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Minimal freeness and commutativity

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Abstract. A *pseudobasis* for an abstract algebra A is a subset X of A such that every mapping X into A extends uniquely to an endomorphism on A. A is *minimally free* if A has a pseudobasis. In this paper we look at how minimal freeness interacts with various notions of commutativity (e.g., "operational" commutativity in the algebra, usual commutativity in the endomorphism monoid of the algebra). One application is a complete classification of minimally free torsion abelian groups.

0. Introduction

Let Ω be an operation type for universal algebras; members of Ω are operation symbols of various finite "arities." Set $\Omega_m = \{\mu \in \Omega : \mu$ has arity $m\}$, $m \geq 0$. If $\mu \in \Omega_m$ and A is an Ω-algebra, the *interpretation* μ_A of μ in A is a mapping from the finite cartesian power A^m into A. (0-ary operations are also called *constants*.) When confusion is unlikely to arise, we drop subscripts from interpretations of operations in algebras. Also the letters μ, ν generally stand for operations of positive arities m and n respectively.

An Ω-algebra A is *minimally free* if there is a subset $X \subseteq A$, called a *pseudobasis*, such that every map $f : X \to A$ extends uniquely to an endomorphism $[f]$ on A. A is κ-*free*, where κ is a cardinal number, if A has a pseudobasis of cardinality κ.

One-element (*trivial*) algebras are clearly both 0-free (*rigid*) and 1-free. A nontrivial 0-free algebra cannot be κ-free for any $\kappa > 0$, but it is quite possible for nontrivial minimally free algebras to possess pseudobases of widely varying cardinalities (finite and infinite [1], [3]; see also 2.10(iii) in the sequel).

The notion of pseudobasis is stronger than that of "independent set" as introduced by E. Marczewski [8]. Recall that $X \subseteq A$ is *independent* if every $f : X \to A$ extends to a homomorphism from the subalgebra $\langle X \rangle$ generated by X. X is a *basis*, in the sense of Marczewski, if X is independent and is a generating set for A. (In that case A is free with respect to some variety containing A.) Pseudobases can fail

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very badly to be generating sets (see [1], [2], [3], [4] for various demonstrations of this fact); however if X is a pseudobasis for A, then it is easy to see that X is a basis for $\langle X \rangle$.

There is a model-theoretic property of pseudobases which, while not exploited in the sequel, is of interest in its own right. If X is a pseudobasis for the Ω-algebra A, then X is a set of "indiscernibles." That is, if $\varphi(v_1, \ldots, v_n)$ is a first order formula over Ω involving at most the distinct free variables $v_1, \ldots, v_n$, and if $\langle x_1, \ldots, x_n \rangle$ and $\langle x'_1, \ldots, x'_n \rangle$ are two sequences from X such that for $1 \leq i, j \leq n$, $x_i = x_j$ if and only if $x'_i = x'_j$, then the substitution instances $\varphi[x_1, \ldots, x_n]$ and $\varphi[x'_1, \ldots, x'_n]$ (where x_i is substituted for v_i, etc.) are either both true in A or both false in A. This follows easily from the fact that there is an automorphism on A taking x_i to x'_i for $1 \leq i \leq n$. To get this automorphism, one needs only a bijection on X taking each x_i to x'_i. But this can be done using a simple combinatoric argument: Let $Y = \{x_1, \ldots, x_n\}$, $Z = \{x'_1, \ldots, x'_n\}$. For $1 \leq i \leq n$, let $\beta(x_i) = x'_i$. Extend β to a permutation on the finite set $Y \cup Z$ by scanning the list $x'_1, \ldots, x'_n$. If these elements have all been assigned a value, we are done. Otherwise, let $1 \leq j \leq n$ be least such that x'_j has not been assigned a value. Then there is a smallest i such that x_i has not yet appeared as a value, so define $\beta(x'_j)$ to x_i. β is clearly one-one on $Y \cup Z$, and is hence a permutation. Now extend β to a permutation on X by setting $\beta(x) = x$ for $x \notin Y \cup Z$.

As mentioned above, the 0-free Ω-algebras are precisely the (endomorphism-) rigid ones, and there is an extensive literature dealing with ways of constructing large examples. (The papers [2], [10] use large rigid algebras to construct large κ-free algebras for any fixed κ.) 1-free groups, always abelian (see below), have also been studied [4], [5], [10] under the guise of "E-rings," those unital (i.e., having a 1) rings all of whose additive endomorphisms arise via left multiplication. As pointed out in [1], E-rings are precisely the endomorphism rings of 1-free groups; the additive group of any E-ring is automatically 1-free. In [4] a machine is developed for constructing arbitrarily large E-rings (hence 1-free groups), and in [2] it is observed that one can then construct arbitrarily large κ-free abelian groups, for $\kappa > 0$, because the direct sum (=weak direct product) of κ copies of any 1-free group is κ-free.

