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Betweenness-Induced Convexity in Hyperspaces of Normed Vector Spaces

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Dedicated to Ralph Tyrell Rockafellar on the occasion of his 90th birthday.

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Using Minkowski addition of sets, we study linear betweenness in the hyperspace $L(X)$ of linearly convex nonempty subsets of a normed real vector space X, as well as in the sub-hyperspace $KL(X)$ of compact elements of $L(X)$. We also study the metric betweenness relation induced by the Hausdorff metric on the latter. While linear betweenness in $L(X)$ behaves reasonably like linear betweenness at the point level, the analogy is not perfect: linear intervals in X are honest line segments; this is no longer the case for $L(X)$, where linear intervals can have exactly two elements. However, when we restrict our focus to $KL(X)$, the Rådström extension theorem allows us to view this hyperspace as a linearly convex cone in a normed vector space $\mathcal{R}(X)$; in particular, all linear intervals are line segments that are contained in the corresponding metric intervals.

We are especially interested in the notions of convexity induced by these two kinds of betweenness relation. While all closed balls and metric intervals in $KL(X)$ are linearly convex, metric convexity has more nuanced behaviour. For example, the metric intervals in $KL(X)$ determined by singletons are all metrically convex if and only if X is strictly convex. When X is one-dimensional, $\mathcal{R}(X)$ is Cartesian 2-space equipped with the *max* norm and $KL(X)$ looks like the half-plane $\{\langle x, y\rangle : x \leq y\}$. In particular, all metric intervals – and no closed balls of positive radius – are metrically convex. When X is multi-dimensional, though, while it is still the case that closed balls are metrically nonconvex, it is now always possible to find a metrically nonconvex metric interval that is determined by a singleton and a line segment.

Keywords: Betweenness, betweenness axioms, convexity, vector spaces, Minkowski addition, metric spaces, normed vector spaces, strict convexity, hyperspaces, Hausdorff metric, Rådström extension.

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1. Introduction

If X is a real vector space – assumed throughout to be nondegenerate (i.e., having more than one point, and hence of positive dimension) – with $a, b \in X$, we say that

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a point $c \in X$ is *linearly between* a and b if there is a scalar $0 \leq t \leq 1$ such that $c = (1-t)a + tb$. Analogously, if $X = \langle X, \varrho \rangle$ is a metric space, we say that c is *metrically between* a and b if $\varrho(a,c) + \varrho(c,b) = \varrho(a,b)$.

Both linear and metric betweenness have been studied extensively in [6] and elsewhere over the years. The points linearly (resp., metrically) between a and b constitute the *linear* (resp., *metric*) *interval* $[\![a,b]\!]$ (resp., $[a,b]$) *bracketed* by a and b. When $X = \langle X, \|\cdot\| \rangle$ is a normed vector space with $\varrho(a,b) := \|a-b\|$, both notions of betweenness are present in the same structure, and we have an opportunity to study them as they interact.

Any time a set X is equipped with a ternary "betweenness" relation $[\cdot,\cdot,\cdot]$ there is an induced notion of *convexity*: $A \subseteq X$ is *convex* if whenever $a, b \in A$ and $[a,x,b]$ holds – i.e., *x lies between* a and b – then $x \in A$ as well.[1] Of major interest to us here are the convexity notions induced by linear and metric betweenness. Classical convexity in vector spaces is, of course, what we call *linear convexity*; metric convexity in normed vector spaces is a much stronger property, as we will see.

Both linear and metric betweenness make sense at the set level. In the linear case, owing to the usual pointwise addition – introduced by Hermann Minkowski – and scalar multiplication of sets of vectors (i.e., $A + B := \{a + b : a \in A \text{ and } b \in B\}$, etc.), we define C to be *linearly between* A and B if there is a scalar $0 \leq t \leq 1$ such that $C = (1-t)A + tB$. In the metric case, if A, B and C are nonempty and compact, the metric ϱ gives rise to the Hausdorff metric ϱ_H for such sets, and we say that C is *metrically between* A and B if $\varrho_H(A,C) + \varrho_H(C,B) = \varrho_H(A,B)$. As at the level of points, we use the same interval notation for sets; being careful to distinguish, say, $[\{a\},\{b\}]$ from $[a,b]$, as these are entirely different objects.

Linear betweenness at the set level has its idiosyncrasies, but behaves more like its counterpart at the point level when we restrict attention to the hyperspace $L(X)$ of nonempty sets that are linearly convex. Furthermore, metric betweenness behaves reasonably well when restricted to the hyperspace $K(X)$ of nonempty sets that are compact. Hence, in this paper we will be most interested in the joint behaviour of these two betweenness notions when restricted to the hyperspace $KL(X) := K(X) \cap L(X)$ of compact, linearly convex, nonempty subsets of a normed vector space X.

A quick summary of what we cover here runs as follows: In Section 2 we define basic concepts; one highlight is that if X is a normed vector space and $A, B \in L(X)$ (resp., $A, B \in KL(X)$), then $\bigcup [\![A,B]\!]$ is also a member of $L(X)$ (resp., $KL(X)$).

Section 3 is about first-order betweenness axioms that all hold for linear betweenness at the point level (where linear intervals are either singletons or real line segments), and we show that most of them hold for $L(X)$. For example, linear betweenness in $L(X)$ satisfies *convexity* (i.e., linear intervals are themselves linearly convex) but it does not satisfy *gap-freeness* (i.e., linear intervals with distinct bracket points always contain at least three elements): one can always find $A, B \in L(X)$ distinct such that $[\![A,B]\!] = \{A, B\}$. In particular, linear intervals in $L(X)$ are not necessarily line segments in the usual sense.

[1] Abstract convex structures induced by suitable notions of betweenness are called *interval structures* in [20].

In Section 4 we move into the metric betweenness relation in hyperspaces – especially in $KL(X)$ – and consider the simplest case, where the bracket points are singletons. We show that for $a, b \in X$ and $A \in KL(X)$, A is in $[\{a\}, \{b\}]$ if and only if $A \subseteq [a, b]$ and all elements of A are the same distance from a (or b). This is then used to show that a normed vector space X is strictly convex if and only if each metric interval with singleton bracket points is metrically convex, if and only if each metric interval with singleton bracket points coincides with its corresponding linear interval.

Section 5 brings linear and metric betweenness together for the first time in $K(X)$. While $[\![A, B]\!]$ need not always be contained in $[A, B]$, as is the case at the point level, the containment does hold if one of the bracket points is a singleton.

Rådström's famous extension theorem, whereby $KL(X)$ is naturally embedded as a linearly convex cone in a normed vector space $\mathcal{R}(X)$, is introduced in Section 6. An immediate consequence of this is the fact that linear intervals are always contained in their respective metric intervals; furthermore, both the linear and metric betweenness structures of $KL(X)$ satisfy the gap-freeness axiom. Another consequence, though not as immediate, is the fact that if $A, B \in KL(X)$, then $\bigcup[A, B]$ is in $L(X)$. And if X is finite-dimensional, then $\bigcup[A, B]$ is in $K(X)$ as well.

The simple case of $\mathcal{R}(X)$ when X is unidimensional is the topic of Section 7. Here $\mathcal{R}(X)$ is the Cartesian 2-space $\mathbb{R}^2$, equipped with the *max* norm (i.e., $\|\langle x, y\rangle\| = \max\{|x|, |y|\}$), and $KL(X)$ embeds as the half-plane $\{\langle x, y\rangle : x \leq y\}$. In particular: (1) $\mathcal{R}(X)$ is a two-dimensional Banach space; (2) the embedded copy of $KL(X)$ has nonempty interior and is metrically convex in $\mathcal{R}(X)$; and (3) every metric interval in $KL(X)$ is metrically convex.

Finally, in Section 8, we examine the basic features of $\mathcal{R}(X)$ from Section 7 when X is multidimensional. In contrast to the unidimensional case we have the following "bad" behaviour: (1) $\mathcal{R}(X)$ is an uncountable-dimensional normed vector space that is not a Banach space; (2) the embedded copy of $KL(X)$ has empty interior and is not metrically convex in $\mathcal{R}(X)$; and (3) it is always possible to find $A, B \in KL(X)$ – where one bracket point is a singleton and the other is a line segment – such that $[A, B]$ is not metrically convex. In this section we also relate metric convexity in closed balls with the relative lengths of line segments on their boundary spheres. In particular, no closed ball of positive radius in $\mathcal{R}(X)$ – regardless of dimension – can be metrically convex because its boundary sphere contains a line segment whose length exceeds that radius.

2. Preliminaries

As stated in the Introduction, this paper is about nondegenerate – i.e., positive-dimensional – vector spaces over the real scalar field $\mathbb{R}$. Given the vector space X and $a, b \in X$, we define the *linear interval* $[\![a, b]\!]$, with *bracket points* a and b, to be the usual line segment $\{(1-t)a + tb : 0 \leq t \leq 1\}$. Points in this line segment are said to be *linearly between* a and b. The doubleton $\{a, b\}$ is also called a *bracket set* for the interval; in the linear case[2] this set is unique and comprises the end points of the line segment. Note that $[\![a, a]\!]$ is always the degenerate set $\{a\}$.

[2] There are many others; see, e.g., [1, 2, 4, 5, 6, 7, 9].

In the metric space $\langle X, \varrho\rangle$, with points $a, b \in X$, we essentially follow Menger [15] and define the *metric interval* $[a,b] = [a,b]_\varrho$, with *bracket points* a and b, to be the set $\{x \in X : \varrho(a,x) + \varrho(x,b) = \varrho(a,b)\}$. This is the zero set of the continuous map $x \mapsto \varrho(a,x) + \varrho(x,b) - \varrho(a,b)$, and is the set of points *metrically between* a and b. It is automatically a closed subset of X in the metric topology. The metric interval $[a,b]$ is also bounded; namely its diameter is $\varrho(a,b)$ [6, Proposition 3.1]. Bracket sets are not necessarily unique, but separate bracket sets are disjoint: Indeed, suppose $[c,d] = [a,b]$. Then $\varrho(a,b) = \varrho(c,d)$. If it so happened that there was intersection, say $b = c$, we would then have $\varrho(a,d) + \varrho(d,b) = \varrho(a,b) = \varrho(b,d)$. This forces $\varrho(a,d) = 0$, and hence $\{a,b\} = \{c,d\}$.

A subset B of a vector (resp., metric) space X is *linearly* (resp., *metrically*) *star-shaped* about $A \subseteq B$ if $[\![a,b]\!] \subseteq B$ (resp, $[a,b] \subseteq B$) for all $a \in A$ and $b \in B$. The set B is *linearly* (resp., *metrically*) *convex* if it is linearly (resp., metrically) star-shaped about itself.

Suppose $\langle X, \|\cdot\|\rangle$ is a normed vector space over $\mathbb{R}$. Then the norm metric is defined, in the usual way, as $\varrho(x,y) := \|x-y\|$. (When the norm metric is complete, X is commonly referred to as a *Banach space*.) It is known [6, Theorem 5.5 (ii)] that metric intervals are linearly convex in this setting.

Clearly $[\![a,b]\!]$ is always contained in $[a,b]$, but the metric interval may be much larger – even with nonempty topological interior in X. Thus, as one might expect, metric convexity implies linear convexity, but the converse is quite false. For example, while metric intervals are always linearly convex, they need not be metrically convex. (See [7, Example 4.2] for a simple three-dimensional normed vector space example.)

For a mathematical structure with underlying set X, a *hyperspace* over X is any family $\mathcal{H}$ of nonempty subsets of X. X is the *base* of $\mathcal{H}$; the connotation being that a hyperspace inherits structure from its base, which in turn somehow "naturally embeds" in the hyperspace. The hyperspace is termed *unary* if it contains all singleton subsets of X, and the natural embedding is $x \mapsto \{x\}$.[3] Well-known examples of unary hyperspaces include: the family $\wp^+(X)$ of all nonempty subsets of X; and, for each $n = 1, 2, \ldots$, the family $F_n(X)$ of nonempty subsets of cardinality $\leq n$. A hyperspace is termed *binary* (*ternary*, etc.) if it contains $F_2(X)$ ($F_3(X)$, etc.). Hyperspaces of interest to us here are all unary.

If X is a vector space, the hyperspace $\wp^+(X)$ inherits some of the linear structure from X as follows: Let $A, B \subseteq X$ be nonempty, with $s \in \mathbb{R}$. Then define

- $sA := \{sa : a \in A\}$; and
- $A + B := \{a + b : a \in A, b \in B\}$.

Under this interpretation of scalar multiplication and addition we easily see that:

(1) For each $s \in \mathbb{R}$, multiplication by s is a one-place operation, and addition a two-place operation, on the hyperspace $\wp^+(X)$. Addition gives us a semigroup operation that is commutative; also $\{0\} \in \wp^+(X)$ is the additive identity. Thus $\wp^+(X)$ may be thought of as a commutative monoid, which is also equipped with scalar multiplication.

[3] See, e.g., [3] for a study of nonunary hyperspaces that still allow a naturally-embedded copy of the base.

(2) A set $A \in \wp^+(X)$ has an additive inverse if and only if A is a singleton.

(3) Scalar multiplication satisfies the associative law $s(tA) = (st)A$ for $\wp^+(X)$; also $0A = \{0\}$ and $1A = A$.

(4) The distributive law $s(A + B) = sA + sB$ holds for $\wp^+(X)$, but the other distributive law $(s + t)A = sA + tA$ does not: indeed, $(-1 + 1)A = \{0\}$, but $(-1)A + 1A$ contains at least two elements if A does. (In general, though, we have the *semidistributive* law, $(s + t)A \subseteq sA + tA$.)

When X is a vector space, $F_1(X)$, as a subset of $\wp^+(X)$, is closed under the operations of scalar multiplication and addition. As an algebraic structure in its own right, it is clearly a vector space that is isomorphic to X under the embedding $x \mapsto \{x\}$. Now let $L(X)$ consist of the nonempty subsets of X that are linearly convex. Then $L(X)$ is a unary (but nonbinary) hyperspace, which is also closed under the principal operations of $\wp^+(X)$. (To check closure under addition, suppose $A, B \in L(X)$, with $a_1 + b_1$ and $a_2 + b_2$ arbitrary elements of $A + B$. Then, for $0 \le t \le 1$, we have $(1 - t)(a_1 + b_1) + t(a_2 + b_2) = ((1 - t)a_1 + ta_2) + ((1 - t)b_1 + tb_2) \in A + B$ because the summands are linearly convex.)

In $L(X)$ one can recover the distributive law $(s + t)A = sA + tA$ for nonnegative scalars, as is well known. This will be useful later on (see Theorems 3.2, 3.4, and Proposition 3.7 below).

Proposition 2.1. *Let X be a vector space, with $A \in L(X)$ and $s, t \ge 0$. Then $(s + t)A = sA + tA$.*

Proof. It is a triviality to see that $(s + t)A \subseteq sA + tA$ always, so assume A is linearly convex and that both s and t are nonnegative. Pick a typical element from $sA + tA$, say it is $sa + ta'$. If s and t are both zero, there is nothing to prove; so assume $s + t > 0$, and set $u = \frac{t}{s+t}$. Then both u and $1 - u = \frac{s}{s+t}$ are nonnegative. Furthermore, $sa + ta' = (s + t)a''$, where $a'' = (1 - u)a + ua'$, an element of A because A is linearly convex. Thus $sA + tA \subseteq (s + t)A$. □

We extend the notion of linear betweenness in a vector space X as follows: given $A, B \in \wp^+(X)$, define the *linear interval* $[\![A, B]\!]$, with *bracket points* A and B, to be the set $\{(1 - t)A + tB : 0 \le t \le 1\}$ – appropriately called a *hyperline segment* – of sets *linearly between* A and B. Clearly if $A, B \in \wp^+(X)$ then $[\![A, B]\!] \subseteq \wp^+(X)$; we define a hyperspace $\mathcal{H} \subseteq \wp^+(X)$ to be *linearly convex* if $[\![A, B]\!] \subseteq \mathcal{H}$ whenever $A, B \in \mathcal{H}$. Note that, because of being closed under the principal operations of $\wp^+(X)$, both $F_1(X)$ and $L(X)$ are linearly convex hyperspaces.

Example 2.2. For any vector space X, the hyperspaces $F_n(X)$ are clearly not linearly convex when $n \ge 2$. For another natural example, let $S(X)$ be the hyperspace of (possibly degenerate) line segments. Then clearly $F_1(X) \subseteq S(X) \subseteq L(X)$. Let X now be the usual Cartesian plane $\mathbb{R}^2$, let $a = \langle 0, 0\rangle$, $b = \langle 1, 0\rangle$, and $c = \langle 0, 1\rangle$. Let $A = [\![a, b]\!]$ and $B = [\![a, c]\!]$. Then, in usual interval notation for the real line, we have $A = [0, 1] \times \{0\}$ and $B = \{0\} \times [0, 1]$. For any $0 \le t \le 1$, then, $C = (1 - t)A + tB$ is easily seen to be the rectangle $[0, 1 - t] \times [0, t]$. So $[\![A, B]\!] \cap S(X) = \{A, B\}$, showing that $S(X)$ is not linearly convex. □

We will see that linear betweenness at the set level is much better behaved when restricted to $L(X)$.

Proposition 2.3. *Let X be a vector space, with $A, B \in \wp^+(X)$.*

(i) $\bigcup [\![A, B]\!] = \bigcup \{[\![a, b]\!] : a \in A, b \in B\}$.

(ii) *If there is a singleton $C \in [\![A, B]\!] \setminus \{A, B\}$, then both A and B are singletons.*

(iii) $\bigcap [\![A, B]\!] = A \cap B$.

