarily equivalent to one for the interval. This gives an alternative proof of the finite inexpressibility of local connectedness.

We divide the proof of Theorem 3.1 into two parts. The first part, of independent interest, is the *elasticity theorem*, a topological/combinatorial result that may be seen (perhaps perversely) as an "anti-Hemmingsen" theorem for 3-chainability. The second part folds the elasticity theorem into an ultracoproduct argument, and is presented in the final section.

4. THE ELASTICITY THEOREM

Our aim in this section is a proof of the following elementary result related to 3-chainablity.

Theorem 4.1 (Elasticity)**.** *Let X be a nondegenerate connected normal space, with N any natural number. Then there is a 3-set open cover $\mathcal{U}$ of X such that no chain open cover refining $\mathcal{U}$ has fewer than N sets.*

Proof. Fix X and $N > 0$. Also fix a small positive $\varepsilon < \frac{1}{4}$, as well as a natural number n such that $3n \geq N$. For each integer k, we denote by I_k the closed interval $[k, k+1]$, and by J_k the slightly larger open interval $(k-\varepsilon, k+1+\varepsilon)$. For each $i \in \{0, 1, 2\}$, we let V_i be the union $\bigcup_{k<n} J_{3k+i}$ of n pairwise disjoint open intervals, so that $\{V_0, V_1, V_2\}$ is a 3-set open cover of the interval $[0, 3n]$.

Next, using the fact that X is normal and connected, we fix a continuous surjection $f : X \to [0, 3n]$ and set $U_i := f^{-1}[V_i]$, for $i \in \{0, 1, 2\}$. Then $\mathcal{U} := \{U_0, U_1, U_2\}$ is a 3-set open cover of X. Our objective is to show that if $\mathcal{W} := \{W_j\}_{j<m}$ is any chain open cover of X that refines $\mathcal{U}$, then $m \geq 3n$.

Using the normality assumption on X once again, we may find a partition of unity $\{\lambda_j : X \to [0,1]\}_{j<m}$ subordinated to $\mathcal{W}$; i.e., $\lambda_j(x) = 0$ for $x \in X \setminus W_j$, $j < m$, and $\sum_{j<m} \lambda_j(x) = 1$ for all $x \in X$. We now define the map $g(x) := \sum_{j<m}(j + \frac{1}{2})\lambda_j(x)$. Noting that the partition of unity is subordinated to a chain open cover, it is easy to verify that g is a map into the interval $[0, m]$, and that, for each $j < m$, $g^{-1}[I_j] \subseteq W_j$. Since $\mathcal{W}$ refines $\mathcal{U}$, we have a function $\xi : \{0, \ldots, m-1\} \to \{0, 1, 2\}$ with the property that $g^{-1}[I_j] \subseteq U_{\xi(j)}$, $j < m$.

Consider the map $h : X \to [0, 3n] \times [0, m]$, given by the assignment $h(x) := \langle f(x), g(x)\rangle$. We set $Y := h[X]$, a connected subset of the rectangle $[0, 3n] \times [0, m]$, which we are trying to prove is at least as high as it is wide.

For each $j < m$, let $Y^j := Y \cap ([0, 3n] \times I_j)$, the jth "horizontal slice" of Y, so that $Y = \bigcup_{j<m} Y^j$. It is easy to check that $Y^j = h[g^{-1}[I_j]]$, and hence that $Y^j \subseteq f[g^{-1}[I_j]] \times I_j \subseteq f[U_{\xi(j)}] \times I_j = V_{\xi(j)} \times I_j$.

Now, for $p < 3n$, $j < m$, we denote by R^j_p the rectangle $J_p \times I_j$. We let R be the union of all such rectangles that intersect Y, and where $p = 3k + \xi(j)$, for some $k < n$. Then R contains Y, and is a union of connected sets, each of which intersects Y. Since Y is connected, so too is R.

For each $j < m$, let $R^j := R \cap ([0, 3n] \times I_j)$, the jth horizontal slice of R. Then any $R^j_{3k+\xi(j)} \subseteq R^j$ must be contained in $V_{\xi(j)} \times I_j$. Thus the distance between any two rectangles in R^j must be at least $2 - 2\varepsilon > 1$. Since the map $f : X \to [0, 3n]$ is onto, the projection of R onto the first coordinate must be all of $[0, 3n]$. Thus, for each $p < 3n$, there exists some $j < m$ with $R^j_p \subseteq R^j$. So if we can show that each slice R^j contains at most one rectangle R^j_p, then we will have shown $m \geq 3n$.