Our interest is this paper lies in the various ways that minimal freeness interacts with notions of commutativity. As an introductory example, consider $\varepsilon = \varepsilon(v_1, \ldots, v_n)$, an equation over the type Ω, in which at most the distinct variables $v_1, \ldots, v_n$ occur. If A is an Ω-algebra that has a (Marczewski) independent set X with at least n elements, and if $x_1, \ldots, x_n$ are distinct elements of X such that the substitution instance $\varepsilon[x_1, \ldots, x_n]$ holds in A, then ε is universally true in A. In particular, suppose Ω contains a binary operation and A has an independent set of two elements that commute with one another. Then all pairs of elements of

A commute. The number of elements in the independent set cannot generally be taken to be less than n above: there are 1-free semigroups (see [1] for an example due to M. Petrich [11]) that are not commutative. However, it turns out that 1-free groups are always commutative [1]. The simple argument uses the full force of the notion of "pseudobasis," and is peculiarly group-theoretic. (If A is 1-free with pseudobasis $\{x\}$, then the inner automorphism $t \mapsto x^{-1}tx$ fixes x, hence must be the identity map. Thus $xt = tx$ for all $t \in A$. For any $a \in A$, then, the automorphism $t \mapsto a^{-1}ta$ also fixes x, and is hence the identity map. Thus $ab = ba$ for all $a, b \in A$.)

A brief summary of the contents of the sequel is as follows. In §1 we consider the variety $\mathbf{Z}_\Omega$ of "zero algebras," Ω-algebras with a unique constant that behaves as an idempotent with respect to all other operations in Ω. We look at conditions under which weak direct powers of κ copies of a 1-free algebra in $\mathbf{Z}_\Omega$ is κ-free. In §2 we introduce the variety $\mathbf{OC}_\Omega$ of "operationally commutative" algebras, those zero algebras in which all operations of positive arity commute with one another in a natural way (e.g., diagonal algebras, modules over a commutative ring, normal bands). We characterize the 1-free members of $\mathbf{OC}_\Omega$ in terms of the commutativity of their endomorphism monoids. Nontrivial 1-free algebras in $\mathbf{OC}_\Omega$ are "uniquely" 1-free, i.e., they have pseudobases only of cardinality 1. In §3 we apply the previous material to abelian groups. In particular, we classify the κ-free torsion groups as the κ-fold weak direct powers of finite cyclic groups. We also show that a nontrivial countable κ-free abelian group is uniquely κ-free. In §4 we look at another general situation, namely at the variety $\mathbf{IOC}_\Omega$ of idempotent operationally commutative Ω-algebras (where there are no constants; every element is an idempotent). This generalizes semilattices and, more generally, normal bands. If $A \in \mathbf{IOC}_\Omega$ is 2-free, then its endomorphism monoid contains a left zero such that any two elements commuting with this left zero must commute with each other.

1. Minimally free zero algebras

For any Ω-algebras A, B, let $Hom(A, B)$ be the set of Ω-homomorphisms from A to B, with $End(A) = Hom(A, A)$, the Ω-endomorphisms on A. $End(A)$ has a natural monoid structure under function composition, the monoid identity being the identity map id_A on A. This monoid we call $End^\circ(A)$. (Under special circumstances, $End(A)$ may be given other algebraic structure; we modify notation as the need arises.) The following is obvious.

1.1 PROPOSITION. *Let A have a pseudobasis X. Then the mapping $f \mapsto [f]$ is a bijection between the cartesian power A^X and $End(A)$ that takes the inclusion map from X into A to id_A.* □

1.2 PROPOSITION. *Let A have a nonempty pseudobasis X. If either $|X| \geq 2$ or A has at least two distinguished constants, then $End^\circ(A)$ is not commutative.*

Proof. Suppose first $|X| \geq 2$, and let f and g be two distinct constant maps from X to X. Then $[f] \circ [g] \neq [g] \circ [f]$. Now suppose $X = \{x\}$, and let A have two distinguished constants a_1, a_2. Then $[a_1] \circ [a_0] \neq [a_2] \circ [a_1]$ (where $[a]$ is the unique endomorphism taking x to $a \in A$). □

In the remainder of this section, Ω is a type that contains exactly one constant, denoted 0. (So when we refer to $\mu \in \Omega$, we intend for μ to be of positive arity m.) An Ω-algebra A is a *zero* algebra if $\{0\}$ is a subalgebra of A. (Equivalently: (i) $\mu(0, \ldots, 0) = 0$ for all $\mu \in \Omega$, i.e., 0 is an *idempotent*; or (ii) the constantly 0 map on A is an endomorphism.) The variety of zero algebras of type Ω is denoted $\mathbf{Z}_\Omega$. For $A \in \mathbf{Z}_\Omega$, $End^\circ(A)$ now has additional algebraic structure, namely the zero map 0. The obvious equation is satisfied here: $v \circ 0 = 0 \circ v = 0$. Call this algebra $End_0^\circ(A)$.

