PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

CLOPEN SETS IN HYPERSPACES[1]

PAUL BANKSTON

ABSTRACT. Let X be a space and let $H(X)$ denote its hyperspace (= all nonempty closed subsets of X topologized via the Vietoris topology). Then X is Boolean (= totally disconnected compact Hausdorff) iff $H(X)$ is Boolean; and if B denotes the characteristic algebra of clopen sets in X then the corresponding algebra for $H(X)$ is the free algebra generated by B modulo the ideal which "remembers" the upper semilattice structure of B.

0. **Introduction.** This note concerns the algebraic topology of the hyperspace $H(X)$ of a compact Hausdorff space X. Two antithetical situations immediately arise, namely when X is connected (i.e., a continuum) and when X is totally disconnected (i.e., a Boolean space). In the first situation one can study the homotopy of $H(X)$. Indeed, in 1931 Borsuk and Mazurkiewicz [2] showed that $H(X)$ is path connected if X is metric; and in an unpublished paper, Banaschewski was able to remove the metrizability condition. Although the calculation of homotopy for $H(X)$ is still an open problem when X is a general continuum (even a metric continuum), the question has long been settled for X a Peano space (= locally connected metric continuum). In this case there is a beautiful succession of increasingly stronger results: $H(X)$ is Peano (Vietoris, 1923); $H(X)$ is contractible (Woydyslawski, 1938); $H(X)$ is an AR (Woydyslawski, 1939); and $H(X)$ is the Hilbert cube (Curtis and Schori, 1974). The historical details up to 1939 are in [2]; the last result is in [1]. Needless to say $H(X)$ has uninteresting homotopy when X is Peano.

Our interest here lies in the second of the above situations, that is where X is Boolean; and the algebraic object we study is the characteristic Boolean algebra $\chi(X)$ of clopen subsets of X. We then have a computational result which takes the following form: Let $B = \chi(X)$. Then $\chi(H(X))$ is the quotient of the free algebra generated by B divided by the ideal which "remembers" the upper semilattice structure of B. Although the theorem makes sense without requiring that $H(X)$ be Boolean, the proof we present requires this property; and in fact it is easy to show (modulo classical results) that X is Boolean iff $H(X)$ is Boolean as well.

Presented to the Society, November 15, 1974; received by the editors November 27, 1974.

AMS (MOS) subject classifications (1970). Primary 02J05, 54B20, 54F50, 54H10; Secondary 54F25, 55E05.

Key words and phrases. Hyperspace, Boolean space, Boolean algebra, clopen sets, characteristic algebra, Stone space.

[1] This work was carried out while the author was a Post-Doctoral Fellow (under the Canadian N.R.C.) at McMaster University, Hamilton, Ontario.

1. Preliminaries on hyperspaces.

1.1 DEFINITION. Let X be a space, $S \subseteq X$. Define $S^1 = \{C: C \subseteq X$ closed nonempty, $C \subseteq S\}$, and $S^2 = \{C: C \subseteq X$ closed nonempty, $C \cap S \neq \emptyset\}$. The *hyperspace* $H(X)$ of X consists of all closed nonempty subsets of X topologized by taking subbasic sets of the form U^1, U^2 for $U \subseteq X$ open. The resulting topology is called the *Vietoris topology*.

An interesting and helpful fact is the following:

1.2 LEMMA. *Let $r = \{U_1, \ldots, U_m\}$ be a finite set of open subsets of X and denote by $r\sharp$ the set $\{C \in H(X)$: $C \subseteq U_1 \cup \cdots \cup U_m$, and for $1 \leqq k \leqq m$, $C \cap U_k \neq \emptyset\}$. Then the collection of all such sets $r\sharp$ forms a basis for the Vietoris topology.*

