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CORRIGENDUM TO 'SOME OBSTACLES TO DUALITY IN TOPOLOGICAL ALGEBRA'

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I wish to correct an error in [**1**]; and, in doing so, improve on a result contained therein. I am grateful to Evelyn Nelson for pointing out the error, which is the following.

Theorem 4.4 of [**1**] states that if $\mathscr{A}$ is a full subcategory of the category of compact Hausdorff spaces and continuous maps satisfying: (i) $\mathscr{A}$ is closed-hereditary and closed under usual (Tichonov) products; and $\mathscr{A}$ is category dual to a class $\mathscr{L}$ of finitary algebras which has fewer than c (the cardinality of the continuum) distinguished operations and which has representable underlying set functor, then every object of $\mathscr{A}$ is totally disconnected.

The proof incorrectly assumes that whenever $\mathscr{L}$ satisfies the above conditions then the free $\mathscr{L}$-algebra $F^{\mathscr{L}}$ over a singleton set (guaranteed by representability) must have fewer than c elements. (This would definitely be true if $\mathscr{L}$ were, say, an S-class, i.e., closed under subalgebras; for then $F^{\mathscr{L}}$ would be a homomorphic image of the corresponding absolutely free algebra F on one generator. The assumption would also be justifiable if $\mathscr{L}$ were an elementary class, i.e., the class of models of a set of sentences in the appropriate first order language. To see this, let $\phi:F \rightarrow F^{\mathscr{L}}$ be the natural homomorphism and let A be the image of F under ϕ. By the Löwenheim-Skolem Theorem there is an elementary subalgebra B of $F^{\mathscr{L}}$ containing A and of the same cardinality as F. Since $B \in \mathscr{L}$, it follows that $B = F^{\mathscr{L}}$.)

The following simple result refutes only the above assumption, not the theorem itself. (However, the validity of that theorem is definitely shrouded in doubt.)

1. Proposition. *Let $\mathscr{L}_C$ be the class of unitary rings $C(X)$ of continuous real-valued functions with topological domains. Then the underlying set functor has a left adjoint (in particular there exists a free $\mathscr{L}_C$-algebra over a singleton); however no nontrivial ring in $\mathscr{L}_C$ has fewer than c elements.*

Proof. The reader is referred to [**2**] for ample background on rings of continuous functions.

Let I be a given set. Then we claim that $C(\mathbf{R}^I)$ is the free $\mathscr{L}$-algebra over I, where $\mathbf{R}^I$ is the I-fold Tichonov power of the real line $\mathbf{R}$. To see this we need two standard facts which can be found in [**2**].

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(i) If $A \in \mathscr{L}_C$ then $A = C(X)$ for some realcompact Tichonov space X.

(ii) If X and Y are realcompact Tichonov and $\phi:C(Y) \to C(X)$ is any unital ring homomorphism then there is a unique continuous $f:X \to Y$ such that, for any $g \in C(Y)$, $\phi(g) = g \circ f$ (i.e., $\phi = C(f)$).

Now let $A \in \mathscr{L}_C$ be $C(X)$ for some realcompact Tichonov space X and let $g:I \to C(X)$ be given. The "insertion of generators" function is $p:I \to C(\mathbf{R}^I)$, given by $p(i) =$ the i^{th} projection map $p_i:\mathbf{R}^I \to \mathbf{R}$. We need a unique unital ring homomorphism $\phi:C(\mathbf{R}^I) \to C(X)$ such that $g = \phi \circ p$. Now $g_i = g(i)$ is a continuous map from X to $\mathbf{R}$ for each $i \in I$, so there is a unique continuous $f:X \to \mathbf{R}^I$ such that $g_i = p_i \circ f$, $i \in I$. Let $\phi = C(f)$. Then

$$(\phi \circ p)(i) = \phi(p_i) = C(f)(p_i) = p_i \circ f = g(i),$$

so ϕ does what we want. The uniqueness of ϕ is ensured by (ii) above.

2 *Remark.* As noted before, if we were to weaken the hypotheses of Theorem 4.4 of [**1**] by insisting that $\mathscr{L}$ also be either a S-class or an elementary class then the original proof would be valid. Moreover, that same proof would work with "compact Hausdorff" replaced by "realcompact Tichonov". Since the category of such spaces is dual to $\mathscr{L}_C$ we see the necessity for some kind of subalgebra condition to obtain the conclusion.

REFERENCES

1. P. Bankston, *Some obstacles to duality in topological algebra*, Can. J. Math. *34* (1982), 80-90.

2. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, 1960).

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