Let $A \in \mathbf{Z}_\Omega$, I a nonempty set. The direct power A^I is again a zero algebra. For each $j \in I$, let $\sigma_j : A \to A^I$ be defined by

$$\sigma_j(a)(i) = \begin{cases} a & \text{if } i \neq j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then each σ_j is an embedding. Let $A_j = \sigma_j(A)$. Then clearly $A_i \cap \langle \bigcup_{j \in I \setminus \{i\}} A_j \rangle = \{0\}$ for $i \in I$. We write $A^{\langle I \rangle} = \bigcup_{i \in I} A_i$, the *wedge power* of A (from the analogous topological construction). $A^{\langle I \rangle}$ is a partial zero algebra in the obvious sense. (One other way of describing $A^{\langle I \rangle}$ is to take the disjoint union of $|I|$ copies of A and then identify all the zeros.) The *weak direct product* (see [8]) is then the subalgebra $A^{[I]} = \langle A^{\langle I \rangle} \rangle$ of A^I. This zero algebra is clearly contained in the subalgebra of A^i consisting of all I-tuples f that take the value 0 for all but finitely many indices i.

The constructions above can easily be extended to I-indexed families of zero algebras. (Just replace the word "power" with the word "product.") We will not need this greater generality here. In the setting of abelian groups (more generally, the setting of R-modules over a commutative ring R), these constructions are quite familiar.

We are interested in using the weak direct power to obtain κ-free algebras from 1-free algebras, where $\kappa > 0$. We start with a 1-free algebra $A \in \mathbf{Z}_\Omega$, and examine when $A^{[I]}$ is $|I|$-free. Define $A \in \mathbf{Z}_\Omega$ to have the *wedge extension property* if for any set I and any homomorphism $\varphi : A^{\langle I \rangle} \to A^{[I]}$, there is an extension of φ to an endomorphism on $A^{[I]}$.

1.3 THEOREM. *Let A be a nontrivial* 1*-free zero algebra with the wedge extension property, I a set of cardinality κ. Then $A^{[I]}$ is κ-free.*

Proof. Let $\{x\}$ be a pseudobasis for A, $B = A^{\langle I \rangle}$, $C = A^{[I]}$, where $|I| = \kappa$. For each $i \in I$, $\pi_i : C \to A$ is the ith projection map. We set $X = \{\sigma_i(x) : i \in I\}$. Since A is nontrivial, we know $x \neq 0$; hence the elements $\sigma_i(x)$ are distinct for distinct i. Thus $|X| = \kappa$. To show X is a pseudobasis for C, let $f : X \to C$ be given, say $\pi_i(f(\sigma_j(x))) = a_{ij} \in A$. For $\langle i, j \rangle \in I^2$, let $\varphi_{ij} \in End(A)$ be unique such that x goes to a_{ij}. Define $\varphi : B \to C$ by $\varphi(\sigma_j(a))(i) = \varphi_{ij}(a)$. One can readily show that φ is a homomorphism from B to C; so by the wedge extension property, φ extends to an endomorphism $\bar{\varphi} \in End(C)$. $\bar{\varphi}$ clearly extends f, so we must show $\bar{\varphi}$ is unique. Indeed if $\psi \in End(C)$ extends f, then for each $\langle i, j \rangle \in I^2$, $\pi_i \circ \psi \circ \sigma_j = \varphi_{ij}$ by the uniqueness of φ_{ij}. Thus $\pi_i \circ \psi \circ \sigma_j = \pi_i \circ \bar{\varphi} \circ \sigma_j$. This implies $\psi \circ \sigma_j = \bar{\varphi} \circ \sigma_j$ for all $j \in I$. Since the σ_j's are embeddings and B generates C, we infer that $\psi = \bar{\varphi}$. □

The following is an analogue of the main result in [2].