Assume, on the contrary, that there is some $j < m$ and two distinct rectangles R^j_p and R^j_q in R^j. Because R is connected, there is a chain of rectangles $\{R^{j_k}_{i_k} : k < l\}$ such that $R^{j_0}_{i_0} = R^j_p$ and $R^{j_{l-1}}_{i_{l-1}} = R^j_q$. We may assume this chain has minimal length; in particular there are no repetitions in the list.

Let s and S be, respectively, the minimum and the maximum element of the set $\{j_k : k < l\}$. Since R^j_p and R^j_q cannot be linked by rectangles in the slice R^j, it must be the case that either $s < j$ or $S > j$. Both situations are treated similarly; we consider the case $s < j$. Fix $k < l$ such that $j_k = s$. Then $j_k = s < j = j_0 = j_{l-1}$, so clearly $0 < k < l - 1$. Consider the three consecutive rectangles, $R^{j_{k-1}}_{i_{k-1}}$, $R^{j_k}_{i_k}$, $R^{j_{k+1}}_{i_{k+1}}$ in our chain. Because of the minimality assumption, all three rectangles are distinct. And because the first two intersect, they cannot be in the same horizontal slice. Consequently, $j_{k-1} > j_k$. For the same reason, $j_{k+1} > j_k$.

Because both $R^{j_{k-1}}_{i_{k-1}}$ and $R^{j_{k+1}}_{i_{k+1}}$ intersect $R^{j_k}_{i_k}$, and both j_{k-1} and j_{k+1} exceed j_k, it must be the case that $j_{k-1} = j_{k+1}$. Hence $R^{j_{k-1}}_{i_{k-1}}$ and $R^{j_{k+1}}_{i_{k+1}}$ are in the same slice and are at distance ≤ 1 from each other. This means they must coincide, contradicting our minimality assumption. This completes the proof.

□

Remarks 4.2. (i) Theorem 4.1 answers affirmatively the question, posed in an earlier version of this paper, whether every nondegenerate chainable continuum is *elastically chainable*. This means that there exists an "elasticity number" $M < \omega$ such that, for every $N < \omega$, there is an M-set open cover with no refining N-set chain open cover. We now know the adverb *elastically* adds nothing new, and we can get away with $M = 3$ (the theoretical minimum) every time.

(ii) The elasticity theorem has gone through three stages of generalization. First we proved it for arcs ($M = 3$), as well as for chainable continua containing arcs ($M \leq 4$). The proof for the arc case was entirely combinatorial (inspired by a brief conversation that the second and fourth authors had had with K. Kunen [15]), and was enough to show that chainability is not finitely expressible. (In their preprint [11], K. P. Hart and B. J. van der Steeg independently show this fact about chainability by using spans of continua to prove that ultracopowers of arcs, via nonprincipal ultrafilters on ω, are not chainable.)

In the second stage, we were able to prove the elasticity theorem for all nondegenerate metrizable chainable continua ($M = 3$), as well as for all chainable continua containing nondegenerate metrizable continua ($M \leq 4$). The proof for the metrizable case involved the celebrated representation theorem, due to J. R. Isbell [14], stating that *all metrizable nondegenerate chainable continua are inverse limits of ω-indexed systems consisting of arcs and piecewise linear surjective bonding maps.* The combinatorial segment of the proof involved a "winding number" argument, and was rather lengthy. This version of the theorem, however, allowed as a corollary the statement of Theorem 3.1, restricted to the metrizable case.

The third stage, which we present here, is both simpler and more general than the previous one. While its proof retains much of the "winding number" flavor of its predecessor, it is not much longer than the proof for the arc case, and affords a good example of how the essence of a theorem is sometimes made more transparent through generalization.

5. PROOF OF THEOREM 3.1

In the sequel it will be convenient to define the collection of complements of a lattice base for a space X to be a **lattice open base** for X. (The class of isomorphic copies of lattice open bases for compacta is also axiomatizable: just replace the axioms for lattice (closed) bases with their lattice-theoretic duals.)