(iv) *If $A \subseteq B$, then $\bigcup [\![A, B]\!] \subseteq B$ if and only if $\bigcup [\![A, B]\!] = B$, if and only if B is linearly star-shaped about A.*

(v) $[\![A, A]\!] = \{A\}$ *if and only if $A \in L(X)$.*

(vi) *If $A, B \in L(X)$, then $\bigcup [\![A, B]\!]$ is the linear convex hull of $A \cup B$.*

Proof. Ad (i). This is straightforward.

Ad (ii). Let $C = \{c\} \in [\![A, B]\!] \setminus \{A, B\}$. Then $C = (1-t)A + tB$ for some $0 < t < 1$. Pick $a \in A$ and $b_1, b_2 \in B$. Then $c = (1-t)a + tb_1 = (1-t)a + tb_2$. Since $t \neq 0$, we infer $b_1 = b_2$. This shows B is a singleton; by the same token (since $t \neq 1$), we conclude that A is a singleton too.

Ad (iii). For any $0 \leq t \leq 1$, let $C = (1-t)A + tB$. If $a \in A \cap B$ we obtain $a = (1-t)a + ta \in C$. The reverse inclusion is equally trivial because $A, B \in [\![A, B]\!]$.

Ad (iv). Suppose B is linearly star-shaped about A. If $c \in \bigcup [\![A, B]\!]$, pick $a \in A$ and $b \in B$, $0 \leq t \leq 1$ such that $c = (1-t)a + tb$. Then $c \in B$; hence $\bigcup [\![A, B]\!] \subseteq B$. Since $B \subseteq \bigcup [\![A, B]\!]$, equality holds.
Suppose B is not linearly star-shaped about some $a \in A$. Then there are $b \in B$ and $0 \leq t \leq 1$ such that $(1-t)a + tb \in [\![a, b]\!] \setminus B$. Hence $\bigcup [\![A, B]\!] \nsubseteq B$.

Ad (v). This follows immediately from (iii) and (iv).

Ad (vi). Let $A, B \in L(X)$, with $U = \bigcup [\![A, B]\!]$, and H the linear convex hull of $A \cup B$. Then $[\![a, b]\!] \subseteq H$ for each $a \in A$ and $b \in B$; hence $U \subseteq H$. For equality, we need to show U – a set containing $A \cup B$ – is linearly convex.
Pick $c_1, c_2 \in U$; we wish to show that $[\![c_1, c_2]\!] \subseteq U$. By (i) above, we have the existence of $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $c_i \in [\![a_i, b_i]\!]$, $i = 1, 2$. Since both A and B are linearly convex, we have $C = \bigcup [\![[\![a_1, a_2]\!], [\![b_1, b_2]\!]]\!] \subseteq U$; hence it suffices to show that C is linearly convex. Let D be the linear convex hull of the set $\{a_1, a_2, b_1, b_2\}$. We are done once we show that $C = D$.
Indeed, it is obvious that $C \subseteq D$, so let $c \in D$ be arbitrary. We need to find $s, t, u \in [0, 1]$ such that $c = (1-u)((1-s)a_1 + sa_2) + u((1-t)b_1 + tb_2)$. Now, because c is in the linear convex hull of $\{a_1, a_2, b_1, b_2\}$, there are scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, where $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 1$, such that $c = \alpha_1 a_1 + \alpha_2 a_2 + \beta_1 b_1 + \beta_2 b_2$. Solving for u, s, t in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2$, we obtain

$$u = \beta_1 + \beta_2, \quad s = \frac{\alpha_2}{1 - (\beta_1 + \beta_2)} = \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad \text{and} \quad t = \tfrac{\beta_2}{\beta_1 + \beta_2}.$$

These solutions make sense as long as both $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$ are positive, and we may then conclude that $c \in C$. But suppose $\alpha_1 + \alpha_2 = 0$. Then $c \in [\![b_1, b_2]\!]$, and is therefore in C. In the case $\beta_1 + \beta_2 = 0$, we have $c \in [\![a_1, a_2]\!] \subseteq C$. So, in any event, we have $c \in C$, showing $C = D$. Hence C is linearly convex, as desired. □

If $\langle X, \varrho \rangle$ is a metric space, we denote by $K(X)$ the hyperspace of nonempty compact subsets of X. Given $a \in X$ and $B \in K(X)$, we define $\varrho(a, B) := \inf\{\varrho(a, x) : x \in B\}$.

Also if $A, B \in K(X)$, we define $\varrho(A, B) := \sup\{\varrho(a, B) : a \in A\}$. Then the *Hausdorff distance* induced by ϱ on $K(X)$ is given by

$$\varrho_H(A, B) := \max\{\varrho(A, B), \varrho(B, A)\},$$

and is well known to be a metric on $K(X)$.[4]

For ball neighborhood notation, we define the *open ball* $N^\circ(a; r)$ (respectively, *closed ball* $N(a; r)$), for $a \in X$ and $r \geq 0$, to be $\{x \in X : \varrho(x, a) < r\}$ (respectively, $\{x \in X : \varrho(x, a) \leq r\}$). The point a is the *centre*, and r the *radius*, of the ball.[5] For $A \in K(X)$ and $r \geq 0$, we define $(A)_r := \bigcup\{N(a; r) : a \in A\}$, the *r-fattening* of A. Then it is known that, for $A, B \in K(X)$,

$$\varrho(A, B) = \inf\{r \geq 0 : A \subseteq (B)_r\}.$$

Remarks 2.4. (i) For metric space X, let $H(X)$ be the hyperspace of all closed nonempty subsets of X that are bounded (i.e., of finite diameter). Then the Hausdorff distance makes sense here too, and is easily shown to be a metric. However, many of the nice arguments that hold for ϱ_H restricted to $K(X)$ do not work in the wider context. In particular, when X is a normed vector space, $H(X)$ is not a linearly convex hyperspace (see Remark 2.7 (iii) below).

(ii) The r-fattening of a compact set, though always closed and bounded, is not necessarily compact: a closed ball of positive radius is just such a set, but in any infinite-dimensional normed vector space, it is well known to be noncompact.

(iii) Intervals in metric spaces are also closed and bounded. But the nondegenerate ones in c_0, the Banach space of all real null sequences, equipped with the supremum norm, are never compact [7, Example 4.19].

(iv) If X is a normed vector space, $A \in K(X)$, and $r \geq 0$, it is easy to see that the r-fattening $(A)_r$ is just $A + rN(0; 1)$; i.e., the (Minkowski) sum of A and the appropriately scaled *closed unit ball* of X. □

Although it is natural to treat a (unary) hyperspace as containing its base, we will continue to use curly brackets when denoting singletons. Thus, by definition, we have $\varrho_H(\{a\}, \{b\}) = \varrho(a, b)$. Note that if A, B in $K(X)$ and either $A \subseteq B$ or A is a singleton, then $\varrho_H(A, B) = \varrho(B, A)$. We refer to $\varrho(A, B)$ as a *one-sided* Hausdorff distance.

The *canonical map* $x \mapsto \{x\}$ embeds X isometrically as the subspace $F_1(X)$ of $K(X)$ (which can easily be shown to be closed in $K(X)$). When X is a normed vector space, this embedding is an isometric isomorphism onto $F_1(X)$. In this setting the combination of compactness and linear convexity of subsets is quite potent, and thus a natural focus in this paper is the hyperspace $KL(X) := K(X) \cap L(X)$ of compact, linearly convex nonempty subsets of X.

Let $\langle X, \varrho \rangle$ be a metric space, with $\mathcal{H}$ a hyperspace of X. If $A, B \in \mathcal{H}$, we denote by $[A, B]_{\mathcal{H}}$ the *metric interval* in $\mathcal{H}$ *bracketed by* A, B; i.e., the set

$$\{C \in \mathcal{H} : \varrho_H(A, C) + \varrho_H(C, B) = \varrho_H(A, B)\}.$$

[4] Since we are dealing with compact sets, all suprema and infima are achieved; e.g., $\varrho(a, B) = \varrho(a, b)$ for some $b \in B$.

[5] While $N^\circ(a; r)$ is an open set in the metric topology, it is not always the interior of $N(a; r)$. It is the interior, however, when X is a normed vector space.

Then we say that $C \in \mathcal{H}$ is *metrically between* $A, B \in \mathcal{H}$ if $C \in [A, B]_{\mathcal{H}}$.

The following adds to Proposition 2.3 in the presence of compactness.

Proposition 2.5. *Let X be a normed vector space, with $A, B \in \wp^+(X)$.*

(i) *If $A \setminus B \neq \emptyset$, $B \in K(X)$, and $C \in [\![A, B]\!]$, then $C \subseteq B$ if and only if $C = B$.*

(ii) *If $A, B \in K(X)$, then $\bigcup [\![A, B]\!] \in K(X)$.*

(iii) *If $A, B \in KL(X)$, then $\bigcup [\![A, B]\!] \in KL(X)$.*

Proof. Ad (i). Suppose $C = (1-t)A + tB$, for some $t \in [0, 1]$. Assuming $C \neq B$, we have $t < 1$. Fix $a \in A \setminus B$. Because B is compact, there is some $b \in B$ with $\|a - b\| = \varrho(a, b) = \varrho(a, B)$. Let $c = (1-t)a + tb$. Then $c \in C$; however, since $\|a - c\| = t\|a - b\| < \|a - b\| = \varrho(a, B)$, we infer that $c \notin B$.

Ad (ii). Let $U = \bigcup [\![A, B]\!]$. We show U to be sequentially compact. Indeed, let $\langle c_n \rangle$ be a sequence in U. Then, for $n = 0, 1, 2, \ldots$, we have $c_n \in [\![a_n, b_n]\!]$, where $a_n \in A$ and $b_n \in B$. Since A is compact, the sequence $\langle a_n \rangle$ has a subsequence converging to $a \in A$. The corresponding subsequence of $\langle b_n \rangle$ itself has a subsequence converging to $b \in B$, since B is compact. Without loss of generality, we may assume $\langle a_n \rangle \to a \in A$ and $\langle b_n \rangle \to b \in B$. For $n = 0, 1, 2, \ldots$, let $c_n = (1-t_n)a_n + t_n b_n$. Then the sequence $\langle t_n \rangle$, being a sequence from the compact set $[0, 1]$, has a convergent subsequence; without loss of generality, we may assume $\langle t_n \rangle \to t \in [0, 1]$. Then we have $\langle c_n \rangle$ converging to $(1-t)a + tb \in [\![a, b]\!] \subseteq U$.

Ad (iii). This follows immediately from (ii) and Proposition 2.3 (vi). □

We have previously seen that for any vector space X, both $F_1(X)$ and $L(X)$ are linearly convex hyperspaces of X. If X is topological as well, we can say more.

Proposition 2.6. *If X is a topological vector space, then both $K(X)$ and $KL(X)$ are linearly convex hyperspaces.*

Proof. To show $K(X)$ is linearly convex, it suffices to show it is stable under scalar multiples, as well as sums. Indeed, because X is a topological vector space, both scalar multiplication and addition, are continuous operations on X. So if A is compact, then so is sA for any $s \in \mathbb{R}$. Furthermore if both A and B are compact, then $A + B$, the image of the compact set $A \times B$ under the continuous function $+ : X \times X \to X$, is also compact.
Since $K(X)$ is linearly convex, and we already know $L(X)$ is linearly convex, so too is $KL(X) = K(X) \cap L(X)$. □

Remarks 2.7. (i) The argument in the proof of Proposition 2.6 can be used to show that such properties as finiteness and connectedness are also stable under scalar multiplication and set addition.

(ii) One may easily show directly that being closed, and being bounded are each stable under scalar multiplication. However, being closed is not stable under addition: Let $X = \mathbb{R}$, with $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, where $a_n = n$ and $b_n = -n + \frac{1}{n+1}$. Then both A and B are closed in X, and $0 \notin A + B$. But the sequence $\langle a_n + b_n \rangle = \langle \frac{1}{n+1} \rangle$ from $A + B$ converges to 0. Hence $A + B$ is not closed in X.

(iii) Although being compact is stable under addition, being both closed and bounded is not. Let $X = c_0$, the space of real null sequences (as in Remark 2.4 (iii)). Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$, where each sequence a_n is 1 in its nth entry and 0 elsewhere, and each b_n is $-1 + \frac{1}{n+1}$ in its nth entry and 0 elsewhere. Then both A and B are closed and bounded in X, and $A + B$ does not contain the zero sequence. But the sequence $\langle a_n + b_n \rangle$ from $A + B$ converges to the zero sequence. Hence $A + B$ is not closed in X (though it is bounded). This supports the claim, made in Remark 2.4 (i) above, that $H(X)$ is not linearly convex: for if $A, B \in H(X)$ and $A + B$ is not closed, then neither is $\frac{1}{2}(A + B)$; so $[\![A, B]\!] \not\subseteq H(X)$.

(iv) The sum of two closed sets may not be closed, but it is closed if one of the summands is compact. For suppose $\langle a_n + b_n \rangle$ is a sequence in $A + B$ that converges to $c \in X$. By compactness of A, there is a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$, converging to $a \in A$. So we have $\langle a_{n_k} + b_{n_k} \rangle \to c$. Then $\langle b_{n_k} \rangle \to c - a$. Since B is closed, we have $c - a \in B$; hence $c \in A + B$. Thus $A + B$ is closed in X. □

3. Betweenness axioms

In [4] (and many other works) betweenness is approached axiomatically by regarding it as an abstract ternary predicate. We place our main focus on the following small list of (universally quantified) betweenness axioms; all are discussed in detail in [6]. The first-order symbol $I(x, y, z)$ should be read, "y lies between x and z," and generalises the statements "$y \in [\![x, z]\!]$" and "$y \in [x, z]$" from the previous section.

(Inclusivity)	$I(x, x, y) \wedge I(x, y, y)$
(Symmetry)	$I(y, x, z) \to I(z, x, y)$
(Uniqueness)	$I(y, x, y) \to x = y$
(Antisymmetry)	$(I(y, x, z) \wedge I(y, z, x)) \to x = z$
(Concentration)	$(I(y, x, z) \wedge I(y, w, x) \wedge I(x, w, z)) \to w = x$
(Transitivity)	$(I(y, w, x) \wedge I(y, x, z)) \to I(y, w, z)$
(Convexity)	$(I(u, w, v) \wedge I(x, u, y) \wedge I(x, v, y)) \to I(x, w, y)$
(Weak Disjunctivity)	$(I(x, u, y) \wedge I(x, v, y)) \to (I(x, u, v) \vee I(v, u, y))$

The first three axioms are referred to in [4] (and later papers) as *basic betweenness axioms*, and a ternary structure satisfying them is called a *basic betweenness structure*.

In interval terms, where $I(a, b) := \{x : I(a, x, b)\}$ in any interpretation of the betweenness predicate, inclusivity says that $I(a, b)$ contains $\{a, b\}$. Symmetry asserts that $I(a, b) = I(b, a)$, and uniqueness says $I(a, a) = \{a\}$. Antisymmetry says that no two distinct points can each lie between the other and a single third point, and implies that any two distinct bracket sets for an interval must be disjoint. Concentration – in the presence of the basic betweenness axioms – says that $I(a, c) \cap I(c, b) = \{c\}$ whenever $c \in I(a, b)$. Transitivity amounts to the condition that intervals are star-shaped about their bracket sets, and convexity asserts that intervals themselves are convex. In any basic betweenness structure in which transitivity holds, antisymmetry is easily seen [6] to follow from concentration. And if weak disjunctivity also holds, then antisymmetry and concentration are equivalent (see [4, Theorem 5.0.5]). Weak disjunctivity itself says that $I(a, b) \subseteq I(a, c) \cup I(c, b)$ for all $c \in I(a, b)$.

All these axioms hold for linear betweenness in vector spaces, and all but the last two hold for metric betweenness [6].[6] By [6, Proposition 2.2], convexity follows from weak disjunctivity in all basic betweenness structures satisfying transitivity. The metric betweenness structure of a normed vector space X satisfies weak disjunctivity if and only if its linear and metric intervals coincide, if and only if X is *strictly convex*[7] [6, Corollary 5.6].[8] The metric betweenness structure of a three-dimensional normed vector space can fail to satisfy convexity [6, Example 4.2], but convexity always holds for metric betweenness in a normed vector space of dimension at most two [6, Theorem 5.14].

Clearly linear betweenness satisfies inclusivity and symmetry for sets in $\wp^+(X)$, but Proposition 2.3 (v) tells us it satisfies uniqueness if and only if it is restricted to $L(X)$. So we need linear convexity in order for linear betweenness at the set level even to be basic. The same is true for transitivity.

Example 3.1. Linear betweenness also fails to satisfy transitivity for sets in $\wp^+(X)$, even if one of the bracket points is a singleton. Indeed, let $A = \{a_1, a_2\}$ and $B = \{b\}$ be subsets of $\mathbb{R}^2$, such that the points $\{a_1, a_2, b\}$ are distinct. Then each $C \in [\![A, B]\!] \setminus \{B\}$ is a doubleton set; hence sets in $[\![A, C]\!]$ typically have four points, and $[\![A, C]\!]$ is therefore not generally contained in $[\![A, B]\!]$. □

If X is a vector space, we know that linear betweenness in $L(X)$ is basic; and, in general, metric betweenness always satisfies the antisymmetry, concentration, and transitivity axioms (along with the basic ones). It is our aim in this section to show that all eight axioms hold in $L(X)$; we first address the issue of transitivity.