PROOF. Let $r = \{U_1, \ldots, U_m\}$, $s = \{V_1, \ldots, V_n\}$, with $U = \bigcup r$, $V = \bigcup s$; and let t be the set $\{U \cap V, U_1 \cap V, \ldots, U_m \cap V, U \cap V_1, \ldots, U \cap V_n\}$. Then $t\,\sharp = r\,\sharp \cap s\,\sharp$, so the collection forms a basis for some topology τ. Now if $U \subseteq X$ is open then $U^1 = \{U\}\,\sharp$, $U^2 = \{U, X\}\,\sharp$, whence τ contains the Vietoris topology. On the other hand if $r = \{U_1, \ldots, U_m\}$ then $r\,\sharp = (U_1 \cup \cdots \cup U_m)^1 \cap U_1^2 \cap \cdots \cap U_m^2$, so τ is in fact the Vietoris topology itself. □

REMARKS. (i) For metrizable X, say with metric d, $H(X)$ is metrizable as well via the well-known Hausdorff metric over d; and the Vietoris topology is precisely the derived metric topology.

(ii) Let $f: X \to Y$ be a continuous closed map and define $H(f)$: $H(X) \to H(Y)$ in the obvious way, i.e. $H(f)(C) = f''C = \{f(x): x \in C\}$. Then $H(f)$ is continuous, indeed

$$H(f)^{-1}(\{U_1, \ldots, U_m\}\,\sharp) = \{f^{-1}(U_1), \ldots, f^{-1}(U_m)\}\,\sharp\,.$$

$H(f)$ need not be a closed map, however.

Assume all spaces henceforth to be T_1. Then for each X there is a natural injection $i: X \to H(X)$ taking $x \in X$ to its singleton $\{x\}$. i is evidently a topological embedding (and in the metric case an isometry). Moreover, if X is Hausdorff then i is closed as well. For suppose $C \in H(X) - i''X$, with $a, b \in C$ distinct. Let $U = U(a)$, $V = V(b)$ be a disjoint pair of open sets. Then $\{U, V, X\}\,\sharp$ is a nbd of C missing $i''X$.

Now a standard result of Vietoris (see [4]) is that X is compact Hausdorff iff $H(X)$ is compact Hausdorff. By extending this theorem we can easily prove that *X is Boolean iff $H(X)$ is Boolean.* To see this, assume $H(X)$ is Boolean. Then, since X embeds via i as a closed subset of $H(X)$, X is Boolean as well. Conversely if X is Boolean then X has a basis of clopen sets so that if C_1, $C_2 \in H(X)$ are distinct with, say, $a \in C_1 - C_2$, then there is a clopen U containing a and missing C_2. Thus $\{U, X\}\,\sharp$ is a clopen nbd of C_1 missing C_2, proving that $H(X)$ is "ultra-Hausdorff" hence (in view of compactness) Boolean.

An alternative description of $H(X)$ for X Boolean goes as follows (the straightforward details being left to the reader): Let B be the algebra of clopen sets in X and let $\sigma^*(B)$ be the set of proper filters in B. For $s = \{b_1, \ldots, b_n\} \subseteq B$ let $s\,\sharp = \{p \in \sigma^*(B)$: $b_1 \vee \cdots \vee b_n \in p$ and for $1 \leqq k \leqq n$, $b \in p$,

$b_k \wedge b \neq 0\}$. Then the sets $s\sharp$ form a topological basis; and the resulting space is precisely $H(X)$. The proof of this fact hinges upon the Stone duality between the closed nonempty subsets of a Boolean space and the proper filters of the corresponding Boolean algebra.

2. **The main theorem.** Let BTop denote the category of Boolean spaces and continuous maps with Boo the category of Boolean algebras and homomorphisms. Let

$$\text{BTop} \underset{\sigma}{\overset{\chi}{\rightleftarrows}} \text{Boo}$$

be the statement of Stone duality; where $\chi(X)$ is the characteristic algebra of clopen subsets of X, and $\sigma(B)$ is the Stone space of ultrafilters of B topologized by taking as basis sets all sets of the form $b^0 = \{u \in \sigma(B)\colon b \in u\}$ as b ranges over B. The Stone Duality Theorem says that χ and σ are contravariant natural equivalences. Now if X is compact, Y is Hausdorff and $f\colon X \to Y$ continuous then f is automatically a closed map. Thus in particular the hyperspace operator H: BTop$\to$BTop is functorial. A corollary of our theorem will be that there is an (algebraically defined) endofunctor H': Boo $\to$ Boo which makes the category-theoretic diagram

$$\begin{array}{ccc} \text{BTop} & \xrightarrow{H} & \text{BTop} \\ \chi \downarrow \uparrow \sigma & & \chi \downarrow \uparrow \sigma \\ \text{Boo} & \xrightarrow{H'} & \text{Boo} \end{array}$$

commutative.