Lemma 5.1. *Let X be a continuum, with $\mathcal{B}$ a lattice open base for X. If $\{U_j\}_{j<n}$ is any chain open cover of X, then there is a chain open cover $\{B_j\}_{j<n}$ of X, consisting of members of $\mathcal{B}$, such that $B_j \subseteq U_j$, $j < n$.*

Proof. We invoke the covering characterization of normality (Theorem 6.1 in [8]) to obtain a finitely-indexed open cover $\{V_j\}_{j<n}$ of X such that the closure $\overline{V_j}$ is contained in U_j for each $j < n$. Because each $\overline{V_j}$ is compact and $\mathcal{B}$ is a lattice open base, we have a family $\{B_j\}_{j<n}$ of $\mathcal{B}$-basic open sets such that $\overline{V_j} \subseteq B_j \subseteq U_j$ for each $j < n$. $\{B_j\}_{j<n}$ is an open cover because it has an open cover for a refinement. If $|i-j| \geq 2$, then clearly $B_i \cap B_j = \emptyset$ because $\{B_j\}_{j<n}$ refines a chain of open sets. Finally, because X is connected, we have $B_j \cap B_{j+1} \neq \emptyset$ for $j < n-1$. This gives us the chain open cover we need.

□

To complete the proof of Theorem 3.1, we need to introduce the topological ultracoproduct construction. The most "topological" definition of this may be phrased in terms of the Stone-Čech compactification, as follows.

Given an indexed family $\{X_i\}_{i\in I}$ of compacta and an ultrafilter $\mathcal{D}$ on I, we let Y be the disjoint union $\bigcup_{i\in I}(X_i \times \{i\})$, with $q : Y \to I$ the map that takes a point in Y to its unique index in I. Letting $q^\beta : \beta(Y) \to \beta(I)$ be the lifting of q to the respective Stone-Čech compactifications, the **ultracoproduct** of the family relative to $\mathcal{D}$ is the inverse image under q^β of $\mathcal{D} \in \beta(I)$. This space, a closed subspace of $\beta(Y)$, is denoted $\sum_{\mathcal{D}} X_i$. When each X_i is the same space X, we have the **ultracopower** of X relative to $\mathcal{D}$, denoted $\sum_{\mathcal{D}} X$. In this case the disjoint union Y becomes the product space $X \times I$.

The connection between ultracoproducts of compacta and ultraproducts of algebraic structures may be made more precise in the following fundamental result (see [1, 3, 10]).

Lemma 5.2. *Let $\{X_i\}_{i\in I}$ be an I-indexed family of compacta, with $\mathcal{D}$ an ultrafilter on I. If, for each $i \in I$, $\mathcal{A}_i$ is a lattice base for X_i, then the ultraproduct lattice $\prod_{\mathcal{D}} \mathcal{A}_i$ is isomorphic to a lattice base for $\sum_{\mathcal{D}} X_i$.*

Let $\{X_i\}_{i\in I}$ be a family of compacta, with $\mathcal{D}$ an ultrafilter on an index set I. If $\mathcal{A}_i$ is a lattice base for X_i, $i \in I$, then, by Lemma 5.2, points in $\sum_{\mathcal{D}} X_i$ may be viewed as maximal filters in the ultraproduct lattice $\prod_{\mathcal{D}} \mathcal{A}_i$. If $S_i \subseteq X_i$ for each $i \in I$, then we denote by $(\prod_{\mathcal{D}} S_i)^\sharp$ the set of points $P \in \sum_{\mathcal{D}} X_i$ such that some member of P is a subset of $\prod_{\mathcal{D}} S_i$. Since ultracoproducts of sets from the lattice bases $\mathcal{A}_i$ form a lattice (closed) base for $\sum_{\mathcal{D}} X_i$, sets of the form $(\prod_{\mathcal{D}}(X \setminus A_i))^\sharp$, where each A_i is from $\mathcal{A}_i$, correspondingly form a lattice open base. Recall that an ultrafilter $\mathcal{D}$ on a set I is **countably incomplete** if there is a countble subfamily of $\mathcal{D}$ whose intersection is empty. All nonprincipal ultrafilters on a countable index set are countably incomplete.

Lemma 5.3. *Let $\mathcal{D}$ be a countably incomplete ultrafilter on I, and suppose that $\{X_i\}_{i\in I}$ is a family of infinite compacta. Then the ultracoproduct $\sum_{\mathcal{D}} X_i$ is not 3-chainable.*

Proof. If the spaces X_i are not connected, then neither is the ultracoproduct, and there is nothing to prove. So assume each X_i is a nondegenerate continuum.

We let $\{J_n\}_{n<\omega}$ be a countable strictly decreasing sequence of members of $\mathcal{D}$ such that $J_0 = I$ and $\bigcap_{n<\omega} J_n = \emptyset$. Then for each $i \in I$, we may define $n(i) := \max\{m < \omega : i \in J_m\}$; and we may also invoke Theorem 4.1 to obtain a 3-set open cover $\mathcal{U}_i$ of X_i that has no refining chain open cover of cardinality $< n(i)$. Set $\mathcal{U}_i := \{U_{i,j}\}_{j<3}$. Then $\{(\prod_{\mathcal{D}} U_{i,j})^\sharp\}_{j<3}$ is a 3-set open cover of $\sum_{\mathcal{D}} X_i$. If $\sum_{\mathcal{D}} X_i$ were 3-chainable, there would be a refining chain open cover (see Lemma 5.1) $\{(\prod_{\mathcal{D}} V_{i,j})^\sharp\}_{j<k}$. But then it would follow that
$\{i \in I : \{V_{i,j}\}_{j<k}$ is a chain open cover of X_i refining $\mathcal{U}_i\} \in \mathcal{D}$. This further implies $\{i \in I : n(i) \leq k\} \in \mathcal{D}$. But since $J_{k+1} \subseteq \{i \in I : n(i) \geq k+1\}$, and each J_m is a member of $\mathcal{D}$, we obtain a contradiction. We therefore conclude that $\sum_{\mathcal{D}} X_i$

is not 3-chainable.

□

We are now ready to finish the proof of Theorem 3.1, whose final argument combines Lemma 5.3 with an application of the Löwenheim-Skolem theorem from model theory.

So let X be any nondegenerate continuum, which we may as well take to be 3-chainable, and let $\mathcal{A}$ be a lattice base for X. Then $\mathcal{A}$ is clearly infinite. Let κ be a fixed infinite cardinal. We let I be a set of cardinality κ, with $\mathcal{D}$ a *κ-regular* ultrafilter on I; i.e., there exists a subfamily $\mathcal{E}$ of $\mathcal{D}$ that has cardinality κ, and such that no $i \in I$ is a member of infinitely many members of $\mathcal{E}$. Regular ultrafilters are specially designed (see [6]) to make ultraproducts large, and that is why we use them here.

Fix $\mathcal{C} \subseteq \mathcal{A}$, an infinite family of proper basic subsets of X, such that each pairwise union of members of $\mathcal{C}$ is X. We let $(\prod_{\mathcal{D}} \mathcal{C})^\sharp$ be the collection of all sets $(\prod_{\mathcal{D}} C_i)^\sharp$, where each C_i is a member of $\mathcal{C}$. then $(\prod_{\mathcal{D}} \mathcal{C})^\sharp \subseteq (\prod_{\mathcal{D}} \mathcal{A})^\sharp \subseteq F(\sum_{\mathcal{D}} X)$; and, because $\mathcal{D}$ is κ-regular, $(\prod_{\mathcal{D}} \mathcal{C})^\sharp$ is a collection of at least 2^κ proper basic closed subsets of $\sum_{\mathcal{D}} X$, such that each two of its members form a cover.

Since κ-regular ultrafilters are countably incomplete, we know, from Lemma 5.3, that $\sum_{\mathcal{D}} X$ is not 3-chainable. Now, in view of Lemma 5.1, the 3-chainability of a continuum may be decided using only the members of a given lattice base. So since $\sum_{\mathcal{D}} X$ is not 3-chainable, there is a finite subset $\mathcal{S}$ of $(\prod_{\mathcal{D}} \mathcal{A})^\sharp$ witnessing the fact; i.e., the complements of members of $\mathcal{S}$ form a 3-set open cover of $\sum_{\mathcal{D}} X$, admitting no refining chain open cover of $\sum_{\mathcal{D}} X$ that consists of sets whose complements are in $(\prod_{\mathcal{D}} \mathcal{A})^\sharp$.

Using the Löwenheim-Skolem theorem, we let $\mathcal{B}$ be an elementary sublattice of $(\prod_{\mathcal{D}} \mathcal{A})^\sharp$, of cardinality κ, such that $\mathcal{B}$ contains $\mathcal{S}$, as well as κ many members of $(\prod_{\mathcal{D}} \mathcal{C})^\sharp$. If Y is any compactum with an isomorphic copy of $\mathcal{B}$ as a lattice base (e.g., the maximal spectrum of $\mathcal{B}$), then Y is a non-3-chainable continuum that has a lattice base elementarily equivalent to $\mathcal{A}$. The weight of Y is at most κ because $\mathcal{B}$ is a lattice base of cardinality κ. The weight of Y is at least κ because Y has a family of κ pairwise disjoint nonempty open sets. This completes the proof of Theorem 3.1.

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