Theorem 3.2. *Let X be a vector space, with $A \in L(X)$ and $B \in \wp^+(X)$. If $0 \leq s, t \leq 1$ are such that $C = (1 - s)A + sB$ and $D = (1 - t)A + tC$, then $D = (1 - st)A + stB$. Hence $[\![A, C]\!] \subseteq [\![A, B]\!]$ whenever $C \in [\![A, B]\!]$.*

Proof. Given $C = (1 - s)A + sB$ and $D = (1 - t)A + tC$, substitute for C in the second equation, obtaining $D = (1-t)A+t(1-s)A+stB$. Since A is linearly convex and $1-t, t(1-s) \geq 0$, we use Proposition 2.1 to get $D = ((1-t)+t(1-s))A+stB = (1 - st)A + stB$. The second sentence of the proposition follows immediately. □

Corollary 3.3. *For a vector space X, the linear betweenness structure in $L(X)$ satisfies transitivity.*

Next comes weak disjunctivity.

Theorem 3.4. *Let X be a vector space, with $A \in L(X)$, $B \in \wp^+(X)$, $0 \leq s \leq t \leq 1$, $C = (1-s)A+sB$, and $D = (1-t)A+tB$. If $t = 0$, we have $C = D = A$; otherwise we have $C = (1 - \frac{s}{t})A + \frac{s}{t}D$. In any case, $C \in [\![A, D]\!]$.*

Proof. Let $0 \leq s \leq t \leq 1$ be such that $C = (1 - s)A + sB$ and $D = (1 - t)A + tB$. The conclusion where $t = 0$ is immediate, so assume $t > 0$.

[6] For metric betweenness, concentration holds even without benefit of weak disjunctivity.

[7] This says that if a and b are nonzero vectors such that $\|a + b\| = \|a\| + \|b\|$, then each of the vectors is a positive scalar multiple of the other. It is well known to be equivalent to the condition that there are no nondegenerate line segments in the unit sphere.

[8] Disjunctivity, a stronger form of weak disjunctivity, asserts $I(a, b) \subseteq I(a, c) \cup I(c, b)$ even if $c \notin I(a, b)$. This axiom holds for many topological interpretations of betweenness (see, e.g., [1]), but rarely for geometric ones (see, e.g., [6, Proposition 5.7]).

Then $C = ((1-\frac{s}{t})+\frac{s}{t}(1-t))A+sB$. Since A is linearly convex and $1-\frac{s}{t}, \frac{s}{t}(1-t) \geq 0$, we obtain from Proposition 2.1 that

$$C = (1-\tfrac{s}{t})A + \tfrac{s}{t}(1-t)A + sB = (1-\tfrac{s}{t})A + (\tfrac{s}{t}(1-t)A + \tfrac{s}{t}tB) = (1-\tfrac{s}{t})A + \tfrac{s}{t}D.$$

Thus $C \in [\![A, D]\!]$, as desired. □

Corollary 3.5. *For a vector space X, the linear betweenness structure in $L(X)$ satisfies weak disjunctivity.*

Proof. Let $A, B \in L(X)$, and fix $0 \leq s, t \leq 1$ so that $C = (1-s)A + sB$ and $D = (1-t)A+tB$. If $s \leq t$, then, because A is linearly convex, we have $C \in [\![A, D]\!]$, by Theorem 3.4. If $s \geq t$, we use Theorem 3.4, with the roles of A and B reversed, to infer that $C \in [\![B, D]\!] = [\![D, B]\!]$. □

From Corollaries 3.3 and 3.5, plus the fact [6, Proposition 2.2] that basic betweenness structures satisfying both transitivity and weak disjunctivity also satisfy convexity, we obtain the following.

Corollary 3.6. *For a vector space X, the linear betweenness structure in $L(X)$ satisfies convexity.*

We now address antisymmetry and concentration. In view of the fact that $L(X)$ satisfies one if it satisfies the other, we concentrate on concentration because a direct proof of this property is somewhat simpler. To this end we first state a kind of converse of convexity.

Proposition 3.7. *Let X be a vector space, with $A, B \in L(X)$, $0 \leq s \leq r \leq t \leq 1$, $C = (1-s)A + sB$, $D = (1-t)A + tB$, and $E = (1-r)A + rB$. Then $E \in [\![C, D]\!]$.*

Proof. Pick $\lambda \in [0, 1]$ such that $r = (1-\lambda)s + \lambda t$. Then, using Proposition 2.1, it is easy to show that $E = (1-\lambda)C + \lambda D$. □

Theorem 3.8. *Let X be a vector space, with $A, B, C \in L(X)$ such that $C \in [\![A, B]\!]$. Then $[\![A, C]\!] \cap [\![C, B]\!] = \{C\}$.*

Proof. Fix $t \in [0, 1]$ such that $C = (1-t)A + tB$. Then, by Theorems 3.2 and 3.4, we have

$$[\![A, C]\!] = \{(1-s)A + sB : s \in [0, t]\} \quad \text{and} \quad [\![C, B]\!] = \{(1-s)A + sB : s \in [t, 1]\}.$$

Let $D \in [\![A, C]\!] \cap [\![C, B]\!]$ be arbitrary. We than have $D = (1-s_1)A + s_1B = (1-s_2)A + s_2B$ for some $s_1 \in [0, t]$ and $s_2 \in [t, 1]$. In particular, we have $t \in [s_1, s_2]$; so by Proposition 3.7, we have $C \in [\![D, D]\!]$. Hence $C = D$, completing the proof. □

Corollary 3.9. *Let X be a vector space. Then $L(X)$ satisfies both antisymmetry and concentration.*

As noted before, any metric betweenness structure satisfies antisymmetry and concentration; the following is an immediate consequence of Corollary 3.9 and Proposition 2.6.

Corollary 3.10. *For a normed vector space X, the linear and metric betweenness structures in $KL(X)$ satisfy antisymmetry and concentration.*

Remark 3.11. Because, from Corollaries 3.9 and 3.10, the linear betweenness structures in $L(X)$ and $KL(X)$ satisfy antisymmetry/concentration, we can use [4, Theorem 5.0.6] to conclude for any A, B, C: (1) $\{A, B\}$ is the unique bracket set for $[\![A, B]\!]$; and (2) $[\![A, B]\!] \cap [\![A, C]\!] \cap [\![B, C]\!]$ is either empty or a singleton. □

For any points a, b in a normed vector space X, the parameterization $t \mapsto (1-t)a+tb$ is a dilation from $[0, 1]$ into X, with dilation constant $\|a - b\|$. As such, it is a constant map when $a = b$, and is a continuous embedding otherwise. Thus each nondegenerate linear interval $[\![a, b]\!]$ is an arc (i.e., a homeomorphic copy of $[0, 1]$) whose two end (i.e., noncut) points are the bracket points a and b.

The situation is more complicated at the set level; we first consider the case of the hyperspace $L(X)$, with the following observation.

Proposition 3.12. *Let X be a vector space. Then there exist distinct $A, B \in L(X)$ such that $[\![A, B]\!] = \{A, B\}$.*

Proof. let $A = \{0\}$, $B = X$, and let $C = (1 - t)A + tB$ for some $t \in [0, 1]$. If $t = 0$, we have $C = A$. Otherwise we have, for any $x \in X$, $x = t(\frac{1}{t}x) \in C$. Hence $C = X$, and we have $[\![A, B]\!] = \{A, B\}$. □

As a consequence of Proposition 3.12, it is impossible to place a Hausdorff-even a T_1-topology on $L(X)$ so that each parameterization $t \mapsto (1-t)A+tB$ is continuous from $[0, 1]$ to $L(X)$. Even without the topology on $L(X)$, though, we can make a statement about the topological nature of these parameterizations.

If X is a set and Y is a topological space, a function $f : Y \to X$ is *monotone* if the point-inverse $f^{-1}[x]$ is a connected subset of Y for each $x \in X$.

Proposition 3.13. *Let X be a vector space, with $A, B \in L(X)$. Then the function $f(t) := (1 - t)A + tB$ from $[0, 1]$ into $L(X)$ is monotone.*

Proof. It suffices to show that every point-inverse of f is an order-convex subset of $[0, 1]$. Suppose $0 \le s \le u \le t \le 1$ with $C = (1 - s)A + sB$, $D = (1 - t)A + tB$, and $E = (1 - u)A + uB$. Assume now that $C = D$; i.e., $s, t \in f^{-1}[C]$ and $u \in [s, t]$. Then, by Proposition 3.7, we have that $E \in [\![C, D]\!]$. By our assumption, plus the uniqueness axiom, we obtain $C = E = D$. Thus $u \in f^{-1}[C]$, as desired. □

Hence, if $\mathcal{H}$ is a linearly convex unary hyperspace, contained in $L(X)$, and $\mathcal{H}$ is endowed with a Hausdorff topology such that each $t \mapsto (1 - t)A + tB$ is continuous, then each linear interval in $\mathcal{H}$ is an arc whose end points are the bracket points of the interval. This is because [16] the Hausdorff image of an arc, under a nonconstant monotone continuous map, is an arc. Moreover, the image of an end point of the domain arc is an end point of the image arc.

Remark 3.14. An important first-order betweenness axiom, of higher quantifier rank than those introduced elsewhere in this section, is

(Gap-freeness) $\forall x \forall y \exists z (x \neq y \to (z \neq x \wedge z \neq y \wedge I(x, z, y)))$

This universal-existential sentence says that between any two distinct elements there exists a third, distinct from the first two. In interval terms it says that each nondegenerate interval has at least three elements. A complete metric space is called *convex* [15] if its metric betweenness structure satisfies gap-freeness. To avoid confusion with the notion of convexity induced by a betweenness relation, we refer to such a metric space as *gap-free* in this paper. Complete gap-free metric spaces are, among other things, locally connected and arc-connected. By Proposition 3.12, if X is a nondegenerate normed vector space, the linear betweenness structure of $L(X)$ is not gap-free. However, in Section 6 we will show that both the linear and the metric betweenness structures of $KL(X)$ are indeed gap-free. □

We finish this section with a study of the following fact from plane Euclidean geometry: if there is a point in the intersection of the three legs of a triangle, then the three vertices are collinear.

In any betweenness structure X, define a triple $\langle a, b, c\rangle$ to be *collinear* if at least one of the statements $a \in I(b, c), b \in I(a, c), c \in I(a, b)$ holds. We consider the following two closely-related betweenness axioms.

(Collinearity) $(I(x, u, y) \wedge I(x, u, z) \wedge I(y, u, z)) \to (I(y, x, z) \vee I(x, y, z) \vee I(x, z, y))$

(Strong Collinearity) $(I(x, u, y) \wedge I(x, u, z) \wedge I(y, u, z)) \to (u = x \vee u = y \vee u = z)$

Collinearity easily follows from strong collinearity: For suppose we have the triple $\langle a, b, c\rangle$ and a point $d \in I(a, b) \cap I(b, c) \cap I(a, c)$. Using strong collinearity, we infer that, say, $d = a$, and hence $a \in I(b, c)$. For the converse, we have the following.

Proposition 3.15. *If a betweenness structure satisfies both collinearity and concentration, then it also satisfies strong collinearity.*

Proof. Suppose $\langle a, b, c\rangle$ is a triple, with $d \in I(a, b) \cap I(b, c) \cap I(a, c)$. Using collinearity, we infer that, say, $b \in I(a, c)$. But then we have both $d \in I(a, b)$ and $d \in I(b, c)$, by hypothesis. By concentration, we have $d = b$. □

In any normed vector space, (strong) collinearity clearly holds for linear betweenness; the following shows it need not hold for the corresponding metric betweenness.

Example 3.16. Consider $X = \mathbb{R}^2_1$, real 2-space $\mathbb{R}^2$ equipped with the *taxicab norm* (i.e., given by $\|\langle x, y\rangle\|_1 := |x| + |y|$), and let $a = \langle 0, 0\rangle$, $b = \langle 1, 1\rangle$, and $c = \langle 1, -1\rangle$. Then intervals in X are rectangles with sides parallel to the coordinate axes; in particlar we have $[a, b] = [0, 1]\times[0, 1]$, $[a, c] = [0, 1]\times[-1, 0]$, and $[b, c] = \{1\}\times[-1, 1]$. The triple $\langle a, b, c\rangle$ is clearly noncollinear; however $[a, b] \cap [a, c] \cap [b, c] = \{\langle 1, 0\rangle\}$ a nonempty set. Hence the collinearity condition does not hold for metric betweenness in $\mathbb{R}^2_1$. □

We currently do not know whether linear betweenness satisfies collinearity in $L(X)$ generally. In Section 6 we delve more deeply into how the linear and metric betweenness structures interact in $KL(X)$; in particular, we find that strong collinearity does indeed hold for linear betweeness in that case.

4. Metric intervals with singleton bracket points

In this section, we refer to a hyperspace over a metric space X as being *compact* if it is contained in $K(X)$.

When X is a vector space, $F_1(X)$ is a linearly convex hyperspace; hence the linear interval $[\![\{a\}, \{b\}]\!]$, in any unary hyperspace, is just $\{\{c\} : c \in [\![a, b]\!]\}$. The situation with metric spaces is different: If X is a metric space and $\mathcal{H}$ is a compact hyperspace of X, $F_1(X)$ is not necessarily a *metrically convex* hyperspace; i.e., it is quite possible for $[\{a\},\{b\}]_{\mathcal{H}}$ to properly contain $[\{a\},\{b\}]_{F_1(X)}$. In this section we analyze this phenomenon more closely.

By way of notation: if $Y \subseteq X$ is a subspace, we let $\mathcal{H}[Y]$, the *restriction* of $\mathcal{H}$ to Y, be $\mathcal{H} \cap \wp^+(Y)$. For example, $K(X)[Y] = K(Y)$ and $F_n(X)[Y] = F_n(Y)$, $n = 1, 2, \ldots$; but in the case of the hyperspace $C(X)$ of closed nonempty subsets of X, we have $C(X)[Y]$ properly contained in $C(Y)$ whenever Y is not closed in X.

In each metric interval $[a,b] \subseteq X$ we define the binary relation $\sim_{ab}$ by saying $x \sim_{ab} y$ just in case $\varrho(a,x) = \varrho(a,y)$ (equivalently, $\varrho(x,b) = \varrho(y,b)$). This is clearly an equivalence relation on $[a,b]$, and is the same relation as $\sim_{ba}$. Also, for $a \in X$ and $r \geq 0$, we let $S(a;r) := N(a;r) \setminus N^\circ(a;r)$ denote the *sphere of radius* r, *centred at* a. Then we may express $[\{a\},\{b\}]_{\mathcal{H}}$ in simple terms involving these binary relations and spheres. This will prove useful at several points in this section.

Lemma 4.1. *Let X be a metric space, with $a, b \in X$ and $\mathcal{H}$ a compact hyperspace over X. Then*

$$\begin{aligned}[\{a\},\{b\}]_{\mathcal{H}} &= \{A \in \mathcal{H} : A \subseteq [a,b] \text{ and } \forall x,y \in A,\ x \sim_{ab} y\} \\ &= \bigcup\{\mathcal{H}[S(a;r) \cap [a,b]] : 0 \leq r \leq \varrho(a,b)\}.\end{aligned}$$

Proof. As mentioned earlier, if $a \in X$ and $A \in \mathcal{H}$ are arbitrary, then

$$\varrho_H(\{a\},A) = \varrho(A,\{a\}) = \sup\{\varrho(a,y) : y \in A\}.$$

To prove the left-hand side of the equality contains the middle, suppose $A \subseteq [a,b]$ is such that $A \in \mathcal{H}$ and A is contained in a $\sim_{ab}$-equivalence class. Then $\varrho_H(\{a\},A) = \varrho(a,x)$ for any $x \in A$. So, since $A \neq \emptyset$, fix $x \in A$. Then

$$\varrho_H(\{a\},A) + \varrho_H(A,\{b\}) = \varrho(a,x) + \varrho(x,b) = \varrho(a,b) = \varrho_H(\{a\},\{b\}).$$

Thus $A \in [\{a\},\{b\}]_{\mathcal{H}}$.

For the reverse inclusion, suppose $A \in \mathcal{H}$ is not in the middle set. If $A \not\subseteq [a,b]$, say $x \in A \setminus [a,b]$, then $\varrho_H(\{a\},A) \geq \varrho(a,x)$ and $\varrho_H(A,\{b\}) \geq \varrho(x,b)$. Hence $\varrho_H(\{a\},A) + \varrho_H(A,\{b\}) \geq \varrho(a,x) + \varrho(x,b) > \varrho(a,b) = \varrho_H(\{a\},\{b\})$, so we have $A \notin [\{a\},\{b\}]_{\mathcal{H}}$.

Now assume $A \in \mathcal{H}$ is contained in $[a,b]$, but is not contained in a $\sim_{ab}$-equivalence class. Then there are $x, y \in A$ that are not equidistant from a; say $\varrho(a,x) < \varrho(a,y)$. Then $\varrho_H(\{a\},A) + \varrho_H(A,\{b\}) \geq \varrho(a,y) + \varrho(x,b) > \varrho(a,x) + \varrho(x,b) = \varrho(a,b) = \varrho_H(\{a\},\{b\})$, and we conclude $A \notin [\{a\},\{b\}]_{\mathcal{H}}$ in this case too. This establishes the equality between the left-hand and middle terms.