We now define H'. Let B be a Boolean algebra. For ease of notation we will assume B to be a field of sets and so use the usual set-theoretic notation for the Boolean algebraic operations. Now let $F(B)$ denote the free Boolean algebra generated by the elements of B. In this context we will use the connectives of elementary logic to denote the operations in $F(B)$ and use square brackets to distinguish the set U in B from its "name" $[U]$ in $F(B)$. So if $B = \chi(X)$ for some $X \in$ BTop then $\{\cup, \cap, X - (\cdot), \varnothing, X\}$ denote the Boolean operations in B, whereas $\{\vee, \wedge, \neg, 0, 1\}$ denote the corresponding "formal" operations in $F(B)$. Typical elements of $F(B)$ include words of the form $[X]$, $[\varnothing]$, $[U]$, $[U \cap (X - V)] \vee \neg[W]$, etc.

Given $B = \chi(X)$ we define the ideals I_1, I_2 in $F(B)$ as follows:

I_1 is generated by the words $\{[U] \wedge \neg[V]\colon U \subseteq V \text{ in } B\} \cup \{([U] \wedge [V]) \wedge \neg[U \cap V]\colon U, V \in B\} \cup \{[\varnothing], \neg[X]\}$. I_2 is generated by the words $\{[U] \wedge \neg[V]\colon U \subseteq V \text{ in } B\} \cup \{[U \cup V] \wedge \neg([U] \vee [V])\colon U, V \in B\} \cup \{[\varnothing], \neg[X]\}$.

Intuitively I_1 (resp. I_2) "remembers" the lower (resp. upper) semilattice structure of B so that $F(B)/I_1$, say, "believes" that $[U] \leqq [V]$ whenever $U \subseteq V$, and that $[U] \wedge [V] = [U \cap V]$.

Now define the homomorphisms h_1, h_2 from $F(\chi(X))$ to $\chi(H(X))$ as follows: Let $U \in \chi(X)$. Then we set $f_1(U) = U^1$, $f_2(U) = U^2$. (*Note.* U^1, U^2 are clopen in $H(X)$ since $H(X) - U^1 = (X - U)^2$, etc.) Since $F(\chi(X))$ is free, f_1, f_2 extend uniquely to homomorphisms h_1, h_2. We can now state our main theorem thusly:

2.1 THEOREM. *Let* $X \in$ BTop *with* $h_1, h_2\colon F(\chi(X)) \to \chi(H(X))$ *given as above. Then*:

(i) *Both* h_1, h_2 *are epimorphisms.*

(ii) Ker $h_1 = I_1$, Ker $h_2 = I_2$ *whence*

$$\chi(H(X)) \cong F(\chi(X))/I_1 \cong F(\chi(X))/I_2.$$

Thus if B is any Boolean algebra, if we let $H'(B)$ be either of the above quotients, and if $g\colon B_1 \to B_2$ is a homomorphism, let $H'(g)$ be the obvious quotient homomorphism. Then

(iii) *The diagram*

$$\begin{array}{ccc} \text{BTop} & \xrightarrow{H} & \text{BTop} \\ \chi \downarrow \uparrow \sigma & & \chi \downarrow \uparrow \sigma \\ \text{Boo} & \xrightarrow{H'} & \text{Boo} \end{array}$$

commutes (*up to natural equivalences*).

(iv) *Let* $j\colon F(\chi(X)) \to \chi(X)$ *be the natural projection. Then* Ker $j = I_1 \vee I_2$ = *the ideal generated by* $I_1 \cup I_2$.