1.4 COROLLARY. *Let $\mathbf{K} \subseteq \mathbf{Z}_\Omega$ be any class closed under subalgebras and direct powers. If $\kappa > 0$ and $A \in \mathbf{K}$ is any* 1*-free algebra with the wedge extension property, then A embeds as a retract of a κ-free algebra in $\mathbf{K}$.* □

1.5 REMARK. Algebras with the wedge extension property include: (i) algebras all of whose operations of positive arity are unary; (ii) R-modules over a commutative ring R; (iii) commutative monoids. The main result of [4] states that there are arbitrarily large 1-free abelian groups; hence there are arbitrarily large κ-free abelian groups for any fixed $\kappa > 0$. (The same result obviously holds for commutative monoids.) It would be interesting to find general conditions that suffice for a commutative (unital) ring to admit large 1-free R-modules. (N.B. R must not be a field, for then every 1-free R-module is module isomorphic to R.)

We turn our attention now to minimally free subalgebras of minimally free algebras.

1.6 THEOREM. *Let $A \in \mathbf{Z}_\Omega$ have a nonempty pseudobasis X, and for each $Y \subseteq X$ let $f_Y : X \to A$ be defined by*

$$f_Y(x) = \begin{cases} x & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}.$$

Consider the power set $\mathbf{P}(X)$ as a bounded meet-semilattice. Then the mapping $Y \mapsto [f_Y]$ is an embedding into $End_0^\circ(A)$.

Proof. $X \mapsto [f_X] = id_A$; $\varnothing \mapsto [f_\varnothing] = 0$.

Let $Y, Z \in \mathbf{P}(X)$. We claim that $[f_Y] \circ [f_Z] = [f_{Y \cap Z}]$. All we have to check is that the two endomorphisms agree on members of X; this is entirely routine. Clearly if $Y \neq Z$ in $\mathbf{P}(X)$, then $f_Y \neq f_Z$. Thus $[f_Y] \neq [f_Z]$. □

1.7 REMARK. Suppose $A \in \mathbf{Z}_\Omega$ has a pseudobasis X with $|X| > 1$. Then 1.2 tells us $End^\circ(A)$ is noncommutative. However, 1.6 tells us that $End^\circ(A)$ may have quite substantial commutative submonoids consisting of idempotents of $End^\circ(A)$.

1.8 THEOREM. *Let $A \in \mathbf{Z}_\Omega$ have a nonempty pseudobasis X, and for each $Y \subseteq X$, let $A_Y = [f_Y](A)$. Then*:

(i) *The map $Y \mapsto A_Y$ is an embedding of $\mathbf{P}(X)$, as a bounded meet-semilattice, into the bounded meet-semilattice of subalgebras of A.*

(ii) *For each $Y \subseteq X$, Y is a pseudobasis for A_Y.*

(iii) *If $Y, Z \subseteq X$ and $|Y| = |Z|$, then $A_Y \cong A_Z$.*

Proof. (i) $A_X = [f_X](A) = A$; $A_\varnothing = [f_\varnothing](A) = \{0\}$. Let $Y, Z \in \mathbf{P}(X)$. Then $A_{Y \cap Z} \subseteq [f_{Y \cap Z}](A) = ([f_Y] \circ [f_Z])(A)$, by 1.6. Thus $A_{Y \cap Z} = [f_Y](A_Z)$. Clearly $A_{Y \cap Z} \subseteq A_Y \cap A_Z$; so suppose $[f_Y](a) = [f_Z](b)$ is an element of $A_Y \cap A_Z$. Then, by 1.6, $[f_Y](a) = [f_Y]([f_Y](a)) = [f_Y]([f_Z](b)) \in A_{Y \cap Z}$. Thus $Y \mapsto A_Y$ is a homomorphism of bounded meet-semilattices. If $y \in Y$, then $y = [f_Y](y)$, so $y \in A_Y$. Suppose $Y \not\subseteq Z$, say $y \in Y \backslash Z$. Then $A_{\{y\}} \cap A_Z = \{0\}$ by (i); so $y \notin A_Z$. Thus $A_Y \subseteq A_Z$ implies $Y \subseteq Z$; hence $A_Y = A_Z$ implies $Y = Z$. Therefore $Y \mapsto A_Y$ is an embedding.

(ii) We already know $Y \subseteq A_Y$ by (i). To show Y is a pseudobasis for A_Y, let $f: Y \to A_Y$ be given. Let $g: Y \to A$ be any map such that $f = [f_Y] \circ g$, and let $h: X \to A$ be defined by the rule

$$h(x) = \begin{cases} g(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}.$$

We then set $\varphi = ([f_Y] \circ [h]) \mid A_Y$. If $x \in Y$, then $\varphi(x) = [f_Y]([h](x)) = [f_Y](h(x)) = [f_Y](g(x)) = f(x)$. Now suppose $\psi \in End(A_Y)$ also extends f. Then $\varphi \circ [f_Y]$ and $\psi \circ [f_Y]$ are endomorphisms on A that agree on X. They are thus equal, hence $\varphi = \psi$.

(iii) Suppose $Y, Z \subseteq X$ have the same cardinality; let $f: Y \to Z$ and $g: Z \to Y$ be inverses of each other. Define

$$\bar{f}: X \to A \text{ by } \bar{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y; \end{cases}$$

similarly define $\bar{g} : X \to A$ by

$$\bar{g}(x) = \begin{cases} g(x) & \text{if } x \in Z \\ 0 & \text{if } x \notin Z. \end{cases}$$

Set $\varphi = ([f_Z] \circ [\bar{f}]) \mid A_Y, \psi = ([f_Y] \circ [\bar{g}]) \mid A_Z$. Then $\varphi \in Hom(A_Y, A_Z)$ and $\psi \in Hom(A_Z, A_Y)$. Suppose $x \in Y$. Then $(\psi \circ \varphi)(x) = \psi([f_Z]([\bar{f}](x))) = \psi([f_Z](f(x))) = \psi(f(x)) = [f_Y]([\bar{g}](f(x))) = [f_Y](x) = x$. Since Y is a pseudobasis for A_Y, we infer $\psi \circ \varphi = id_{A_Y}$. Similarly, $\varphi \circ \psi = id_{A_Z}$. □

1.9 COROLLARY. *Let $A \in \mathbf{Z}_\Omega$ have a nonempty pseudobasis X. Then there exists a* 1*-free $B \in \mathbf{Z}_\Omega$ and an embedding of the wedge power $B^{(Y)}$ into A.* □

In §2 we give conditions under which the result in 1.8 can be considerably improved (see 2.11).

2. Operational commutativity

It was shown in [1], using inspiration from the lore on E-rings (see [5]), that $End^\circ(A)$ is a commutative monoid when A is a 1-free commutative monoid. Here we provide a general setting in which this result is true.

For any Ω-algebras A and B, $Hom(A, B)$ is a subset of B^A, but not generally a subalgebra. Clearly necessary and sufficient conditions for $Hom(A, B)$ to be a subalgebra are: (i) at most one element $b_0 \in B$ is a distinguished constant; (ii) b_0 is an idempotent; and (iii) whenever $\mu \in \Omega_m$ and $\varphi_1, \ldots, \varphi_m \in Hom(A, B)$, then $\mu(\varphi_1, \ldots, \varphi_m) \in Hom(A, B)$.

In this section, Ω is an operation type containing at most one constant symbol, which we continue to denote 0 (when it exists). An Ω-algebra A is *operationally commutative* if: (i) $A \in \mathbf{Z}_\Omega$; and (ii) whenever $\mu \in \Omega_m$, $\nu \in \Omega_n$, and (v_{ij}) is an $m \times n$ array of variables, the following equation holds in A.

$$\mu(\nu(v_{11}, \ldots, v_{1n}), \nu(v_{21}, \ldots, v_{2n}), \ldots, \nu(v_{m1}, \ldots, v_{mn}))$$
$$= \nu(\mu(v_{11}, \ldots, v_{m1}), \mu(v_{12}, \ldots, v_{m2}), \ldots, \mu(v_{1n}, \ldots, v_{mn})).$$

The variety of operationally commutative Ω-algebras is denoted $\mathbf{OC}_\Omega$.

2.1 PROPOSITION. *Let A and B be Ω-algebras, with $B \in \mathbf{OC}_\Omega$. Then $Hom(A, B)$ is an operationally commutative subalgebra of B^A.*

Proof. $\mathbf{OC}_\Omega \subseteq \mathbf{Z}_\Omega$, so the zero map from A to B is a homomorphism. Let $\mu \in \Omega_m$, with $\varphi_1, \ldots, \varphi_m \in Hom(A, B)$. Then $\mu(\varphi_1, \ldots, \varphi_m) \in Hom(A, B)$ because: (i) $\mu(\varphi_1, \ldots, \varphi_m)(0) = \mu(\varphi_1(0), \ldots, \varphi_m(0)) = \mu(0, \ldots, 0) = 0$; and (ii) whenever $\nu \in \Omega_n$, then we have

$$\begin{aligned}\mu(\varphi_1, \ldots, \varphi_m)(\nu(a_1, \ldots, a_n)) &= \mu(\varphi_1(\nu(a_1, \ldots, a_n)), \ldots, \varphi_m(\nu(a_1, \ldots, a_n)))\\ &= \mu(\nu(\varphi_1(a_1), \ldots, \varphi_1(a_n)), \ldots, \nu(\varphi_m(a_1), \ldots, \varphi_m(a_n)))\\ &= \nu(\mu(\varphi_1(a_1), \ldots, \varphi_m(a_1)), \ldots, \mu(\varphi_1(a_n), \ldots, \varphi_m(a_n)))\\ &= \nu(\mu(\varphi_1, \ldots, \varphi_m)(a_1), \ldots, \mu(\varphi_1, \ldots, \varphi_m)(a_n)).\end{aligned}$$