Proving the equality between the middle and right-hand terms is straightforward and left to the reader. □

Remark 4.2. We saw above that metric intervals can have two or more bracket sets (all disjoint from one another). It is tempting to conjecture that if $\mathcal{H}$ is a compact hyperspace over X, with $[a,b] = [c,d]$, then it follows that $[\{a\},\{b\}]_{\mathcal{H}} = [\{c\},\{d\}]_{\mathcal{H}}$. This is not true, as the following example shows: start with $X = \mathbb{R}^2_1$, real 2-space with the taxicab norm, as in Example 3.16. If $a = \langle 0,0\rangle$ and $b = \langle 1,1\rangle$, then $[a,b] = [0,1]^2$. If $c = \langle 1,0\rangle$ and $d = \langle 0,1\rangle$, then $[c,d] = [a,b]$. However, we have $[\![c,d]\!] \in [\{a\},\{b\}]_{\mathcal{H}} \setminus [\{c\},\{d\}]_{\mathcal{H}}$ when $\mathcal{H} = K(X)$. □

As in [6], we define a pair $\langle a, b\rangle$ in a metric space X to be *narrow* if $\sim_{ab}$ is trivial; i.e., if no two points in $[a, b]$ can be equidistant from a (or from b). The space is *narrow* if each of its pairs is narrow. We say that a metric space X is *metrically convex in* a unary hyperspace $\mathcal{H}$ of X if $F_1(X)$ is metrically convex in $\mathcal{H}$; i.e., if $[\{a\}, \{b\}]_{\mathcal{H}} = [\{a\}, \{b\}]_{F_1(X)} = \{\{x\} : x \in [a, b]\}$.[9]

Theorem 4.3. *Let X be a metric space.*

(i) *If X is metrically convex in some binary compact hyperspace over X, then X is narrow.*

(ii) *If X is narrow, then X is metrically convex in every compact hyperspace over X.*

Proof. Ad (i). Fix binary compact hyperspace $\mathcal{H}$ and assume X is metrically convex in $\mathcal{H}$, with $a, b \in X$ and $x, y \in [a, b]$ equidistant from a. Then $\{x, y\} \in F_2(X) \subseteq \mathcal{H}$; hence $\{x, y\} \in [\{a\}, \{b\}]_{\mathcal{H}}$, by Lemma 4.1. By convexity, each element of $[\{a\}, \{b\}]_{\mathcal{H}}$ is a singleton, and so we have $x = y$. Thus the pair $\langle a, b\rangle$ is narrow.

Ad (ii). If X is not metrically convex in some compact hyperspace $\mathcal{H}$, then there is a pair $\langle a, b\rangle$ such that $[\{a\}, \{b\}]_{\mathcal{H}}$ is not contained in $F_1(X)$. By Lemma 4.1, there is some $A \subseteq [a, b]$ such that A has at least two points and every element of A is the same distance from a. This says that $\langle a, b\rangle$ is not a narrow pair. □

Corollary 4.4. *For a metric space X, the following statements are equivalent:*

(a) *X is metrically convex in $F_2(X)$.*

(b) *X is metrically convex in at least one of its binary compact hyperspaces.*

(c) *X is narrow.*

(d) *X is metrically convex in each of its compact hyperspaces.*

By [6, Theorem 5.5 (iii)], a normed vector space is narrow if and only if each of its nondegenerate metric intervals is a line segment, if and only if it is strictly convex. Thus in Proposition 4.4 – with X assumed to be a normed vector space – we may add: (e) X is strictly convex; and (f) for each compact hyperspace $\mathcal{H}$, $[\{a\}, \{b\}]_{\mathcal{H}} = [\![\{a\}, \{b\}]\!]$.

When X is a normed vector space, the hyperspace $KL(X)$ is clearly compact, but not binary. Is the strict convexity of X equivalent to X being metrically convex in $KL(X)$? The answer is *yes*, but first we need the following fact.

Lemma 4.5. *Let X be a normed vector space, $a, b \in X$ distinct, and $0 \le r \le \|a-b\|$. Then both $[a, b]$ and $[a, b] \cap S(a; r)$ are linearly convex.*

Proof. The proof follows the argument of [6, Theorem 5.5 (ii)]; we repeat it here for completeness. First let $x, y \in [a, b]$, with $z = sx + ty$ for some $s, t \ge 0$ such that $s + t = 1$. It suffices to show that $\|a - z\| + \|z - b\| \le \|a - b\|$. Indeed, the left-hand side is $\|a - (sx + ty)\| + \|(sx + ty) - b\|$, which equals

$$
\begin{aligned}
&\|s(a - x) + t(a - y)\| + \|s(x - b) + t(y - b)\| \\
&\qquad \le s\|a - x\| + t\|a - y\| + s\|x - b\| + t\|y - b\| \\
&\qquad = s(\|a - x\| + \|x - b\|) + t(\|a - y\| + \|y - b\|) = (s + t)\|a - b\|) = \|a - b\|,
\end{aligned}
$$

since $x, y \in [a, b]$. This shows that $[a, b]$ is linearly convex.

[9] Every vector space is *linearly convex* in each of its unary hyperspaces.

Next let $0 \leq r \leq \|a-b\|$, with $x, y \in [a,b] \cap S(a;r)$ and $z = sx+ty$ as above. Then we have $\|a-z\| \leq s\|a-x\|+t\|a-y\| = (s+t)r = r$; and similarly, $\|z-b\| \leq \|a-b\|-r$. If either of these inequalities were strict, we would have $\|a-z\| + \|z-b\| < \|a-b\|$, contradicting the linear convexity of $[a,b]$. Thus $z \in [a,b] \cap S(a;r)$, showing that $[a,b] \cap S(a;r)$ is linearly convex as well. □

Our advertised characterisation of strict convexity is the following.

Theorem 4.6. *A normed vector space X is strictly convex if and only if X is metrically convex in $KL(X)$.*

Proof. Assume X is strictly convex. Then, as mentioned above, X is narrow, and hence convex in $KL(X)$, by Corollary 4.4.

For the converse, suppose that X is not strictly convex. Then there are distinct $a, b \in X$ such that $\langle a, b\rangle$ is not narrow. Thus there is some $0 \leq r \leq \|a-b\|$ and distinct $x, y \in [a,b] \cap S(a;r)$. But $[\![x,y]\!] \in KL(X)$, and by Lemma 4.5, we have $[\![x,y]\!] \subseteq [a,b] \cap S(a;r)$. Hence $[\![x,y]\!] \in [\{a\},\{b\}]_{KL(X)} \setminus [\![\{a\},\{b\}]\!]$, showing that X is not metrically convex in $KL(X)$. □

In the sequel we will be concerned with explicit computations of Hausdorff distances between linearly convex compact sets. Invaluable to this endeavor is what is known as the *Bauer maximum principle*: If X is a normed vector space, $A \in KL(X)$, and $f : A \to \mathbb{R}$ is continuous and *convex* – in the sense that for any $a, b \in A$ and $t \in [0,1]$, $f((1-t)a+tb) \leq (1-t)f(a)+tf(b)$ – then f takes its maximum value over A at an *extreme point* of A; i.e., a point of A not in the relative interior of any line segment in A. For details, see [10] (especially Section 3.2.5).[10]

For any point b in a normed vector space X, the map $x \mapsto \|x-b\|$ is well known to be both continuous and convex. Moreover, if $B \in KL(X)$, then the map $\varrho(x,B) := \inf\{\|x-b\| : b \in B\}$ is also continuous and convex (see page 87 of [10]). Note that the one-sided Hausdorff distance $\varrho(A,B)$ is the maximum value that $\varrho(x,B)$ takes as x ranges over A. For any $A \in KL(X)$, let $\varepsilon(A)$ be the set of extreme points of A. Then after applying the Bauer maximum principle, we immediately obtain the following.

Lemma 4.7. *Let X be a normed vector space, with $A, B \in KL(X)$. Then*

$$\varrho(A,B) = \sup\{\varrho(a,B) : a \in \varepsilon(A)\}.$$

Of course, where this lemma is most useful is when the sets A and B are *polytopes*; i.e., where the number of extreme points is finite (see, especially, Theorems 4.9, 8.5, 8.6, and 8.11 below). Then $\varrho_H(A,B)$ is the maximum of a finite number of terms of the form $\varrho(x,A)$ and $\varrho(x,B)$.

In later results, we will make use of the following simplification, what in [6] is referred to as the *rescaled translation* principle: Given $r > 0$ and two distinct points a, b in a normed vector space X, the mapping f, given by $x \mapsto \frac{r}{\|b-a\|}(x-a)$, is an invertible affine transformation that maps a to 0, and b to a point in $S(0;r)$.

[10] By the well-known Krein-Milman theorem, each linearly convex compact set is the closed convex hull of its set of extreme points.

In particular, for any $x \in X$, we have $x \in [a, b]$ if and only if $f(x) \in [0, f(b)]$; both $[a, b]$ and $[0, f(b)]$ are rescaled translations of each other and share the same qualitative geometric properties.

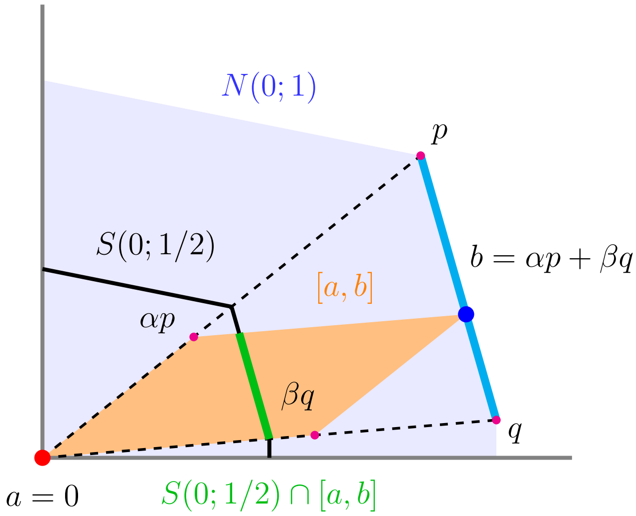

Figure 1: Schematic of the interval $[\{a\}, \{b\}]$ in two dimensions (only positive quadrant shown). The point b is located on some face of the unit ball $N(0; 1)$. By Lemma 4.8, the orange parallelogram is the set $[a, b]$. The parallelogram is partitioned into slices, with each slice being the intersection of an r-sphere with the parallelogram. By Lemma 4.1, a set is an element of $[\{a\}, \{b\}]$ exactly if it is a subset of some slice.

Also we will have use for the following *parallelogram lemma*, a slight paraphrasing of [6, Theorem 5.9]. Note that, in the finite-dimensional case, every point on the unit sphere $S(0; 1)$ is either an extreme point of the ball $N(0; 1)$ or it lies in the relative interior of a line segment in $S(0; 1)$ whose end points are extreme points of $N(0; 1)$. Hence this lemma, in conjunction with the rescaled translation principle, allows us to describe the metric interval between any two points in a normed plane.

Lemma 4.8. *Let $X = \langle \mathbb{R}^2, \|\cdot\| \rangle$ be a two-dimensional normed vector space, with $p, q \in S(0; 1)$ distinct extreme points of $N(0; 1)$ such that $[\![p, q]\!] \subseteq S(0; 1)$. Fix $a \in [\![p, q]\!]$, along with $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and $a = \alpha p + \beta q$. Let P be the parallelogram $\{\alpha' p + \beta' q : 0 \leq \alpha' \leq \alpha, 0 \leq \beta' \leq \beta\}$ (a line segment if and only if $a \in \{p, q\}$). Then $[0, a] = P$; in particular, when $a \notin \{p, q\}$ we have that $[0, a]$ is a proper parallelogram with $[\![0, a]\!]$ as one of its two diagonals.*

The following relates strict convexity to the metric convexity of a metric interval with singleton bracket points.

Theorem 4.9. *Let X be a normed vector space, with $a, b \in X$. Then $[\{a\}, \{b\}]_{KL(X)}$ is metrically convex if and only if $[a, b] = [\![a, b]\!]$.*

Proof. For the moment, let $[A, B]$ abbreviate $[A, B]_{KL(X)}$ whenever $A, B \in KL(X)$. If $[a, b] = [\![a, b]\!]$, then $[\{a\}, \{b\}] = [\![\{a\}, \{b\}]\!]$, by Lemma 4.1, and is hence metrically convex.

As to the converse: if $\dim(X) \leq 1$ the equality $[a, b] = [\![a, b]\!]$ always holds; so assume $\dim(X) > 1$ and suppose $a, b \in X$ are such that there is some $x \in [a, b] \setminus [\![a, b]\!]$. Let $P \subseteq X$ be a plane containing $\{a, b, x\}$. In order to show that $[a, b]$ is metrically nonconvex, it suffices to show the same for $[a, b] \cap P$. Using rescaled translation, we may take a as the origin – so P is a vector subspace – and $b \in S(a; 1)$, the unit sphere in P. From here on, we are working in a two-dimensional normed vector space; hence we have justified the assertion that we lose no generality in assuming $\dim(X) = 2$, say $X = \langle \mathbb{R}^2, \|\cdot\| \rangle$.

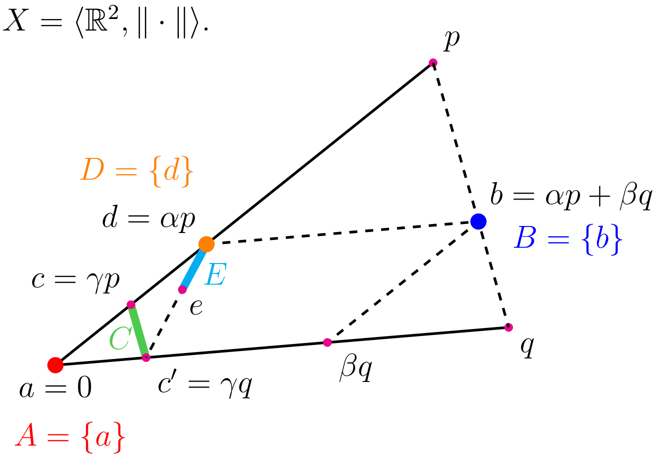

Figure 2: Schematic for Theorem 4.9. The set $[A, B] = [\{a\}, \{b\}]$ is described in Figure 1 above. In this case we have $C, D \in [A, B]$ and $E \in [C, D]$ but $E \notin [A, B]$. Hence $[\{a\}, \{b\}]$ is not metrically convex.

Since $a = 0$, $b \in S(a; 1)$, and $[0, b]$ properly contains $[\![0, b]\!]$, we know from Lemma 4.8 that there are distinct $p, q \in S(0; 1)$, extreme points of the closed unit ball $N(0; 1)$, such that $[\![p, q]\!] \subseteq S(0; 1)$ and $b \in [\![p, q]\!] \setminus \{p, q\}$. Hence there are scalars $0 < \alpha \leq \beta < 1$, with $\alpha + \beta = 1$, such that $b = \alpha p + \beta q$. Then – again from Lemma 4.8 – $[0, b]$ is the proper parallelogram with vertices $0, \alpha p, \beta q, b$. Fix $c = \gamma p$, where $0 < \gamma < \alpha$. We also single out the points $d = \alpha p$ and $c' = \gamma q$, as well as the sets $A = \{0\}$, $B = \{b\}$, $C = [\![c, c']\!]$, and $D = \{d\}$. Note that C is nondegenerate, and equals $[0, b] \cap S(0; \gamma)$, a line segment parallel to $[\![p, q]\!]$. Hence, by Lemma 4.1, both C and D are in $[A, B]$. We aim to find $E \in [C, D] \setminus [A, B]$. To this end, note that $[\![c', d]\!]$ is nondegenerate; so fix $e \in [\![c', d]\!] \setminus \{c', d\}$, and let $E =[\![e, d]\!]$. Then E is a line segment that is not parallel to $[\![p, q]\!]$; hence $E \notin [A, B]$, by Lemma 4.1. It thus remains to show that if e is judiciously chosen, then $E \in [C, D]$; i.e.,

$$\varrho_H(C, E) + \varrho_H(E, D) = \varrho_H(C, D).$$

First, using Lemma 4.7, we immediately obtain $\varrho_H(E, D) = \|e - d\|$ and $\varrho_H(C, D) = \max\{\|c - d\|, \|c' - d\|\}$. In order to to determine which extreme point of C is farther from d, note that since p is an extreme point of $N(0; 1)$, we know [6, Lemma 5.8] that $[0, d] = [\![0, d]\!]$. Thus, for any $x \in C \setminus \{c\}$, we have $\|c\| + \|c - d\| = \|d\| < \|x\| + \|x - d\|$. But $\|c\| = \|x\| = \gamma$; hence we have $\|c - d\| < \|x - d\|$. In particular, $\varrho_H(C, D) = \|c' - d\|$.

It thus remains to show that, for suitable e, we have $\varrho_H(C, E) = \|c' - e\|$. Now, again using Lemma 4.7, we know that

$$\varrho_H(C, E) = \max\{\varrho(c', E), \varrho(c, E), \varrho(e, C), \varrho(d, C)\}.$$

The first term is exactly $\|c' - e\| = \|c' - d\| - \|e - d\|$, which is our aspirational value for $\varrho_H(C, E)$; so we need to choose e such that the other three terms are relatively small. Note that both the second and third terms are $\leq \|c - e\|$, while the fourth is exactly $\|c-d\|$. Thus if e is "close" to d, then the first term is "close" to $\|c'-d\|$, while the others are "close to being dominated by" $\|c-d\|$, which is strictly less than $\|c'-d\|$. To make this intuition precise, let r be the value $\|c'-d\|-\|c-d\|$, which we determined earlier to be positive. It is enough to choose $e \in [\![c', d]\!]$ such that $0 < \|e-d\| \leq r/2$. For then

$$\max\{\varrho(c, E), \varrho(e, C)\} \leq \|c-e\| \leq \|e-d\| + \|c-d\| \leq \|c'-d\| - \|e-d\| = \varrho(c', E)$$

because $2\|e-d\| \leq r$. Finally

$$\varrho(d, C) = \|c-d\| = \|c'-d\| - r < \|c'-d\| - r/2 \leq \|c'-d\| - \|e-d\| = \varrho(c', E),$$

completing the proof. □

We then have the following immediate characterisation of strict convexity.

Corollary 4.10. *For X a normed vector space, the following three statements are equivalent.*

(i) *X is strictly convex (i.e., $[a, b] = [\![a, b]\!]$ for all $a, b \in X$).*

(ii) *For all $a, b \in X$, we have $[\{a\}, \{b\}]_{KL(X)} = [\![\{a\}, \{b\}]\!]$.*

(iii) *For all $a, b \in X$, $[\{a\}, \{b\}]_{KL(X)}$ is metrically convex.*

Remarks 4.11. (i) Corollary 4.10 makes essential use of the nature of hyperspaces. Just having metric convexity in each interval $[a, b]$ is too weak a condition and does not imply strict convexity in a normed vector space X: by [6, Corollary 5.16], each such interval is metrically convex as long as all nonextreme points of the unit ball of X are coplanar (which is always the case in dimension two).

(ii) Let X be any normed vector space. By Lemma 4.8, if $a \in S(0; 1)$, then $[0, a] = [\![0, a]\!]$ if and only if $a \in \varepsilon(N(0; 1))$. By [7, Example 4.19], every nondegenerate metric interval in $X = c_0$ (see Remark 2.4 (iii)) is noncompact; hence $\varepsilon(N(0; 1)) = \emptyset$ in this case. However, in general, any bracket point of a metric interval – always closed, bounded, and linearly convex – is an extreme point of that interval, regardless of whether it is compact. To see this, suppose $a, b, c, d \in X$ with $b \in [\![c, d]\!] \subseteq [a, b]$. If X is one-dimensional, the result is trivial; otherwise, let P be a plane containing a, c, d. Then P also contains b; hence by rescaled translation and Lemma 4.8, we know that $[a, b] \cap P$ is either a singleton, a line segment, or a parallelogram with a and b as diagonally opposite vertices. In any case we cannot have $[\![c, d]\!] \subseteq [a, b] \cap P$ without either $b = c$ or $b = d$. Hence b is an extreme point of $[a, b]$.

Extreme points of intervals need not be bracket points; for example, let X be the Cartesian plane, where the norm agrees with the taxicab norm in the first and third quadrants (both coordinates have the same sign) and with the Euclidean norm elsewhere. Let $a = \langle 0, 0\rangle$ and $b = \langle 1, 1\rangle$. Then $[a, b]$ is the square $[0, 1]^2$. However, the extreme points $c = \langle 1, 0\rangle$ and $d = \langle 0, 1\rangle$ are not bracket points of $[a, b]$ since $[c, d] = [\![c, d]\!]$. □

5. Linear and metric intervals compared in $K(X)$

As mentioned above, each linear interval $[\![a, b]\!]$ in a normed vector space is contained in its corresponding metric interval $[a, b]$. The issue we wish to address in this section is the extent to which this remains true at the level of compact sets. Specifically, if

X is a normed vector space and $A, B, C \in K(X)$ are such that $C \in [\![A, B]\!]$, when is it true that $\varrho_H(A, B) = \varrho_H(A, C) + \varrho_H(C, B)$? From Proposition 2.3 (v) above, we know that an affirmative answer is not assured without a degree of linear convexity: if A is any two-point set, then $[\![A, A]\!]$ is not equal to $\{A\}$; hence $[\![A, A]\!] \not\subseteq [A, A]_{K(X)}$.

The following shows that $[\![A, B]\!]$ may be contained in $[A, B]_{K(X)}$, even when one of the bracket points is not linearly convex.

Theorem 5.1. *Let X be a normed vector space, with $A \in F_1(X)$ and $B \in K(X)$. Then $[\![A, B]\!] \subseteq [A, B]_{K(X)}$.*

Proof. By Proposition 2.6, we know that $[\![A, B]\!] \subseteq K(X)$, so we must now show that for any $C \in [\![A, B]\!]$ it follows that $\varrho_H(A, C) + \varrho_H(C, B) \leq \varrho_H(A, B)$. Let $A = \{a\}$. By rescaled translation we lose no generality in assuming $a = 0$. Since B is compact, there is some $b_0 \in B$ such that $\varrho_H(A, B) = \|a - b_0\| = \|b_0\|$; thus for all $b \in B$, we have $\|b\| \leq \|b_0\|$. Fix arbitrary $C \in [\![A, B]\!]$, say $C = tB$ for some $t \in [0, 1]$. It is enough to show that $\varrho_H(A, C) \leq t\|b_0\|$ and that $\varrho_H(C, B) \leq (1 - t)\|b_0\|$.

The first inequality is immediate because $\|tb\| = t\|b\| \leq t\|b_0\|$ for every $b \in B$. As for the second, we have

$$\begin{aligned}\varrho_H(C, B) &= \max\{\sup\{\varrho(c, B) : c \in C\}, \sup\{\varrho(b, C) : b \in B\}\} \\ &= \max\{\sup\{\varrho(tb, B) : b \in B\}, \sup\{\varrho(b, tB) : b \in B\}\}.\end{aligned}$$

For each $b \in B$, both $\varrho(tb, B)$ and $\varrho(b, tB)$ are at most $\|tb - b\| = (1-t)\|b\| \leq (1-t)\|b_0\|$; hence we immediately have $\varrho_H(C, B) \leq (1 - t)\|b_0\|$, as desired. □

Remark 5.2. From Example 3.1, we know that transitivity may not hold for $[\![A, B]\!]$ when both sets are compact and one of them is also a singleton. However, since metric betweenness always satisfies transitivity, Theorem 5.1 shows that even if $C \in [\![A, B]\!]$ and $D \in [\![A, C]\!] \setminus [\![A, B]\!]$, it still follows that $D \in [A, B]_{K(X)}$.

6. The Rådström extension of $KL(X)$

Hans Rådström's construction [18] shows how we may treat certain hyperspaces of a normed vector space as linearly convex cones in other normed vector spaces. This gives us a very powerful tool in the study of both linear and metric betweenness in such hyperspaces. We outline the process for $KL(X)$ as follows (with details being found in [8, 11, 18]).

Step 1. Start with a normed vector space X. As we saw in Section 2, we may treat $KL(X) = \langle KL(X), +, \{0\}\rangle$ as a commutative monoid under Minkowski addition. And while we may not have additive inverses, we have the next best thing, namely the following *cancellation* property [18, Lemma 2]: *Let $A, B, C \in KL(X)$ be such that $A + C = B + C$. Then $A = B$.*

Step 2. With any commutative monoid $M = \langle M, +, 0\rangle$ satisfying the cancellation property, we may extend M to an Abelian group by the so-called "method of differences," in a manner analogous to how we extend the natural numbers to the integers. That is, we first define the equivalence relation $\sim$ on $M \times M$ by stipulating that $\langle a, b\rangle \sim \langle c, d\rangle$ precisely when $a + d = c + b$. We denote the set of equivalence classes by $M^{\sim} := \{\langle a, b\rangle^{\sim} : a, b \in M\}$, and think of a representative

$\langle c,d\rangle \in \langle a,b\rangle^{\sim}$ as the "difference" $c-d$. Addition in $M^{\sim}$ is defined in the obvious way; i.e., $\langle a,b\rangle^{\sim} + \langle c,d\rangle^{\sim} := \langle a+c, b+d\rangle^{\sim}$, and it is straightforward to show that this addition is well defined and $M^{\sim}$ is an Abelian group. (The additive identity is $\langle 0,0\rangle^{\sim}$, and $-\langle a,b\rangle^{\sim} := \langle b,a\rangle^{\sim}$.) The function $\varphi_M : M \to M^{\sim}$, given by $a \mapsto \langle a,0\rangle^{\sim}$, is a monoid embedding; in this way we view $M^{\sim}$ as an extension of M. Moreover, if G is any Abelian group and $\psi : M \to G$ is a monoid embedding, then the assignment $\langle a,b\rangle^{\sim} \mapsto \psi(a) - \psi(b)$ defines a group embedding $\mu : M^{\sim} \to G$ such that $\mu \circ \varphi_M = \psi$.

Step 3. Back to our monoid $KL(X)$, we denote $KL(X)^{\sim}$ by $\mathcal{R}(X)$, which we call the *Rådström extension* of $KL(X)$. We extend the scalar multiplication on $KL(X)$ to one on $\mathcal{R}(X)$ in the obvious manner: if $\langle A,B\rangle^{\sim} \in \mathcal{R}(X)$ and $t \geq 0$, set $t\langle A,B\rangle^{\sim} := \langle tA, tB\rangle^{\sim}$; and if $t < 0$, set $t\langle A,B\rangle^{\sim} := \langle (-t)B, (-t)A\rangle^{\sim}$. When there is no danger of confusion, we let φ abbreviate $\varphi_{KL(X)}$. Then, for $t \geq 0$, we have $\varphi(tA) = \langle tA, \{0\}\rangle^{\sim} = t\varphi(A)$, so φ respects multiplication by nonnegative scalars. This embedding does *not* respect multiplication by negative scalars, however; so we must not confuse, say, $(-1)A$ with the additive inverse of A viewed as a vector in $\mathcal{R}(X)$. We disambiguate this situation by explicit mention of the embedding φ: indeed, $(-1)A \in KL(X)$, while $(-1)\varphi(A) = -\varphi(A) \in \mathcal{R}(X)$. Since no nonsingleton in $KL(X)$ has an additive inverse in $KL(X)$, we also know that $(-1)\varphi(A) \notin \varphi[KL(X)]$ whenever A is nondegenerate. (In general, we have $\varphi[KL(X)] + (-1)\varphi[KL(X)] = \mathcal{R}(X)$.)

This step allows us to treat $\mathcal{R}(X)$ as a vector space in which $KL(X)$ is embedded as a submonoid. Moreover, since $KL(X)$ is also closed under multiplication by nonnegative scalars, φ embeds $KL(X)$ as a *linearly convex cone* in $\mathcal{R}(X)$. Except for the degenerate case, this cone contains the nonzero vector subspace $\varphi[F_1(X)]$.[11] Furthermore, since no nonsingleton $A \in KL(X)$ has an additive inverse in $KL(X)$, we know that no subspace of $\mathcal{R}(X)$ containing $\varphi[F_1(X)]$ is contained in $\varphi[KL(X)]$.

Step 4. Our final step is to endow $\mathcal{R}(X)$ with a norm, and this is also done quite naturally: for any $\langle A,B\rangle \in KL(X) \times KL(X)$, define the norm $\|\langle A,B\rangle^{\sim}\|$ to be the Hausdorff distance $\varrho_H(A,B)$. This is well defined because of the invariance of the Hausdorff metric under translations: $\varrho_H(A+C, B+C) = \varrho_H(A,B)$ (see [18, Lemma 3] and Remark 2.7 (iv) above). Thus φ is an isometric monoid isomorphism that preserves multiplication by nonnegative scalars.

From here on we we will be placing our focus on $KL(X)$. Hence we drop subscripts when talking of metric intervals in this hyperspace and let $[A,B]$ abbreviate $[A,B]_{KL(X)}$ whenever $A,B \in KL(X)$. And, since we are viewing $KL(X)$, with its Hausdorff metric, as the linearly convex cone $\varphi[KL(X)]$ in the normed vector space $\mathcal{R}(X)$, each hyperline segment $[\![\varphi(A), \varphi(B)]\!]$ is a *bona fide* line segment.

For any elements $V,W \in \mathcal{R}(X)$, let $[V,W]_{\mathcal{R}}$ be the metric interval in $\mathcal{R}(X)$ bracketed by V and W. For $A,B \in KL(X)$, we let $[\varphi(A), \varphi(B)]$ be shorthand for $[\varphi(A), \varphi(B)]_{\mathcal{R}} \cap \varphi[KL(X)]$. Then φ is an isometry between $[A,B]$ and $[\varphi(A), \varphi(B)]$. Because $\varphi[KL(X)]$ is linearly convex in $\mathcal{R}(X)$, we know $[\![\varphi(A), \varphi(B)]\!] \subseteq [\varphi(A), \varphi(B)]$. We need to make a clear distinction between $[\varphi(A), \varphi(B)]$ and $[\varphi(A), \varphi(B)]_{\mathcal{R}}$, however, because $\varphi[KL(X)]$ is not at all guaranteed to be metrically convex in $\mathcal{R}(X)$, as

[11] Cones that contain nonzero vector subspaces are often termed *flat*.

we shall see in Theorem 8.5 below. The following is a description of $[\varphi(A), \varphi(B)]_{\mathcal{R}}$ that will be useful in the proof of that theorem.

Proposition 6.1. *Let X be a normed vector space, with $A, B \in KL(X)$ and $V \in \mathcal{R}(X)$. Then $V \in [\varphi(A), \varphi(B)]_{\mathcal{R}}$ if and only if whenever $C, D \in KL(X)$ are such that $V = \langle C, D\rangle^{\sim}$, it follows that $C \in [A + D, B + D]$.*

Proof. Start with $A, B \in KL(X)$. For any $C, D \in KL(X)$, we have obviously $\langle C, D\rangle^{\sim} \in [\varphi(A), \varphi(B)]_{\mathcal{R}}$ if and only if

$$\|\langle A, \{0\}\rangle^{\sim} - \langle C, D\rangle^{\sim}\| + \|\langle C, D\rangle^{\sim} - \langle B, \{0\}\rangle^{\sim}\| = \|\langle A, \{0\}\rangle^{\sim} - \langle B, \{0\}\rangle^{\sim}\|.$$

By definition of the norm in $\mathcal{R}(X)$, this is equivalent to the following equality of Hausdorff distances: $\varrho_H(A + D, C) + \varrho_H(C, B + D) = \varrho_H(A, B)$.

And, by translation invariance [18, Lemma 3], this is equivalent to the equality

$$\varrho_H(A + D, C) + \varrho_H(C, B + D) = \varrho_H(A + D, B + D);$$

i.e., the condition that $C \in [A + D, B + D]$. □

With this brief discussion in mind, we have the following easy consequences of the Rådström extension construction.

Corollary 6.2. *Let X be a normed vector space, with $A, B \in KL(X)$.*

(i) $[A, B]$ *is linearly convex. In particular,* $[\![A, B]\!] \subseteq [A, B]$.

(ii) *The parameterization* $t \mapsto (1 - t)A + tB$ *is a dilation from* $[0, 1]$ *into* $KL(X)$, *with dilation constant* $\varrho_H(A, B)$.

(iii) *Linear betweenness in $KL(X)$ satisfies the strong collinearity axiom.*

(iv) *Both linear and metric betweenness in $KL(X)$ satisfy the gap-freeness axiom.*

Proof. Ad (i). The metric interval $[\varphi(X), \varphi(B)]_{\mathcal{R}}$ in $\mathcal{R}(X)$ is linearly convex, by Lemma 4.5. Since $\varphi[KL(X)]$ is also linearly convex, so too is the intersection; hence $[A, B]$ is linearly convex in $KL(X)$.

Ad (ii). The parameterization $t \mapsto (1 - t)\varphi(A) + t\varphi(B)$ is a dilation from $[0, 1]$ into $\mathcal{R}(X)$, with dilation constant $\varrho_H(A, B) = \|\varphi(A) - \varphi(B)\|$. The conclusion follows, since $\varphi[KL(X)]$ is linearly convex in $\mathcal{R}(X)$.

Ad (iii). This follows because strong collinearity holds for linear betweenness in vector spaces, and hence in linearly convex subsets of vector spaces.

Ad (iv). This follows from (i) and (ii). □

Another consequence, though not as immediate, is the following companion to Proposition 2.5 (iii).

Theorem 6.3. *Let X be a normed vector space, with $A, B \in KL(X)$. Then $\bigcup[A, B] \in L(X)$. If X is also finite-dimensional, then $\bigcup[A, B] \in KL(X)$.*

Proof. By Corollary 6.2 (i), $[A, B]$ is linearly convex in $KL(X)$. Suppose now that $c, d \in \bigcup[A, B]$, say $c \in C \in [A, B]$ and $d \in D \in [A, B]$. If $0 \leq t \leq 1$, we have $(1 - t)c + td \in (1 - t)C + tD \in [\![C, D]\!] \subseteq [A, B]$. Hence $(1 - t)c + td \in \bigcup[A, B]$, showing that $\bigcup[A, B]$ is linearly convex.

We next show that $\bigcup[A,B]$ is bounded in X. Metric intervals are always bounded in their respective metric spaces, so that $[A,B]$ is bounded, say by $M > 0$, as a Hausdorff metric subspace of $KL(X)$. Let $c, d \in \bigcup[A,B]$, where $c \in C \in [A,B]$ and $d \in D \in [A,B]$. Then $\varrho(c,D) \leq M$; so $\|c-d\| = \varrho(c,d) \leq M + \delta$, where δ is the diameter of D in X. Thus $\bigcup[A,B]$ is bounded in X.

Finally we assume X is finite-dimensional and show that $\bigcup[A,B]$ is closed in X. Then, since $\bigcup[A,B]$ has already been shown to be bounded, we will be able to infer that it is compact. Suppose $\langle c_n\rangle$ is a sequence of elements of $\bigcup[A,B]$, with $\langle c_n\rangle \to c \in X$. For each $n = 0,1,2,\ldots$, we have $c_n \in C_n \in [A,B]$. The sequence $\langle C_n\rangle$ is bounded as a sequence in $KL(X)$ because each term lies in the bounded subset $[A,B]$. Thus, by the Blaschke Selection Theorem [11, Proposition IX.9], there is a subsequence of $\langle C_n\rangle$ that converges to some $C \in KL(X)$. But $[A,B]$ is closed in $KL(X)$, so $C \in [A,B]$. The corresponding subsequence of $\langle c_n\rangle$ also converges to c, so it remains to show that $c \in C$.

Indeed, suppose $\langle D_n\rangle \to D \in KL(X)$ and $\langle d_n\rangle \to d \in X$, where $d_n \in D_n$ for all $n \geq 0$, but $d \notin D$. Since D is closed, we may fix $r > 0$ such that the open ball $N^\circ(d;r)$ is disjoint from D. Since $\langle d_n\rangle \to d$, we may also fix $n_0 \geq 0$ such that for all $m \geq n_0$, $d_m \in N^\circ(d;r/2)$. Thus, for each $m \geq n_0$ we have $\varrho(d_m,D) \geq r/2$; hence $\varrho_H(D_m,D) \geq \varrho(D_m,D) \geq r/2$. This contradicts the fact that $\langle D_n\rangle \to D$, and completes the proof. □

Remarks 6.4. (i) Since it is possible for $[A,B]$ to contain $[\![A,B]\!]$ properly (see Theorem 4.6 above), it is also possible for $\bigcup[A,B]$ to contain $\bigcup[\![A,B]\!]$ properly. The latter, by Proposition 2.3 (vi), is the linear convex hull of $A \cup B$; thus, even though $\bigcup[A,B]$ is a linearly convex set containing $A \cup B$ (Theorem 6.3), it is not necessarily the linear convex hull of $A \cup B$. As an easy example, suppose X is a normed plane, $A = \{a\}$ and $B = \{b\}$, where $[a,b]$ is a proper parallelogram (see Lemma 4.8). Then $\bigcup[A,B] = [a,b]$, while $\bigcup[\![A,B]\!] = [\![a,b]\!]$, a diagonal of $[a,b]$.

(ii) Suppose X is such that no nondegenerate metric interval is compact (such as $X = c_0$, see Remark 2.4 (iii)). Then, for $a, b \in X$ distinct, we have that $[a,b] \subseteq \bigcup[\{a\},\{b\}]$ is closed in X, hence closed in $\bigcup[\{a\},\{b\}]$. This shows that the larger set is not compact; hence the dimension assumption in Theorem 6.3 cannot be dropped.

(iii) In [8] and [11], $\mathcal{R}(X)$ refers to the Rådström extension of $HL(X)$, the hyperspace of closed bounded linearly convex subsets of X. The technical issue that $HL(X)$ need not be closed under Minkowski addition is mitigated by defining $A \oplus B$ to be the closure of $A + B$ – noting that the closure of a linearly convex set is still linearly convex – and the rest of the construction proceeds without further difficulties. In the case where X is infinite-dimensional, $HL(X)$ is much larger than $KL(X)$: indeed, each nondegenerate closed ball is in $HL(X) \setminus KL(X)$.

7. $\mathcal{R}(X)$ in dimension one

In this section we examine the simple case where the dimension of the base normed vector space is one, with an eye toward comparisons with the multi-dimensional situation in the next section. As above, when X is a normed vector space, we have the isometric embedding $\varphi : KL(X) \to \mathcal{R}(X)$.

Assume that $\dim(X) = 1$, say $u \in X$ is a unit vector spanning X. Then each $A \in KL(X)$ is a closed linear interval $[\![\alpha u, \beta u]\!]$, uniquely specified by the ordered pair $\psi(A) := \langle \alpha, \beta \rangle \in \mathbb{R}^2$, where $\alpha \leq \beta$. Let H be the cone $\{\langle x, y \rangle \in \mathbb{R}^2 : x \leq y\}$, with L the bounding line $\{\langle x, x \rangle : x \in \mathbb{R}\}$ of H. Then ψ is a bijection from $KL(X)$ onto H, with members of $F_1(X)$ assigned to L. Furthermore, ψ is a monoid homomorphism that respects multiplication by nonnegative scalars. Because of the straightforward calculation of the Hausdorff distance between two closed bounded subsets of $\mathbb{R}$ in terms of end points (see Lemma 4.7), we see that

$$\varrho_H([\![\alpha u, \beta u]\!], [\![\gamma u, \delta u]\!]) = \max\{|\alpha - \gamma|, |\beta - \delta|\}.$$

Hence ψ is an isometry from $KL(X)$ onto $H \subseteq \mathbb{R}^2_\infty$, real 2-space with the *max norm*: $\|\langle x, y \rangle\|_\infty := \max\{|x|, |y|\}$. As mentioned in Step 2 of Section 6 above, we have the map $\mu : \mathcal{R}(X) \to \mathbb{R}^2_\infty$, given by $\mu(\langle A, B \rangle^\sim) = \psi(A) - \psi(B)$ (so $\mu \circ \varphi = \psi$, and μ may be seen as *extending* ψ to $\mathcal{R}(X)$). Now one readily verifies that μ is an isometric isomorphism, and we may treat $\mathcal{R}(X)$ as $\mathbb{R}^2_\infty$, with $KL(X)$ and $F_1(X)$ identified with H and L, respectively. We collect a few easy consequences of this discussion.

Proposition 7.1. *Let X be a one-dimensional normed vector space.*

(i) *$\mathcal{R}(X)$ is a Banach space, of dimension two.*

(ii) *$\varphi[KL(X)]$ and $\varphi[F_1(X)]$ are closed in $\mathcal{R}(X)$.*

(iii) $\varphi[KL(X)] \cap (-1)\varphi[KL(X)] = \varphi[F_1(X)]$.

(iv) $\varphi[KL(X)] \cup (-1)\varphi[KL(X)] = \mathcal{R}(X)$.

(v) *$\varphi[KL(X)]$ is metrically convex in $\mathcal{R}(X)$.*

(vi) *No closed ball of positive radius in $KL(X)$ is metrically convex.*

(vii) *Every metric interval in $KL(X)$ is metrically convex.*

Proof. Ad (i–iv). These statements follow immediately from the discussion above.

Ad (v). All we need to show is that H is metrically convex in $\mathbb{R}^2_\infty$. Indeed, any metric interval $[a, b] \in \mathbb{R}^2_\infty$ is a solid rectangle with points a and b at opposite corners, and two of its sides parallel to the line L. Thus, if two opposite corners of such a rectangle lie in H, the entire rectangle also lies in H.

Ad (vi). Any closed ball in $\mathbb{R}^2_\infty$ is a solid square, with sides parallel to the coordinate axes, and with the centre of the ball being the intersection of the two diagonals. Hence a closed ball with centre in H retains the entire northernmost side of the original square. If a, b are the end points of that side, then $[a, b]$, the square whose diagonal is the line segment $[\![a, b]\!]$, does not lie in the ball.

Ad (vii). If $c, d \in [a, b]$, a metric interval in $\mathbb{R}^2_\infty$, then the rectangle with opposite corners c and d has sides parallel to the sides of the rectangle $[a, b]$. Hence $[c, d] \subseteq [a, b]$. (This also follows from [6, Theorem 5.14], which says that metric intervals are always metrically convex in a normed vector space of dimension ≤ 2.) □

8. $\mathcal{R}(X)$ in higher dimensions

In this section we remove the assumption that our normed vector space is of dimension one, and consider the seven clauses of Proposition 7.1 in that light.

We first make the observation that if Y is a vector subspace of X, then clearly $KL(Y)$ is a subset of $KL(X)$. However, when we pass to Rådström extensions, a subtlety emerges, and we need to use a subscript when we speak of equivalence classes in Y. So when $A, B \in KL(Y)$,

$$\langle A, B\rangle_Y^\sim := \{\langle C, D\rangle \in KL(Y) \times KL(Y) : A + D = C + B\}.$$

Clearly we have $\langle A, B\rangle_Y^\sim \subseteq \langle A, B\rangle^\sim$, and the mapping $\theta : \mathcal{R}(Y) \to \mathcal{R}(X)$, given by $\langle A, B\rangle_Y^\sim \mapsto \langle A, B\rangle^\sim$, respects the vector space operations as well as the norm, and has trivial kernel. Hence θ is an isometric linear embedding. In this relative case, let $\varphi_Y : KL(Y) \to \mathcal{R}(Y)$ be the canonical embedding $\varphi_{KL(Y)}$, $\eta : KL(Y) \to KL(X)$ denoting the inclusion map. Then we have the commutative relationship $\theta \circ \varphi_Y = \varphi \circ \eta$. In particular, if $\mathcal{P}$ is a normed vector space property that is inherited by subspaces, and we are able to show that $\mathcal{P}$ fails in $\mathcal{R}(X)$ for all X of, say, dimension two, then we know it must fail in the Rådström extensions of all higher-demensional spaces.

Remark 8.1. In [8], where $\mathcal{R}(X)$ is $HL(X)^\sim$ (see Remark 6.4 (iii)), it is no longer the case that $HL(Y) \subseteq HL(X)$; one must deal with the closure operation when Y is not necessarily closed in X (infinite-dimensional case only). Hence θ is now defined by the assignment $\langle A, B\rangle_Y^\sim \mapsto \langle \overline{A}, \overline{B}\rangle^\sim$, where closure is relative to X. (This works because closure respects both boundedness and linear convexity.)

In [8, Theorem 2.2] the authors show that θ is a closed map in this setting, and it is also surjective whenever Y is dense in X. Neither of these assertions is true, however, in the situation where $\mathcal{R}(X)$ is $KL(X)^\sim$; consider the following example.

Let $X = c_0$ be the Banach space of real null sequences *à la* Remark 2.7 (iii), with $Y = c_{00}$ the subspace of all null sequences that are eventually zero. Then Y is well known to be dense in X. If $\theta : \mathcal{R}(Y) \to \mathcal{R}(X)$ were surjective, then for each $\langle A, B\rangle \in KL(X) \times KL(X)$ there would be a pair $\langle A', B'\rangle \in KL(Y) \times KL(Y)$ such that $A' + B = A + B'$. Since adding a nontrivial null sequence to a trivial one results in a nontrivial null sequence, this becomes an impossibility if A consists of a single nontrivial null sequence and the only element of B is the zero sequence.

On the other hand, if $A = \{a\}$ and $B = \{0\}$, let $b \in Y$ be arbitrary. Then $\|\langle\{a\}, \{0\}\rangle^\sim - \langle\{b\}, \{0\}\rangle^\sim\| = \|\langle\{a\}, \{0\}\rangle^\sim + \langle\{0\}, \{b\}\rangle^\sim\| = \|\langle\{a\}, \{b\}\rangle^\sim\| = \|a - b\|$. Since Y is dense in X, this shows that $\langle A, B\rangle^\sim$, while not in $\theta[\mathcal{R}(Y)]$ itself, is in its closure. □

We are now ready to consider Proposition 7.1 in the multi-dimensional case.

Re (i). When $\dim(X) > 1$, $\mathcal{R}(X)$ is no longer a Banach space [8, Theorem 2.1].[12] Because all finite-dimensional normed vector spaces are Banach spaces, we immediately infer that $\mathcal{R}(X)$ is infinite-dimensional.

We also get infinite-dimensionality from another route: By [17, Theorem 7.3], when $1 < \dim(X) < \aleph_0$, $KL(X)$ is homeomorphic to the complement of any point in the Hilbert cube. (In the one-dimensional case, $KL(X)$ is homeomorphic to the complement of a point on the boundary of a two-cell.) This makes $KL(X)$ infinite-dimensional as a topological space; hence $\mathcal{R}(X)$ is infinite-dimensional as a vector space. (The case when X itself is infinite-dimensional is trivial because $\varphi[F_1(X)]$ is an embedded copy of X in $\mathcal{R}(X)$.)

[12] See also [12], where there is an example – attributed to R. J. Aumann and S. Kakutani – of a nonconvergent Cauchy sequence in $\mathcal{R}(X)$, where $\dim(X) = 2$.

Now, every infinite-dimensional Banach space has vector space dimension at least $\mathfrak{c} = 2^{\aleph_0}$, the cardinality of the continuum [14]. But even in the absence of metric completeness on the part of $\mathcal{R}(X)$, we can place $\mathfrak{c}$ as a lower bound for dimension.

Theorem 8.2. *Let X be a normed vector space of dimension greater than one. Then there is a set $\mathcal{L} \subseteq KL(X)$, of cardinality $\mathfrak{c}$, such that $\varphi[\mathcal{L}]$ is a linearly independent family of vectors in $\mathcal{R}(X)$. Hence the vector space dimension of $\mathcal{R}(X)$ is at least $\mathfrak{c}$. Moreover, there is a basis for $\mathcal{R}(X)$ consisting entirely of vectors from $\varphi[KL(X)]$.*

Proof. By the remarks above, we lose no generality in assuming that X is two-dimensional, say $X = \langle \mathbb{R}^2, \|\cdot\|\rangle$. (The norm on X is irrelevant here.)

For each $\theta \in \mathbb{R}$, let $l(\theta) = \langle \cos\theta, \sin\theta\rangle$ be the point on the Euclidean unit circle of angle argument θ, with $L(\theta)$ the line segment $[\![0, l(\theta)]\!] \in KL(X)$. We set $\mathcal{L} = \{L(\theta) : \theta \in [0, \frac{\pi}{2}]\}$. Clearly $|\varphi[\mathcal{L}]| = \mathfrak{c}$; we claim that $\varphi[\mathcal{L}]$ is linearly independent in $\mathcal{R}(X)$.

If this is not true, there is a linear combination $\sum_{i=1}^n c_i\varphi[L(\theta_i)]$ that equates to zero in $\mathcal{R}(X)$, where the angles are distinct and the coefficients are all nonzero. It is safe to assume that at least some of the coefficients are positive. If they are all positive, then our linear combination is

$$\sum_{i=1}^n c_i\langle L(\theta_i), \{0\}\rangle^\sim = \Big\langle \sum_{i=1}^n c_i L(\theta_i), \{0\}\Big\rangle^\sim,$$

which, when equated to zero in $\mathcal{R}(X)$, means that the plane set $\sum_{i=1}^n c_i L(\theta_i)$ is $\{0\}$. But the linear combination on the lefthand side clearly contains nonzero elements; in particular it contains $c_1 l(\theta_1) \neq 0$.

Thus some of the original coefficients must be negative; so there are $1 \leq m < n$ and a linear combination

$$\sum_{i=1}^m d_i\langle L(\theta_i), \{0\}\rangle^\sim - \sum_{i=m+1}^n d_i\langle L(\theta_i), \{0\}\rangle^\sim$$

that equates to zero, where the θ_i are all distinct, and the new coefficients are all positive. This time the linear combination on the left becomes

$$\Big\langle \sum_{i=1}^m d_i L(\theta_i), \sum_{i=m+1}^n d_i L(\theta_i)\Big\rangle^\sim.$$

And for this to be zero in $\mathcal{R}(X)$, it follows that

$$\sum_{i=1}^m d_i L(\theta_i) = \sum_{i=m+1}^n d_i L(\theta_i).$$

Without loss of generality, assume that $\theta_1 > \theta_i$ for $m+1 \leq i \leq n$. Then clearly $d_1 l(\theta_1)$ belongs to the lefthand linear combination. We claim it does not belong to the right; i.e., that it is not of the form $\sum_{i=m+1}^n t_i l(\theta_i)$, where $t_i \in [0, d_i]$ for each $m+1 \leq i \leq n$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional $\langle x, y\rangle \mapsto x\sin\theta_1 - y\cos\theta_1$. Then the kernel of f is the one-dimensional linear subspace of $\mathbb{R}^2$ spanned by $l(\theta_1)$.

For $m+1 \leq i \leq n$ we have $\cos\theta_i > \cos\theta_1$ and $\sin\theta_i < \sin\theta_1$; hence $f(l(\theta_i)) > 0$. Thus, by linearity, and since each $t_i \geq 0$, $f(\sum_{i=m+1}^{n} t_i l(\theta_i))$ is positive, unless each t_i is zero. But $f(d_1 l(\theta_1)) = 0$, while $d_1(\theta_1) \neq 0$. This shows that $d_1 l(\theta_1) \notin \sum_{i=m+1}^{n} d_i L(\theta_i)$, as claimed. Hence $\varphi[\mathcal{L}]$ is linearly independent, and therefore the vector space dimension of $\mathcal{R}(X)$ is at least $\mathfrak{c}$.

Since $\varphi[\mathcal{L}]$ is a linearly independent set of vectors, of cardinality $\mathfrak{c}$, a Zorn's lemma argument allows for the existence of a collection $\mathcal{M} \supseteq \mathcal{L}$ such that $\varphi[\mathcal{M}]$ is maximal with respect to being linearly independent. Thus $\varphi[\mathcal{M}]$ spans $\varphi[KL(X)]$. It also spans $(-1)\varphi[KL(X)]$ and forms a basis for $\mathcal{R}(X) = \varphi[KL(X)] + (-1)\varphi[KL(X)]$. □

Re (ii). This still holds, as long as X is a Banach space, and is easy to prove.

Proposition 8.3. *Let X be a Banach space. Then $\varphi[KL(X)]$ and $\varphi[F_1(X)]$ are closed in $\mathcal{R}(X)$.*

Proof. If X is Banach, then $KL(X)$ is complete in its Hausdorff metric [11, Proposition IX.1]; hence $\varphi[KL(X)]$ is a complete metric subspace of $\mathcal{R}(X)$. Any sequence in $\varphi[KL(X)]$ converging to something in $\mathcal{R}(X)$ is Cauchy, hence it converges to something in $\varphi[KL(X)]$. Thus $\varphi[KL(X)]$ is closed in $\mathcal{R}(X)$. Since $F_1(X)$ is closed in $KL(X)$, we have that $\varphi[F_1(X)]$ is closed in $\mathcal{R}(X)$ as well. □

Re (iii). In Section 2 above, We made the simple observation that for any vector space X, the only nonempty subsets with additive inverses under Minkowski addition are the singletons. Hence Proposition 7.1 (iii) holds without the dimension restriction.

Re (iv). This fails, fairly dramatically, in higher dimensions.

Theorem 8.4. *Let X be a normed vector space of dimension greater than one. Then $\varphi[KL(X)] \cup (-1)\varphi[KL(X)]$ has empty interior in $\mathcal{R}(X)$.*

Proof. Any member of $\varphi[KL(X)]$ (resp.,$(-1)\varphi[KL(X)]$) looks like $\langle A, \{0\}\rangle^{\sim}$ (resp., $\langle \{0\}, A\rangle^{\sim}$), where $A \in KL(X)$. So given $A \in KL(X)$ and $r > 0$ we aim to show that the ball $N(\langle A, \{0\}\rangle^{\sim}; r)$ is not contained in $\varphi[KL(X)] \cup (-1)\varphi[KL(X)]$. (The case where the centre of the ball is $\langle \{0\}, A\rangle^{\sim}$ is handled similarly.)

To meet our aim, we need to find some $\langle A', B\rangle^{\sim} \in \mathcal{R}(X)$ such that :

(1) $\|\langle A, \{0\}\rangle^{\sim} - \langle A', B\rangle^{\sim}\| \leq r$;

(2) there is no $C \in KL(X)$ such that $\langle A', B\rangle^{\sim} = \langle C, \{0\}\rangle^{\sim}$; and

(3) there is no $C \in KL(X)$ such that $\langle A', B\rangle^{\sim} = \langle \{0\}, C\rangle^{\sim}$.

In terms of $KL(X)$ alone: condition (1) amounts to saying $\varrho_H(A + B, A') \leq r$; conditions (2) and (3) say, respectively, that there is no $C \in KL(X)$ such that either $A' = B + C$ or $B = A' + C$.

There are two main cases to consider, depending upon whether or not A is a singleton. So assume first that $A = \{a\}$. Then we fix $a' \in S(a; r)$ and let $A' = [\![a, a']\!]$. Next we use the assumption that $\dim(X) > 1$ to fix $b \in S(0; r)$, such that the points $a, a + b, a'$ are noncollinear (hence b and $a' - a$ are nonparallel) and set $B = [\![0, b]\!]$. For any $C \in KL(X)$, $B + C$ is a union of translates of B, and so cannot equal A'.

Likewise, nothing of the form $A' + C$ can equal B; hence conditions (2) and (3) are satisfied. To address condition (1), we use Lemma 4.7, noting that $A+B = [\![a, a+b]\!]$ and $A = [\![a, a']\!]$ are two line segments, each of length r and having a common end point: $\varrho_H(A+B, A') = \max\{\varrho(a+b, A'), \varrho(a', A+B)\}$. Since $a \in A'$, the first term is $\leq \|b\| = r$; likewise, since $a \in A+B$, the second term is $\leq \|a' - a\| = r$. Hence $\varrho_H(A+B, A') \leq r$, and condition (1) is satisfied.

So henceforth we assume our centre A is not a singleton, and let δ be its positive diameter. We will be able to pick $A' = A$ in this case, so condition (1) reduces to $\|\langle A+B, A\rangle^{\sim}\| = \varrho_H(A+B, A) \leq r$, which by translation invariance [18, Lemma 3], amounts to saying that $\varrho_H(B, \{0\}) \leq r$; i.e., that $\|x\| \leq r$ for all $x \in B$. Conditions (2) and (3), as before, amount to saying that A (resp., B) is not the Minkowski sum of B (resp., A) and any element of $KL(X)$. We will be able to set $B = [\![0, b]\!]$, where b is suitably chosen from $S(0; r)$, automatically satisfying condition (1). Note that we lose no generality in assuming upper bounds for r; in particular, we take $0 < r < \delta$. Then, since the diameter of B is $r < \delta$, it cannot be the case that $B = A+C$ for any $C \in KL(X)$. (Otherwise there exist $x, y \in A$ with $\delta = \|x-y\| = \|(x+c)-(y+c)\|$, where $c \in C$ is arbitrary. But both $x+c$ and $y+c$ are in B, a contradiction.) This takes care of condition (3), so it remains to address condition (2).

Assume X is finite-dimensional. By the Krein-Milman Theorem , as well as Straszewicz's Theorem [19, Corollary 18.5.1 and Theorem 18.6], there is a point $p \in A$ that is *exposed*. This means there is a linear functional $f : X \to \mathbb{R}$ such that for all $x \in A \setminus \{p\}$, $f(x) < f(p)$. Let Y be the kernel of f. Then, by elementary linear algebra, $\dim(Y) = \dim(X) - 1 \geq 1$. So fix $b \in Y \cap S(0; r)$, with $B = [\![0, b]\!]$. Then $f(x) = 0$ for all $x \in B$. To show that there is no $C \in KL(X)$ such that $B+C = A$, assume otherwise. Then there is some $x \in [\![0, b]\!]$ and $c \in C$ with $p = x + c$. By linearity, $f(p) = f(x) + f(c) = f(c)$. Since $c = c + 0 \in C + [\![0, b]\!] \subseteq A$, we have $c = p$ because p is an exposed point of A. On the other hand, $c + b \in A$ too; and $f(c+b) = f(c)$. Thus $c + b = p$; i.e., $b = 0$, a contradiction. Thus condition (2) is satisfied as well as conditions (1) and (3).

Finally assume X is infinite-dimensional; we lose no generality in also assuming $0 < r < \min\{\delta, 1\}$. Then, by Riesz's lemma [13, 2.5-4] we may inductively produce a sequence $\langle u_0, u_1, \ldots\rangle$ of unit vectors in X such that for each $0 \leq m \leq n$, $\|u_{n+1} - u_m\| \geq r$. Then clearly there is no subsequence of $\langle u_n\rangle$ that is Cauchy, let alone convergent to some point in X.

We claim that for every $\varepsilon > 0$ there is some $n \geq 0$ such that no line segment contained in A and parallel to u_n can have length $\geq \varepsilon$. Indeed, if this is not the case, we may fix $\varepsilon_0 > 0$ and a sequence $\langle x_n\rangle$ in A such that $x_n + \varepsilon_0 u_n \in A$ for all $n \geq 0$. By the compactness of A, there is an increasing sequence of indices such that the correspoining subsequences of $\langle x_n\rangle$ and $\langle x_n + \varepsilon_0 u_n\rangle$ converge in A. But that implies that $\langle u_n\rangle$ has a convergent subsequence, a contradiction.

So pick $n \geq 0$ such that any line segment in A that is parallel to u_n cannot have length $\geq r/2$, and let $B = [\![0, ru_n]\!]$. Then B satisfies condition (1); also, given any $C \in KL(X)$ the sum $B + C$ is a union of translates of B, none of which can lie in A. Hence $A \neq B + C$, and condition (2) is satisfied. Condition (3) is satisfied, as above, since $r < \delta$. □

Re (v). In higher dimensions, the linearly convex cone $\varphi[KL(X)]$ is not metrically convex in $\mathcal{R}(X)$; indeed the following is true.

Theorem 8.5. *Let X be a normed vector space of dimension greater than one, with $a, b \in X$. Then $[\varphi(\{a\}), \varphi(\{b\})] = [\varphi(\{a\}), \varphi(\{b\})]_{\mathcal{R}}$ if and only if $a = b$.*

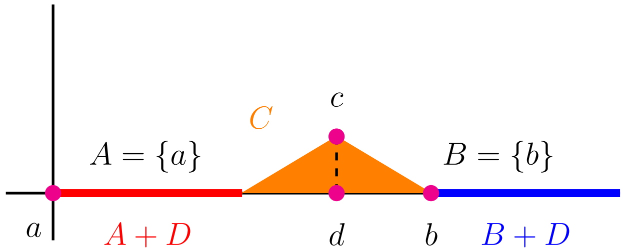

Figure 3: Schematic for Theorem 8.5. Here $D = [0, 1] \times \{0\}$.
We have $\langle C, D\rangle^{\sim} \in [\varphi(A), \varphi(B)]_{\mathcal{R}} = [\varphi(\{a\}), \varphi(\{b\})]_{\mathcal{R}}$ but $\langle C, D\rangle^{\sim} \notin \varphi(KL(X))$.

Proof. The "if" direction is obvious, so assume a and b are distinct points in X. By the remarks above pertaining to subspaces of normed vector spaces, we lose no generality in assuming X is two-dimensional; say $X = \langle \mathbb{R}^2, \|\cdot\|\rangle$.

Let $A = \{a\}$ and $B = \{b\}$. We need to show that $[\varphi(A), \varphi(B)]_{\mathcal{R}} \not\subseteq \varphi[KL(X)]$. By Propisition 6.1, the condition that $\langle C, D\rangle^{\sim}$ is in $[\varphi(A), \varphi(B)]_{\mathcal{R}}$ is equivalent to saying that, back in $KL(X)$, $C \in [A + D, B + D]$. Thus we need to find suitable $C, D \in KL(X)$ such that $\langle C, D\rangle^{\sim} \in [\varphi(A), \varphi(B)]_{\mathcal{R}} \setminus \varphi[KL(X)]$; i.e., that in addition to the condition that $C \in [A + D, B + D]$, there is no $E \in KL(X)$ such that $C = D + E$.

By rescaled translation we may take a to be the origin and $b \in S(0; 2)$. Let p be the unit vector $\frac{1}{2}b$. For our C and D, we let $D = [\![0, p]\!]$. Then $A + D = D$ and $B + D = [\![2p, 3p]\!]$ are collinear line segments of unit length. Next let $d = \frac{3}{2}p$, the midpoint of the segment $[\![p, 2p]\!]$, and choose $c \in X$ so that: (1) the points $p, 2p, c$ are noncollinear; and (2) $\|c - d\| \le \frac{1}{4}$. Finally let C be the proper triangle that is the convex hull of $\{p, 2p, c\}$.

To show that $C \in [A + D, B + D]$, we invoke the Bauer maximum principle in the form of Lemma 4.7. Both $A + D$ and $B + D$ are line segments, with extreme points $0, p$ and $2p, 3p$, respectively. Hence

$$\begin{aligned}\varrho_H(A + D, B + D) &= \max\{\varrho(0, B + D), \varrho(p, B + D), \varrho(2p, A + D), \varrho(3p, A + D)\} \\ &= \max\{2, 1, 1, 2\} = 2.\end{aligned}$$

If we show $\max\{\varrho_H(A+D, C), \varrho_H(C, B+D)\} \le 1$ we are clearly done by the triangle inequality. Indeed, we have

$$\begin{aligned}\varrho_H(A + D, C) &= \max\{\varrho(0, C), \varrho(p, C), \varrho(p, A + D), \varrho(2p, A + D), \varrho(c, A + D)\} \\ &\le \max\{\|p\|, 0, 0, 1, \varrho(c, A + D)\} = \max\{1, \varrho(c, A + D)\}.\end{aligned}$$

But now, by the choice of c, we have

$$\varrho(c, A + D) \le \|c - p\| \le \|c - d\| + \|d - p\| \le \tfrac{1}{4} + \tfrac{1}{2} < 1.$$

Hence $\varrho_H(A + D, C) \le 1$. Since d is the midpoint of $[\![p, 2p]\!]$, a symmetric argument in support of $\varrho_H(C, B + D) \le 1$ may be used. Hence $C \in [A + D, B + D]$ as desired.

To complete the proof, we must show that C is not of the form $D + E$ for any $E \in KL(X)$. Because D is a nondegenerate line segment parallel to the vector p, it follows that a set of the form $D + E$ is a union of nondegenerate line segments parallel to p. But C is a proper triangle whose side opposite the vertex $c \in C$ is parallel to p; hence no segment parallel to p can lie in C and contain c. □

Re (vi). This result holds in higher dimensions, and quickly implies that no Rådström extension is strictly convex.

Theorem 8.6. *Let X be a normed vector space. Then no closed ball of positive radius in $KL(X)$ is metrically convex.*

Proof. Let $K \in KL(X)$, $r > 0$, with $\mathcal{N}$ denoting the closed ball in $KL(X)$, centred at K and of radius r. Fix $a \in X$ where $\|a\| = r$, and consider the sets $A = K + [\![-a, a]\!]$, $B = K + \{a\}$, and $C = K + [\![0, 2a]\!]$. By translation invariance [18, Lemma 3], we have $\varrho_H(A, K) = \varrho_H([\![-a, a]\!], \{0\}) = r$ and $\varrho_H(B, K) = \varrho_H(\{a\}, \{0\}) = r$, so $A, B \in \mathcal{N}$. But $\varrho_H(C, K) = \varrho_H([\![0, 2a]\!], \{0\}) = 2r$; hence $C \notin \mathcal{N}$. On the other hand, we have $\varrho_H(A, B) = \varrho_H([\![-a, a]\!], \{a\})$, $\varrho_H(A, C) = \varrho_H([\![-a, a]\!], [\![0, 2a]\!])$, and $\varrho_H(C, B) = \varrho_H([\![0, 2a]\!], \{a\})$. By Lemma 4.7, these distances are $2r$, r, and r, respectively; hence $C \in [A, B]$.[13] Therefore $\mathcal{N}$ is not metrically convex. □

Remark 8.7. An immediate corollary of Theorem 8.6 is that no closed ball of positive radius in a Rådström extension is metrically convex. Another way to see this is the following. With $\|a\| = r$, $A = [\![-a, a]\!]$, $B = \{a\}$, and $0 \le t \le 1$, we have $(1 - t)A + tB = [\![(2t - 1)a, a]\!]$. Hence $[\![\varphi(A), \varphi(B)]\!]$ is a line segment in $\mathcal{R}(X)$, of length $2r$, which is contained in the sphere of radius r, centred at the origin. This is the maximum length possible, and more than enough to counter metric convexity, as the next proposition shows.

Theorem 8.8. *Let X be a normed vector space such that the closed unit ball is metrically convex. Then every line segment contained in a sphere in X has length at most the radius of the sphere.*

Proof. By rescaled translation, we lose no generality in concentrating on the closed unit ball and sphere centred at the origin. Let $N = N(0; 1)$, $S = S(0; 1)$, and assume there is a line segment $[\![p, q]\!] \subseteq S$ such that $\|p - q\| > 1$. Then $p - q \notin N$. Letting $\mathcal{A}$ be the family of line segments containing the nondegenerate segment $[\![p, q]\!]$ and contained within N, note that the closure of $\bigcup \mathcal{A}$ in X is a line segment in N that is *maximal* in N; i.e., not properly contained in any line segment itself contained in N. So without loss of generality, we may assume our original segment $[\![p, q]\!]$ is maximal in N. Since $-q \in N$, it suffices to show that $p - q \in [-q, p]$. For by our assumption that N is metrically convex, this will give $p - q \in N$, contradicting the assumption that $\|p - q\| > 1$.

Let Y be the span of the linearly independent vectors p, q. We first claim that p and q are extreme points of $N \cap Y$. Indeed if p is not an extreme point of $N \cap Y$, there is a line segment $I = [\![a, b]\!] \subseteq N \cap Y$ such that $p \in I \setminus \{a, b\}$. If $I \cap [\![p, q]\!]$ contains a point other than p, then $I \cup [\![p, q]\!]$ is a line segment that is contained in $N \cap Y$ and that also properly contains $[\![p, q]\!]$. This contradicts the maximality of $[\![p, q]\!]$; hence it must be the case that $I \cap [\![p, q]\!] = \{p\}$.

[13] Since we are just computing Hausdorff distances between closed bounded intervals in the real line, we do not really need the full force of Lemma 4.7.

With the assumption that p is not an extreme point of $N \cap Y$, we seek to contradict the fact that $[\![p, q]\!]$ is contained in the topological boundary of $N \cap Y$ in Y. To this end, let L be the line containing $[\![p, q]\!]$, with H and H' the associated open half-planes in Y. Suppose $0 \in H$, and let T be the proper triangle with vertices $0, p, q$. Then $T \subseteq N \cap Y$ because $N \cap Y$ is linearly convex. Since $I = [\![a, b]\!]$ intersects L in a single point that is neither end point of I, it follows that one end point of I is in H and the other is in H'. Suppose $a \in H'$, and let T' be the proper triangle with vertices a, p, q. Since all three vertices of T' are in $N \cap Y$, T' is also in $N \cap Y$. Since $T' \subseteq L \cup H'$, we have $T \cap T' = [\![p, q]\!]$. Hence $T \cup T'$ is a proper quadrilateral in $N \cap Y$ that contains $[\![p, q]\!] \setminus \{p, q\}$ in its interior. This implies that $[\![p, q]\!]$ is not contained in the boundary of $N \cap Y$, and we have our desired contradiction. We thus conclude that both p and q are extreme points of $N \cap Y$.

Next we show that $p - q \in [-q, p]$. Since $p - q \notin N$, this vector witnesses that N is not metrically convex.

Now, $p - q \in [-q, p]$ if and only if $p \in [0, p + q]$, if and only if $\frac{1}{2}p \in [0, \frac{1}{2}(p + q)]$. By Lemma 4.8, $[0, \frac{1}{2}(p+q)] \cap Y$, the metric interval in Y bracketed by 0 and $\frac{1}{2}(p+q)$, is the proper parallelogram whose vertices are $0, \frac{1}{2}p, \frac{1}{2}q$, and $\frac{1}{2}(p + q)$; therefore $p - q$ is indeed in $[-q, p] \setminus N$, as desired. □

In the following examples, we show: (1) that the converse of Theorem 8.8 is false; and (2) that a ball of radius $r > 0$ can be metrically convex and still contain line segments of length r in its bounding sphere.

Examples 8.9. (i) In the Cartesian plane $\mathbb{R}^2$, let N_r be the intersection of the usual unit disk $\{\langle x, y\rangle : x^2 + y^2 \leq 1\}$ with the square $[-r, r]^2$, where $\frac{1}{2}\sqrt{2} \leq r \leq 1$. Then the *Minkowski functional* $\| \cdot \|_r : \mathbb{R}^2 \to \mathbb{R}$, given by $\|u\|_r = \inf\{t > 0 : t^{-1}u \in N_r\}$, defines a norm on $\mathbb{R}^2$ whose closed unit ball is N_r. When $r = 1$, $\| \cdot \|_r$ is the Euclidean norm and N_r is metrically convex. But for values of r less than unity, N_r is a "truncated disc" whose boundary sphere S_r contains four maximal line segments of equal $\| \cdot \|_r$-length. (When $r = \frac{1}{2}\sqrt{2}$, for example, this $\| \cdot \|_r$-length is 2.) If p and q are the end points of one of these segments, then $[p, q]$ is a rescaled translate of $[0, a]$, where $a = \frac{p-q}{\|p-q\|}$. This point a is also the midpoint of another maximal line segment in S_r, perpendicular to $[\![p, q]\!]$; hence – by Lemma 4.8 – $[p, q]$ is a proper parallelogram with p and q at opposite corners. Thus $[p, q] \not\subseteq N_r$. As r gets ever closer to 1, however, the $\| \cdot \|_r$-length of each of these segments gets ever closer to zero, and certainly less than one. So having all maximal line segments on the unit sphere be of arbitrarily small positive length does not imply that the unit ball is metrically convex.

(ii) Still in the Cartesian plane, let N be a regular hexagon centred at the origin, with $\| \cdot \|$ its associated Minkowski functional. Then the unit sphere S bounding N has six maximal line segments, each of unit $\| \cdot \|$-length. If $a, b \in N$ are distinct points and $[\![a, b]\!]$ is parallel to one of the six sides of the hexagon, then $[a, b] = [\![a, b]\!] \subseteq N$. Alternatively, $[a, b]$ is a proper parallelogram, each of whose sides is parallel to a side of the hexagon. Such a parallelogram is contained in N as long as one of its diagonals lies in N. This shows that the metric convexity of N does not imply that every line segment contained in a sphere in X has length *strictly less than the radius of the sphere.* □

Remark 8.10. The method of proof of Theorem 8.8 may be easily adapted to show the following condition is a consequence of the metric convexity of $N(0;1)$:

Let $p, q \in X$ be distinct such that $[\![p, q]\!] \subseteq S(a;r)$. Then $r\frac{p-q}{\|p-q\|}$ is an extreme point of $N(a;r) \cap P$, where P is the plane containing the noncollinear points a, p, q.

When X is two-dimensional, this condition is equivalent [6, Theorem 5.12] to the condition that whenever $a, b \in X$, the *midset* $[a, b] \cap S(a; \frac{1}{2}\|a - b\|)$ is metrically convex. However, it is not equivalent to the metric convexity of the unit ball, as the following example demonstrates: Let X be the Cartesian plane from the second paragraph of Remark 4.11 (ii); in particular,

$$N(0;1) = \{\langle x, y\rangle \in \mathbb{R}^2 : \max\{x^2 + y^2, |x + y|\} \leq 1\}.$$

Then the consequence above is clearly satisfied; but when $a = \langle -1, 0\rangle$ and $b = \langle 0, 1\rangle$, both in $N(0;1)$, the metric interval $[a, b]$ is the square $[-1, 0] \times [0, 1]$, and is not contained in $N(0;1)$. □

Re (vii). The short answer is that the metric betweenness structure of $KL(X)$ never satisfies the convexity axiom in the higher-dimensional case.

If X is a normed vector space that is of dimension > 1, then Theorem 4.9 shows that when X fails to be strictly convex, we can find $A, B \in KL(X)$, both singletons, such that $[A, B]$ is not metrically convex. If X is indeed strictly convex, this conclusion fails; however, regardless of the strict convexity of X, it is always possible to find a metrically nonconvex interval with one of its bracket points a singleton, and the other a line segment (the next simplest thing).

Theorem 8.11. *Let X be a normed vector space of dimension greater than one. Then there are $A, B \in KL(X)$, where A is a line segment and B is a singleton, such that $[A, B]$ is not metrically convex.*

Proof. Once again we lose no generality in assuming X is two-dimensional, say $X = \langle\mathbb{R}^2, \|\cdot\|\rangle$. We pick $a \in S(0;1)$, and set $A = [\![0, a]\!]$ and $B = \{b\}$, where $b = -a \in S(0;1)$. In this proof we will be repeatedly making use of Lemma 4.7, which allows us to compute Hausdorff distances between compact linearly convex sets in terms of their extreme points. In particular we easily compute $\varrho_H(A, B)$ to be 2.

Let L be the linear span of a, with H one of the two half-planes whose intersection is the line L. Then, in the topological space H, $S(b;1) \cap H$ is a topological arc that meets the disjoint open sets $N^\circ(0;1)$ and $H \setminus N(0;1)$ at the points 0 and $2b$, respectively. Since the arc is connected, there is some point $c \in (S(b;1) \cap H) \cap S(0;1)$. Observe that $S(0;1) \cap L = \{a, b\}$, and neither point belongs to $S(b;1)$. Hence $c \in H \setminus L$. The same argument holds for the other half-plane H', so there is a point $d \in (S(0;1) \cap S(b;1) \cap H') \setminus L$. Let $C = [\![0, c]\!]$ and $D = [\![0, d]\!]$. It remains to show:

(1) that $C, D \in [A, B]$; and

(2) that $[C, D] \not\subseteq [A, B]$.

Note that in order to use Lemma 4.7, we need only consider extreme points of one set that are not members of the other. To show $C \in [A, B]$, we first have that

$$\varrho_H(C, B) = \max\{\varrho(b, C), \varrho(c, B), \varrho(0, B)\}.$$

Since $\|c - b\| = \|c\| = 1$, and $N(b; 1)$ is linearly convex, we know the first term is ≤ 1. Since B is a singleton, the second and third terms are both equal to 1; hence $\varrho_H(C, B) = 1$. To show $C \in [A, B]$, it suffices to show $\varrho_H(A, C) \leq 1$, and invoke the triangle inequality. Indeed, $\varrho_H(A, C) = \max\{\varrho(a, C), \varrho(c, A)\}$. These two terms are at most $\|a - 0\| = 1$ and $\|c - 0\| = 1$, respectively, showing that $\varrho_H(A, C) = \varrho_H(C, B) = 1$. The exact same argument shows that $D \in [A, B]$; i.e., that $\varrho_H(A, D) = \varrho_H(D, B) = 1$.

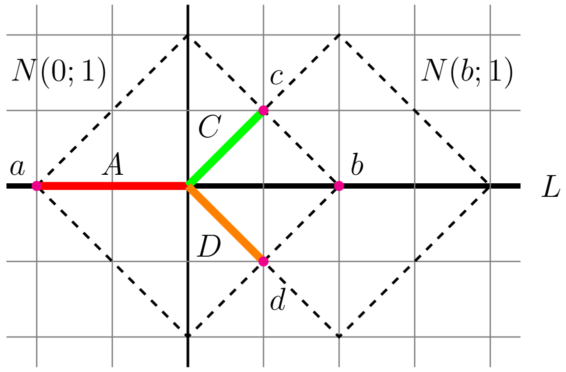

Figure 4: First schematic for Theorem 8.11, the taxicab norm is depicted.

Up to now we have assumed nothing about our starting point a, other than that it belongs to $S(0; 1)$. However, in order to show $[C, D] \not\subseteq [A, B]$, we need to assume further that a is an extreme point of $N(0; 1)$, and hence that 0 is an extreme point of $N(b; 1)$. Then the points $0, c, d$ are noncollinear because otherwise $[\![c, d]\!]$ would witness that 0 is not an extreme point of $N(b; 1)$. We first put $E' = \frac{1}{2}C + \frac{1}{2}D$. Then, because the points $0, c, d$ form the vertices of a proper triangle, E' is the proper parallelogram with vertices $0, \frac{1}{2}c, \frac{1}{2}d$, and $\frac{1}{2}(c + d)$. An important consequence of this is that E' has nonempty topological interior. This set is not yet what we need, however, as it is an element of $[\![C, D]\!] \subseteq [\![A, B]\!] \subseteq [A, B]$. The plan is to enlarge E' slightly to obtain $E \in KL(X)$ so that:

(2.1) $\varrho_H(C, E) = \varrho_H(C, E')$ and $\varrho_H(E, D) = \varrho_H(E', D)$; and

(2.2) $\varrho_H(A, E) + \varrho_H(E, B) > 2 = \varrho_H(A, B)$.

Satisfying the first condition will ensure that $E \in [C, D]$; satisfying the second will gives us $E \notin [A, B]$, completing the proof.

Our set E will be the linear convex hull of $E' \cup \{e\}$, where e is chosen outside E' but close to the origin (which is contained in both C and D). Since $E' \in KL(X)$, we know that $E = \bigcup\{[\![x, e]\!] : x \in E\}$ (see, e.g., the proof of Proposition 2.3 (vi)).

Hence any extreme point of E not equal to e is already an extreme point of E'; i.e., $\varepsilon(E) \subseteq \varepsilon(E') \cup \{e\}$.

For a clue as to how to pick this point e, we have

$$\varrho_H(C, E') = \max\{\varrho(c, E'), \varrho(\tfrac{1}{2}d, C), \varrho(\tfrac{1}{2}(c + d), C)\},$$

and

$$\varrho_H(C, E) = \max\{\varrho(c, E), \varrho(\tfrac{1}{2}d, C), \varrho(\tfrac{1}{2}(c + d), C), \varrho(e, C)\}.$$

This tells us that in order to satisfy (2.1), we need:

(2.1.1) $\varrho(c, E') = \varrho(c, E)$; and

(2.1.2) $\varrho(e, C)$ is sufficiently small; i.e., that e is sufficiently close to $0 \in C$.

To ensure (2.1.1), first note that $c \notin E'$; otherwise there would be $0 \leq s, t \leq \frac{1}{2}$ such that $c = sc + td$. But then $c = \frac{t}{1-s}d$. Since $C \cap D = \{0\}$ and $1 - s \neq 0$, we have a contradiction. With this in mind, we may fix $r = \varrho(c, E') > 0$. Then $N^{\circ}(c; r)$ is disjoint from E', and $N(c; r) \cap E'$ is the nonempty set of points of E' closest to c. If $x \in E'$, plane analytic geometry ensures that $[\![c, x]\!]$ intersects the edge $[\![\frac{1}{2}c, \frac{1}{2}(c + d)]\!]$. It follows that $N(c; r) \cap E'$ is contained in this edge. Now, by [19, Theorem 6.1], the interiors $N^{\circ}(c; r)$ and E'° are linearly convex. Since they are disjoint, the hyperplane separation theorem [19, Theorem 11.3] says there exists an affine functional $f : \mathbb{R}^2 \to \mathbb{R}$ taking negative values in one interior and positive values in the other. By [19, Theorem 6.2], any compact linearly convex set with nonempty interior is the closure of that interior. Hence any point $y \in N(c; r) \cap E'$ is a limit point of both $N^{\circ}(c; r)$ and E'°. This implies that our functional f takes points of $N(c; r) \cap E'$ to zero; i.e., there is a line M_c that separates $N^{\circ}(c; r)$ and E'°, and contains $N(c; r) \cap E'$.

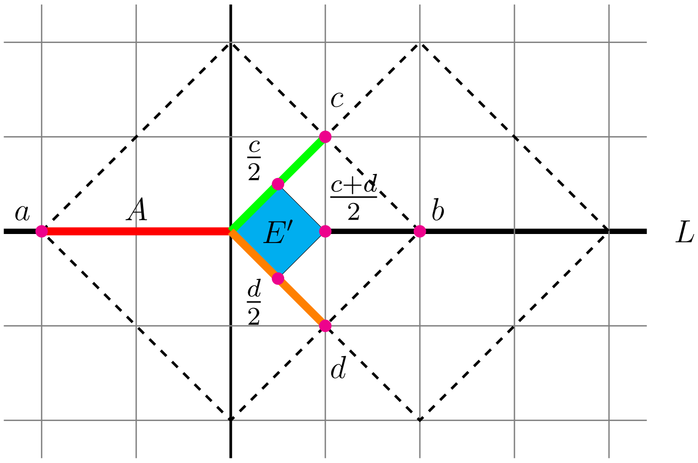

Figure 5: Second schematic for Theorem 8.11. The set $E \in [A, B] \setminus [C, D]$ will be a small enlargement of the parallelogram $E' = \frac{1}{2}C + \frac{1}{2}D$.

We claim that $0 \notin M_c$. Indeed, since M_c intersects the edge $[\![\frac{1}{2}c, \frac{1}{2}(c+d)]\!]$ and misses the interior of the parallelogram E', it must contain either $\frac{1}{2}c$ or $\frac{1}{2}(c + d)$. In the first case, M_c cannot contain 0 because it does not contain c; in the second case it cannot contain 0 because $[\![0, \frac{1}{2}(c + d)]\!]$ is a diagonal of E' and therefore intersects E'°.

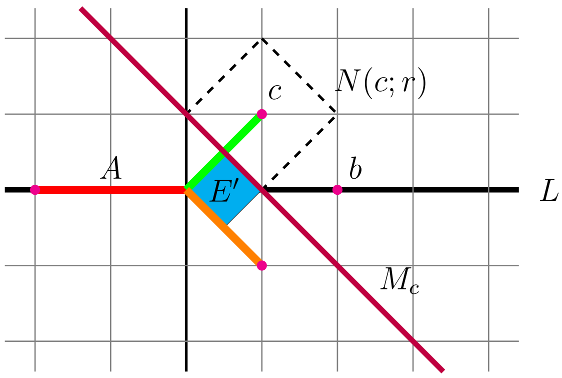

Figure 6: Third schematic for Theorem 8.11. The line M_c separates $N(c; r)^{\circ}$ from E'° and contains $E' \cap N(c; r)$.

Let H_c be the closed half-plane that contains E' and has boundary line M_c. Then H_c contains 0 in its interior and is disjoint from $N^\circ(c;r)$. If $e \in H_c$ and E is the convex hull of $E' \cup \{e\}$, then $E \subseteq H_c$ too; so $\varrho(c,E) \geq r$. Since $N(c;r)$ meets E', it also meets E and we see that $\varrho(c,E) = r = \varrho(c,E')$, as required.

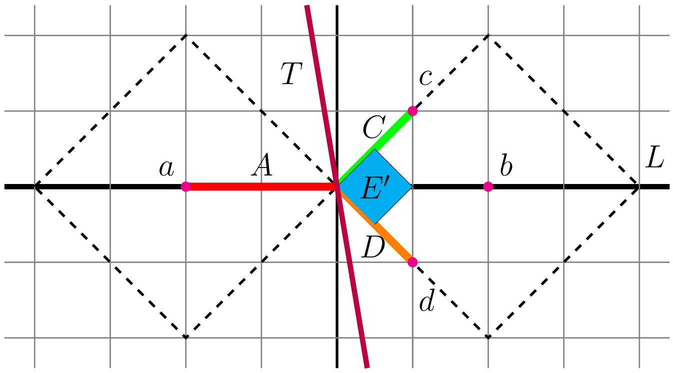

Figure 7: Fourth schematic for Theorem 8.11. The line T separates the two open balls. Letting E be the linearly convex hull of E' plus any point of $T \setminus \{0\}$ sufficiently close to the origin will preserve the distance from A but increase the distance from B.

If we repeat the argument just given, but with D replacing C, we have an analogous line M_d and half-plane H_d containing E' (and 0 in its interior), with boundary line M_d. Thus if $e \in H_c \cap H_d$, then the corresponding set E also is in $H_c \cap H_d$, and we infer that $\varrho(d,E) = \varrho(d,E')$. If, in addition, we take e to be small – say $0 < \|e\| \leq \min\{\varrho(\frac{1}{2}(c+d),C), \varrho(\frac{1}{2}(c+d),D)\}$ – then we will have ensured that condition (2.1) above holds; i.e., that $E \in [C,D]$.

Now, $N(a;1)$ and $N(b;1)$ both contain 0 and have disjoint interiors. Arguing as above, we then have a line T that contains 0 and separates $N^\circ(a;1)$ and $N^\circ(b;1)$. Let T_c (resp., T_d) be the part of T that lies in the open half-plane that contains c (resp., d) and is bounded by the line L (containing $0,a,b$). Since 0 is an extreme point of $N(b;1)$, $N(b;1)$ cannot intersect both T_c and T_d; say it misses T_c. If $e \in T_c$, then, we have $\varrho(e,b) > 1$; hence $\varrho(E,B) > 1$.

It remains to show that $\varrho_H(A,E) \geq 1$. By construction, E is contained in one of the half-planes determined by T and $N(a;1)$ is contained in the other. Thus $1 \leq \varrho(a,E) \leq \varrho(A,E) \leq \varrho_H(A,E)$.

Finally we note that $H_c \cap H_d$ is a neighborhood of 0 and that 0 is a limit point of T_c. Thus, to choose our point e, it suffices to pick any point from $T_c \cap H_c \cap H_d$ such that $\|e\| \leq \min\{\varrho(\frac{1}{2}(c+d),C), \varrho(\frac{1}{2}(c+d),D)\}$. This completes the proof. □

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