PROOF. (i) We show that the algebra A generated by the sets U^1, $U \in \chi(X)$, is all of $\chi(H(X))$. Indeed by compactness the sets $s\sharp$ form a basis for $H(X)$ as s ranges over the finite subsets of $\chi(X)$. Also since

$$s\,\sharp = \{U_1, \ldots, U_m\}\,\sharp = (U_1 \cup \cdots \cup U_m)^1 \cap (U_1^2 \cap \cdots \cap U_m^2)$$

we have that the U^1's generate a basis for the Vietoris topology. Since every clopen set in $H(X)$ is compact and is a union of elements from A, it is a union of finitely many elements from A and is thus itself in A. Thus $A = \chi(H(X))$.

(ii) Since $\varnothing^1 = \varnothing$, $X^1 = X$, $U^1 \subseteq V^1$ for $U \subseteq V$, and $(U \cap V)^1 = U^1 \cap V^1$ for all $U, V \in \chi(X)$, we have Ker $h_1 \supseteq I_1$. Similarly Ker $h_2 \supseteq I_2$. Let $w \in F(\chi(X))$ and assume w is represented as a disjunction of conjunctions of generators and complements of generators. Such a conjunction we refer to as a *minterm.* If $w = w_1 \vee \cdots \vee w_n$ is a disjunction of minterms, and if the minterms of Ker h_1 are in I_1 then Ker $h_1 \subseteq I_1$. For $w \in$ Ker $h_1 \Rightarrow w_k \in$ Ker h_1, each $1 \leqq k \leqq n$. Thus $w \in I_1$. So it suffices to prove the inclusion for minterms, of which there are three kinds: positive (only unnegated generators occur), negative, and mixed.

Positive. $w = [U_1] \wedge \cdots \wedge [U_m] \in$ Ker h_1. Then $U^1 \cap \cdots \cap U_m^1 = (U_1 \cap \cdots \cap U_m)^1 = \varnothing$ iff $U_1 \cap \cdots \cap U_m = \varnothing$. Now

$$([U_1] \wedge \cdots \wedge [U_m]) \wedge \neg[U_1 \cap \cdots \cap U_m]$$
$$= [U_1] \wedge \cdots \wedge [U_m] \wedge \neg[\varnothing] \in I_1.$$

But $[\varnothing] \in I_1$, so $[U_1] \wedge \cdots \wedge [U_m] \in I_1$ as well.

Negative. $w = \neg[V_1] \wedge \cdots \wedge \neg[V_n] \in$ Ker h_1. Then $(X - V_1)^2 \cap \cdots \cap (X - V_n)^2 = \varnothing$ iff some $V_k = X$. But $\neg[X] \in I_1$, so $w \in I_1$ too.

Mixed. $w = [U_1] \wedge \cdots \wedge [U_m] \wedge \neg[V_1] \wedge \cdots \wedge \neg[V_n] \in$ Ker h_1. Then

$$(U_1 \cap \cdots \cap U_m)^1 \cap (X - V_1)^2 \cap \cdots \cap (X - V_n)^2 = \varnothing$$

whence $U_1 \cap \cdots \cap U_m \subseteq V_l$ for some $1 \leqq l \leqq n$. Thus $[U_1 \cap \cdots \cap U_m] \wedge \neg[V_l] \in I_1$ so $([U_1] \wedge \cdots \wedge [U_m]) \wedge \neg[V_l] \in I_1$ and therefore $w \in I_1$. The proof that $\operatorname{Ker} h_2 = I_2$ is similar.

(iii) This follows straightforwardly from (ii).

(iv) This is proved in the same way as (ii). □

References

1. D. Curtis and R. Schori, *2^X and $C(X)$ are homeomorphic to the Hilbert cube*, Bull. Amer. Math. Soc. **80** (1974), 927–931.

2. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36. MR **3**, 315.

3. J. Nagata, *Modern dimension theory*, Bibliotheca Math., vol. 6, Interscience, New York, 1965. MR **34** #8380.

4. K. Kuratowski, *Topology*. Vol. 1, 5th ed., PWN, Warsaw; Academic Press, New York, 1966. MR **36** #840.

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada