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DEFINING TOPOLOGICAL PROPERTIES VIA INTERACTIVE MAPPING CLASSES

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Abstract. We show that a compactum is locally connected if and only if every semimonotone mapping onto it is also monotone. If we put *open* in place of *monotone*, we obtain the finite compacta; if we put in *confluent*, we obtain a large class of compacta, the connected members of which are connected *im kleinen* at each of their cut points.

1. introduction

In what follows, a *compactum* is a nonempty compact Hausdorff space and a *continuum* is a compactum that is connected. We use the words *map* and *mapping* to refer to continuous functions between topological spaces; a mapping $f : X \to Y$ between compacta is:

- *open* if the image $f[U]$ of an open subset U of X is open in Y;
- *monotone* if the pre-image $f^{-1}[K]$ of a subcontinuum K of Y is a subcontinuum of X;
- *semimonotone* if whenever K is a subcontinuum of Y, there is a subcontinuum C of X such that $f[C] = K$ and such that $f^{-1}[U] \subseteq C$ for every open set U contained in K; and
- *confluent* if whenever K is a subcontinuum of Y, each component of $f^{-1}[K]$ is mapped by f onto K.

Remark 1.1. Clearly monotone mappings are semimonotone and confluent; and it is well known [11], if not obvious, that open mappings are confluent. The definition of semimonotonicity seems to be new, and is motivated by model-theoretic considerations. (See the discussion preceding Proposition 2.1.) Note that once the condition on open sets is removed, we obtain the classical notion of *weak confluence*.

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Continuum theorists (see, e.g., [6] and [11]) have long been interested in characterizing when a continuum is the image of other continua under mappings of only a certain kind. For example, there is the very satisfying result that a (metrizable) continuum Y is only a confluent image of other (metrizable) continua if and only if Y is hereditarily indecomposable [10]. In this paper, we extend this idea a little and introduce the notation Class $(\mathfrak{K};\mathfrak{F},\mathfrak{G})$, where $\mathfrak{K}$ is a class of compacta and $\mathfrak{F}$, $\mathfrak{G}$ are classes of mappings, to indicate those compacta $Y \in \mathfrak{K}$ such that whenever $X \in \mathfrak{K}$ and $f : X \to Y$ is a surjective mapping in $\mathfrak{F}$, then f is also in $\mathfrak{G}$. In continuum theory, attention has been traditionally confined to when $\mathfrak{K}$ is the class of (metrizable) continua, $\mathfrak{F}$ consists if the continuous functions, and $\mathfrak{G}$ varies among important subclasses of $\mathfrak{F}$ that contain the monotone maps. [For example the notation Class (C) is typically used to indicate this situation when $\mathfrak{G}$ is the class of confluent mappings.] In this paper we partially maintain the tradition; the only deviation we make is to fix $\mathfrak{F}$ as the class of semimonotone mappings. In the next three sections, we then consider $\mathfrak{G}$ to be the classes of monotone, open, and confluent mappings, respectively.

2. When semimonotone mappings are monotone

Since we are taking $\mathfrak{F}$ to be the semimonotone mappings, and since even weakly confluent mappings both preserve and reflect connectedness, we need only concern ourselves with fixing $\mathfrak{K}$ to be the class of all compacta. In this section we show that when $\mathfrak{G}$ is the class of monotone mappings, Class $(\mathfrak{K};\mathfrak{F},\mathfrak{G})$ is the class of locally connected compacta.

In [1] we introduced the co-existential mappings between compacta, in precise dualized analogy with the existential embeddings between relational structures in model theory. While existential embeddings are *defined* in terms of the satisfaction relation, they may be *characterized* in terms of ultrapowers (see, e.g., [5]). The ultrapower construction has a unique dual construction in the compact Hausdorff context, namely the topological ultracopower; and this enables the re-interpretation of *existential embedding* as a special property of surjective mappings.

To begin, we may define the ultracopower in a familiar topological fashion as follows. Given a compactum Y and a topologically discrete space I (the *index set*), we let $p : Y \times I \to Y$ and $q : Y \times I \to I$ be the standard projection maps. Then there are the Stone-Čech liftings $p^\beta : \beta(Y \times I) \to Y$ and $q^\beta : \beta(Y \times I) \to \beta(I)$; and if $\mathcal{D}$ is an ultrafilter on I, i.e., a point in $\beta(I)$, we define the *$\mathcal{D}$-ultracopower* $Y_\mathcal{D}$ to be the pre-image of $\{\mathcal{D}\}$ under the mapping q^β. A closed-set ultrafilter $\mu \in \beta(Y \times I)$ is in $Y_\mathcal{D}$ just in case $\bigcup_{i\in I}(C_i \times \{i\}) \in \mu$ for each I-indexed sequence $\langle C_i : i \in I\rangle$ of closed subsets of Y for which $\{i \in I : C_i = Y\} \in \mathcal{D}$. The restriction of p^β to $Y_\mathcal{D}$ is the associated *codiagonal mapping*, denoted $p_\mathcal{D}$, and is indeed surjective.

$Y_{\mathcal{D}}$ is very much "like" Y in certain ways (e.g., same covering dimension), but remarkably "unlike" Y in certain others (e.g., higher weight). See [3] for details.

A mapping $f : X \to Y$ is then called *co-existential* if there is an ultracopower $Y_{\mathcal{D}}$ of Y and a surjective map $g : Y_{\mathcal{D}} \to X$, such that $f \circ g = p_{\mathcal{D}}$. Trivially, codiagonal mappings are co-existential; the reader is referred to [4] for a summary of the many ways in which co-existential mappings "occur in nature." A weak corollary of Theorem 2.4 in [2] is the following.

Proposition 2.1. *Co-existential mappings are semimonotone.*

The fact that co-existential mappings to locally connected compacta are monotone has been known for years (see Theorem 2.7 in [2]); it is precisely semimonotonicity that one needs. So the first half of our characterization of local connectedness is as follows.

Proposition 2.2. *Let $f : X \to Y$ be a semimonotone mapping between compacta. If Y is locally connected, then f is monotone.*

Proof. First note that, since Y is a compactum, all we have to do is show $f^{-1}(y)$ (abbreviating $f^{-1}[\{y\}]$) is connected for each $y \in Y$. So if $\mathcal{U}_y$ is an open neighborhood base for $y \in Y$ consisting of connected sets and $U \in \mathcal{U}_y$, we may use semimonotonicity to choose a subcontinuum $C_U \supseteq f^{-1}[U]$ such that $f[C_U] = \overline{U}$ (overline denoting closure in Y). Clearly $f^{-1}(y) \subseteq \bigcap\{C_U : U \in \mathcal{U}_y\}$; if $x \notin f^{-1}(y)$ then there is some $V \in \mathcal{U}_y$ with $f(x) \notin \overline{V}$. Thus $x \notin C_V$, and we have $x \notin \bigcap\{C_U : U \in \mathcal{U}_y\}$. $f^{-1}(y) = \bigcap\{C_U : U \in \mathcal{U}_y\}$ is therefore connected, since $\{C_U : U \in \mathcal{U}_y\}$ is a family of subcontinua of X that is directed under reverse inclusion. Indeed, if $U, V \in \mathcal{U}_y$, we may pick $W \in \mathcal{U}_y$ such that $\overline{W} \subseteq U \cap V$. Then $C_W \subseteq f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V] \subseteq C_U \cap C_V$.

□

For the second half of our characterization, we need some notation. Suppose $\langle S_i : i \in I\rangle$ is an I-indexed family of subsets of a compactum Y, with $\mathcal{D}$ an ultrafilter on I. Then $\sum_{\mathcal{D}} S_i$ denotes the set of all $\mu \in Y_{\mathcal{D}}$ such that some member of μ is contained in $\bigcup_{i\in I}(S_i \times \{i\})$. When each S_i is closed in Y, $\sum_{\mathcal{D}} S_i$ is the closed set $Y_{\mathcal{D}} \cap \overline{\bigcup_{i\in I}(S_i \times \{i\})}$ (where overline indicates closure in $\beta(Y \times I)$). If each S_i is open in Y, so too is $\sum_{\mathcal{D}} S_i = Y_{\mathcal{D}} \setminus \sum_{\mathcal{D}}(Y \setminus S_i)$. If all sets S_i are equal to one set S, then $\sum_{\mathcal{D}} S_i$ is denoted $S_{\mathcal{D}}$; if each S_i is a singleton consisting of one point x_i, then $\sum_{\mathcal{D}} S_i$ is denoted $\sum_{\mathcal{D}} x_i$. ($x_{\mathcal{D}} \in Y_{\mathcal{D}}$, then, has its obvious meaning for $x \in Y$.) Finally, if $\mu \in Y_{\mathcal{D}}$ and $x \in Y$, then $x = p_{\mathcal{D}}(\mu)$ if and only if, for each open neighborhood U of x in Y, we have $\mu \in U_{\mathcal{D}}$. So, not unexpectedly, $p_{\mathcal{D}}(x_{\mathcal{D}}) = x$.

Proposition 2.3. *Let Y be a compactum that is not locally connected. Then there is an ultracopower of Y whose codiagonal mapping is not monotone.*

Proof. Suppose Y is a compactum that is not locally connected. Then there is a point $x \in Y$ at which Y is not connected *im kleinen* (see, e.g., [9]). This means there is an open neighborhood U of x such that for any open neighborhood V of x contained in U, there is some $y \in V$ such that no subcontinuum of U contains both x and y. Fix open W such that $x \in W \subseteq \overline{W} \subseteq U$. Then for any open V with $x \in V \subseteq \overline{W}$, there is some $y_V \in V$ such that no subcontinuum of $\overline{W}$ contains both x and y_V. Now, since $\overline{W}$ is a compactum, these points y_V are not in the same quasicomponent of $\overline{W}$ as is x. Thus for each V as above, there is a set H_V, clopen in $\overline{W}$, such that $y_V \in H_V$ and $x \notin H_V$.

Now let $\langle V_i : i \in I\rangle$ be an indexed collection of all open neighborhoods of x in X. Then, by the argument above, we have an open neighborhood W of x and indexed collections $\langle H_i : i \in I\rangle$ and $\langle y_i : i \in I\rangle$ such that, for each $i \in I$: H_i is clopen in $\overline{W}$; $y_i \in H_i \cap V_i$; and $x \notin H_i$.

For each $i \in I$ let $i^+ := \{j \in I : V_j \subseteq V_i\}$. Then clearly, by the fact that the sets V_i form a neighborhood base at x, the collection $\{i^+ : i \in I\}$ satisfies the finite intersection property and is hence contained in an ultrafilter $\mathcal{D}$ on I. It is now straightforward to show the following four assertions:

(1) $\sum_{\mathcal{D}} H_i$ is a clopen subset of $\overline{W}_{\mathcal{D}}$.

(2) $p_{\mathcal{D}}^{-1}(x) \subseteq \overline{W}_{\mathcal{D}}$.

(3) $x_{\mathcal{D}} \in p_{\mathcal{D}}^{-1}(x) \setminus \sum_{\mathcal{D}} H_i$.

(4) $\sum_{\mathcal{D}} y_i \in p_{\mathcal{D}}^{-1}(x) \cap \sum_{\mathcal{D}} H_i$.

Except for the assertion that $\sum_{\mathcal{D}} y_i \in p_{\mathcal{D}}^{-1}(x)$, all the others hold just because $\mathcal{D}$ is an ultrafilter on I. We infer that $p_{\mathcal{D}}(\sum_{\mathcal{D}} y_i) = x$ because if U is any open neighborhood of x, say $U = V_{i_0}$, then $\{i \in I : y_i \in U\} \supseteq i_0^+$, and thus is a member of $\mathcal{D}$. So $\sum_{\mathcal{D}} y_i \in U_{\mathcal{D}}$.

These four assertions immediately imply that $p_{\mathcal{D}}^{-1}(x)$ is disconnected; hence the codiagonal mapping cannot be monotone.

□

3. WHEN SEMIMONOTONE MAPPINGS ARE OPEN

Every mapping to a discrete space is open, and in the compact Hausdorff setting, discrete means finite. Since this is a very restrictive class of spaces, it is natural to ask whether infinite compacta always admit semimonotone mappings that are not open. There are several possible arguments to show an affirmative answer; one of the simplest is the proof of the following.

Proposition 3.1. *Let Y be an infinite compactum. Then there is an ultracopower of Y whose codiagonal mapping is not open.*

Proof. Suppose Y is an infinite compactum. Then there is a point $x \in Y$ that is not isolated. Let $\langle V_i : i \in I\rangle$ be an indexed collection of all open neighborhoods of x in Y; for each $i \in I$, let $i^+ := \{j \in I : V_j \subseteq V_i\}$. As in the proof of Proposition 2.3, let $\mathcal{D}$ be an ultrafilter on I extending $\{i^+ : i \in I\}$. Then $\sum_{\mathcal{D}} V_i$ is an open neighborhood of $x_{\mathcal{D}}$ in $Y_{\mathcal{D}}$. Suppose $\mu \in \sum_{\mathcal{D}} V_i$. Let U be any open neighborhood of x; say $U = V_i$. Then $\{j \in I : V_j \subseteq U\} = i^+ \in \mathcal{D}$, so $\sum_{\mathcal{D}} V_j \subseteq U_{\mathcal{D}}$. This tells us that $p_{\mathcal{D}}(\mu) = x$; hence the codiagonal map takes an open set to a nonisolated point and is therefore not an open mapping.

□

4. WHEN SEMIMONOTONE MAPPINGS ARE CONFLUENT

When $\mathfrak{K}$ is the class of all compacta, $\mathfrak{F}$ the class of semimonotone mappings, and $\mathfrak{G}$ the class of confluent mappings, Class $(\mathfrak{K}; \mathfrak{F}, \mathfrak{G})$ is a very large subclass of $\mathfrak{K}$, which includes: (i) the locally connected compacta (from Propositions 2.2 and 2.3); (ii) the zero-dimensional compacta (easy exercise: a compactum is zero-dimensional if and only if every mapping from a compactum onto it is confluent); and (iii) the hereditarily indecomposable continua [10]. While we do not at present have anything like a characterization of Class $(\mathfrak{K}; \mathfrak{F}, \mathfrak{G})$, we do know it is not all of $\mathfrak{K}$. This is a trivial consequence of the following analogue of Propositions 2.3 and 3.1, which itself is both a strengthening and a simplification of Theorem 5.1 in [4]. Recall that a point c of a continuum Y is a *cut point* if $Y \setminus \{c\}$ is disconnected.

Proposition 4.1. *Let Y be a continuum that is not connected* im kleinen *at some of its cut points. Then there is an ultracopower of Y whose codiagonal mapping is not confluent.*

Proof. Suppose Y is a continuum that is not connected *im kleinen* (abbreviated c.i.k.) at a cut point $c \in Y$. If B is a clopen subset of $Y \setminus \{c\}$, then $B \cup \{c\}$ is connected; hence we may write $Y = M \cup N$, where M and N are nondegenerate subcontinua of Y, with $M \cap N = \{c\}$.

Suppose, for the moment, that both M and N are c.i.k. at c. If U is an open neighborhood of c in Y, then there are sets $V_M \subseteq U \cap M$ and $V_N \subseteq U \cap N$, open neighborhoods of c in M and N respectively, such that for any $x \in V_M$ (resp., $x \in V_N$), there is a subcontinuum of $U \cap M$ (resp., $U \cap N$) that contains both c and x. But $V_M \cup V_N \subseteq U$ is an open neighborhood of c in Y; hence we have shown that Y is c.i.k. at c.

So if Y fails to be c.i.k. at the cut point c, then either M or N does as well; say it is M. By the proof of Proposition 2.3, there is an ultracopower $Y_{\mathcal{D}}$ of Y such that $p_{\mathcal{D}}^{-1}(c) \cap M_{\mathcal{D}}$ is disconnected. Now it is easy to show that $Y_{\mathcal{D}} = M_{\mathcal{D}} \cup N_{\mathcal{D}}$ and $M_{\mathcal{D}} \cap N_{\mathcal{D}} = \{c_{\mathcal{D}}\}$. Hence $p_{\mathcal{D}}$ maps a component of $p_{\mathcal{D}}^{-1}[N]$ to $\{c\}$, and thus cannot be confluent.

□

Remark 4.2. As mentioned earlier, ultracopowers of a compactum Y are similar to Y in many respects, with one major exception being weight. In particular, $Y_{\mathcal{D}}$ is almost never metrizable. This situation may be remedied in Propositions 2.3, 3.1 and 4.1, however, with the aid of model-theoretic techniques—particularly the Löwenheim-Skolem theorem—applied to lattices of closed sets (see, e.g., Theorem 3.1 in [2]; also [7] and [8]). Instead of ultracopower codiagonal maps, we obtain mappings $f : X \to Y$, where X is a compactum of the same weight as Y and f is "just as good as" $p_{\mathcal{D}}$, in the sense that it is a *co-elementary* map: there is a homeomorphism $h : X_{\mathcal{D}} \to Y_{\mathcal{E}}$ of ultracopowers such that $f \circ p_{\mathcal{D}} = p_{\mathcal{E}} \circ h$.

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