Thus $Hom(A, B)$ is a subalgebra of B^A. Since $\mathbf{OC}_\Omega$ is a variety and $B \in \mathbf{OC}_\Omega$, we have $B^A \in \mathbf{OC}_\Omega$, and hence $Hom(A, B) \in \mathbf{OC}_\Omega$. ($Hom(A, B)$ satisfies all identities true in B, in fact, and usually a lot more.) □

2.2 EXAMPLES. (i) Let $\Omega = \{\mu\}$, where μ is m-ary, $m > 0$. A *diagonal algebra* [8] is an Ω-algebra satisfying the identity $\mu(\mu(v_{11}, \ldots, v_{1m}), \ldots, \mu(v_{m1}, \ldots, v_{mm})) = \mu(v_{11}, v_{22}, \ldots, v_{mm})$. Such an algebra is clearly operationally commutative.

(ii) Any monoid is operationally commutative if and only if it is commutative; a groupoid (i.e., any Ω-algebra where Ω consists of one binary operation $\cdot$) is operationally commutative if and only if it satisfies the "medial law" $(v_1 \cdot v_2) \cdot (v_3 \cdot v_4) = (v_1 \cdot v_3) \cdot (v_2 \cdot v_4)$. Semigroups satisfying the medial law are commonly referred to as *bands* (see [11]).

(iii) Every R-module over a commutative ring R is operationally commutative.

(iv) Every operationally commutative lattice $\langle A, \vee, \wedge \rangle$ is trivial. [Every nontrivial lattice contains a two-element chain $\{a, b\}$ with, say, $a < b$. But $(a \wedge b) \vee (b \wedge a) = a$ and $(a \vee b) \wedge (b \vee a) = b$.]

(v) Every operationally commutative algebra $\langle A, \cdot, +, 0\rangle$, where $\langle A, +, 0\rangle$ is a group, $\langle A, \cdot \rangle$ is a groupoid, and $\cdot$ distributes over $+$ both on the left and on the right, must be additively commutative and have trivial multiplication. [$\langle A, \cdot, +, 0\rangle$ must satisfy the identity $(v_1 + v_2) \cdot (v_3 + v_4) = (v_1 \cdot v_3) + (v_2 \cdot v_4)$. But it must also satisfy $(v_1 + v_2) \cdot (v_3 + v_4) = (v_1 \cdot v_3) + (v_2 \cdot v_3) + (v_1 \cdot v_4) + (v_2 \cdot v_4)$. Hence $(v_2 \cdot v_3) + (v_1 \cdot v_4) = 0$. Substituting $v_1 = 0$, we get the equation $v_2 \cdot v_3 = 0$. Of course the medial law holds for $+$; in the presence of 0 this yields commutativity for $+$.]

Let $\Omega \subseteq \Omega'$ be two types (arbitrary for the moment), with A an Ω'-algebra. The reduct of A to the type Ω is denoted $A \mid \Omega$. Now suppose Ω is a type with at most one constant 0, and let $\circ$ and *id* be two new operations, the first binary and the second nullary. Let $\Omega^* = \Omega \cup \{\circ, id\}$. The variety $\mathbf{OC}_\Omega^\circ$ consists of those Ω^*-algebras A satisfying the following: (i) $A \mid \{\circ, id\}$ is a monoid; (ii) $A \mid \Omega \in \mathbf{OC}_\Omega$; and (iii) A satisfies the identities: $v \circ 0 = 0 \circ v = 0$, and for each $\mu \in \Omega_m$, $v \circ \mu(v_1, \ldots, v_m) = \mu(v \circ v_1, \ldots, v \circ v_m)$ and $\mu(v_1, \ldots, v_m) \circ v = \mu(v_1 \circ v, \ldots, v_m \circ v)$. (In particular, left or right multiplication by a fixed element is an Ω-endomorphism.)

If $A \in \mathbf{OC}_\Omega$, then 2.1 allows us to regard $End(A)$ as an operationally commutative Ω-algebra when operations are computed pointwise. We may also view $End(A)$ as an Ω^*-algebra when $\circ$ is interpreted as function composition and id is interpreted as id_A. Denote this endomorphism algebra by $End_\Omega^\circ(A)$, and set $End_\Omega(A) = End_\Omega^\circ(A) \mid \Omega$. The following is automatic.

2.3 PROPOSITION. *Let* $A \in \mathbf{OC}_\Omega$. *Then* $End_\Omega^\circ(A) \in \mathbf{OC}_\Omega^\circ$. □

2.4 REMARK. When A is an abelian group, $End_\Omega^\circ(A)$ is the well known endomorphism ring $E(A)$ of A. If A is an R-module over a commutative ring R, then $End_\Omega^\circ(A)$ is an R-algebra (in the pre-Birkhoff sense).

When $A \in \mathbf{OC}_\Omega$, we can expand on 1.1.

2.5 PROPOSITION. *Let* $A \in \mathbf{OC}_\Omega$ *have pseudobasis* X. *Then the mapping* $f \mapsto [f]$ *is an isomorphism between* A^X and $End_\Omega(A)$.

Proof. The bracket mapping is a bijection by 1.1. The constantly zero map from X to A extends uniquely to the zero endomorphism on A; if $\mu \in \Omega_m$ and $f_1, \ldots, f_m \in A^X$ then both $[\mu(f_1, \ldots, f_m)]$ and $\mu([f_1], \ldots, [f_m])$ are endomorphisms on A since $A \in \mathbf{OC}_\Omega$. For any $x \in X$, both endomorphisms applied to x give the value $\mu(f_1(x), \ldots, f_m(x))$. They therefore agree at every element of A; whence $f \mapsto [f]$ is an isomorphism. □

The following notion generalizes that of "E-ring," coined by P. Schultz (see [5]). Let $A \in \mathbf{OC}_\Omega^\circ$. A has *property E* if every member of $End(A \mid \Omega)$ can be obtained via left multiplication by a member of A. That is, the map from A to $End_\Omega^\circ(A \mid \Omega)$ taking $a \in A$ to the endomorphism $t \mapsto a \circ t$, generally an embedding of A into $End_\Omega^\circ(A \mid \Omega)$, is an isomorphism.

2.6 PROPOSITION. *Suppose* $A \in \mathbf{OC}_\Omega^\circ$ *has property E. Then* $A \mid \{\circ, id\}$ *is a commutative monoid.*

Proof. Fix $a \in A$. The map $t \mapsto t \circ a$ is an endomorphism on A, so there is some $b \in A$ such that $b \circ t = t \circ a$ for all $t \in A$; in particular for $t = id$. Thus $b = a$, and $A \mid \{\circ, id\}$ is commutative. □

The next result, a straightforward generalization of Theorem 2.3 in [1], tells us how to equate the 1-free algebras in $\mathbf{OC}_\Omega$ and the algebras in $\mathbf{OC}_\Omega^\circ$ with property E.

2.7 THEOREM. *Let $A \in \mathbf{OC}_\Omega$. The following are equivalent*:

(i) *A is 1-free*

(ii) *$A \cong End_\Omega(A)$, and $End_\Omega^\circ(A)$ has property E.*

(iii) *There is an element $x \in A$ such that $A = \{\varphi(x) : \varphi \in End(A)\}$, and $End^\circ(A)$ is a commutative monoid.*

Proof. (i) $\Rightarrow$ (ii). Assume A has pseudobasis $\{x\}$. Then $A \cong End_\Omega(A)$ via the bracket mapping, by 2.5. Suppose Φ is an endomorphism on $End_\Omega(A)$. Define $\varphi : A \to A$ by $\varphi(a) = \Phi([a])(x)$, where $[a]$ is the unique endomorphism on A taking x to a. It is a routine matter to check that $\varphi \in End(A)$; in fact $\varphi = [\Phi(id_A)(x)]$. To verify that $\Phi(\psi) = \varphi \circ \psi$ for any $\psi \in End(A)$, let $\psi = [a]$, and apply both $\Phi(\psi)$ and $\varphi \circ \psi$ to x. The result in each case is $\varphi(a)$.

(ii) $\Rightarrow$ (iii). $End^\circ(A)$ is a commutative monoid by 2.6. Suppose $\eta : End_\Omega(A) \to A$ is an isomorphism, and let $x = \eta(id_A)$. Fix $a \in A$ and define $\Phi : End_\Omega(A) \to End_\Omega(A)$ by $\Phi(\psi) = \eta^{-1}(a) \circ \psi$. Then Φ is an endomorphism. Let $\varphi = \eta \circ \Phi \circ \eta^{-1}$. Then $\varphi(x) = \eta(\Phi(id_A)) = \eta(\eta^{-1}(a) \circ id_A) = a$.

(iii) $\Rightarrow$ (i). Let $x \in A$ be such that $A = \{\varphi(x) : \varphi \in End(A)\}$. To show $\{x\}$ is a pseudobasis for A, we need to show that two distinct endomorphisms on A must disagree at x. Suppose $\varphi, \psi \in End(A)$ agree at x, and let $a \in A$ be arbitrary, say $a = \theta(x)$ for some $\theta \in End(A)$. Then $\varphi(a) = \varphi(\theta(x)) = \theta(\varphi(x)) = \theta(\psi(x)) = \psi(\theta(x)) = \psi(a)$. Thus $\varphi = \psi$. □

2.8 REMARK. We have established that whenever A is a 1-free algebra in $\mathbf{OC}_\Omega$, there is a naturally defined $B \in \mathbf{OC}_\Omega^\circ$, satisfying property E, whose reduct to Ω is A. Conversely, given any $B \in \mathbf{OC}_\Omega^\circ$ satisfying property E, the reduct of B to Ω is a 1-free algebra in $\mathbf{OC}_\Omega$. (Note that every $B \in \mathbf{OC}_\Omega^\circ$ satisfying property E is itself a 0-free Ω^*-algebra.)

The following is an obvious consequence of 1.2 and 2.7.

2.9 COROLLARY. *Let $A \in \mathbf{OC}_\Omega$ be nontrivial and 1-free. Then A is not κ-free for any $\kappa \neq 1$.* □

An algebra that is κ-free for exactly one cardinal κ is called *uniquely κ-free* (or *uniquely minimally free*, if we wish to suppress κ). Thus 2.9 above says that any nontrivial 1-free algebra in $\mathbf{OC}_\Omega$ is uniquely 1-free. Other examples illustrating this phenomenon appear in the following list.

2.10 EXAMPLES. (i) If A is a vector space over a field F, then any pseudobasis for A is also a vector space basis. [A pseudobasis is linearly independent, hence is contained in a basis. But no nonempty pseudobasis can ever be properly contained in another.] Thus every vector space A is uniquely $\dim_F(A)$-free.

(ii) For each cardinal κ, let $\mathbb{R}^\kappa$ be the topological (Tichonov) product of κ copies of the usual real line. Then the unital ring $C(\mathbb{R}^\kappa)$ of continuous real-valued functions on $\mathbb{R}^\kappa$ is uniquely κ-free [1].

(iii) Let I be a set of cardinality continuum, and let $\mathbb{R}$ now be the real field. Then the direct power $\mathbb{R}^I$ is a κ-free unital ring for all positive κ such that $2^\kappa \leq 2^{\aleph_0}$ [1].

(iv) Every nontrivial finite minimally free algebra (of any type) is uniquely minimally free by 1.1.

A question we have spent a fair amount of time trying to answer is whether every nontrivial minimally free abelian group is uniquely minimally free. We take up this issue in the next section. Before doing so, we record the following helpful codicil to 1.8.

2.11 THEOREM. *Let Ω be a type containing both a binary operation + and a (unique) constant* 0, *and suppose $A \in \mathbf{OC}_\Omega$ has a nonempty pseudobasis X and satisfies the identity $v + 0 = 0 + v = v$. Then*:

(i) *For $Y, Z \subseteq X$, $[f_Y]$ and $[f_Z]$ commute with respect to +, and* $[f_Y] + [f_Z] = [f_{Y \cup Z}] + [f_{Y \cap Z}]$.

(ii) *For $Y, Z \subseteq X$ disjoint, $\langle A_Y \cup A_Z \rangle = A_{Y \cup Z}$.*

Proof. (i) Because $A \in \mathbf{OC}_\Omega$, the sum of two endomorphisms is an endomorphism. The desired equalities hold, then, because both sides respectively agree at each element of X.

(ii) Clearly $\langle A_Y \cup A_Z \rangle \subseteq A_{Y \cup Z}$ in general by 1.8(i). By 1.8(ii), we know $[f_W] \mid A_W = id_{A_W}$ for any $W \subseteq X$. Thus, for any $a \in A_{Y \cup Z}$, we have $a = [f_{Y \cup Z}](a) = [f_{Y \cup Z}](a) + [f_{Y \cap Z}](a)$, since $Y \cap Z = \varnothing$ and $[f_\varnothing](a) = 0$. This now becomes $[f_Y](a) + [f_Z](a)$ by (i) above, and is an element of $\langle A_Y \cup A_Z \rangle$. □

3. Applications to Abelian groups

Our work thus far can be applied in the setting of abelian groups with some satisfactory results. First we collect what we already know (or can easily deduce) from §1 and §2.

3.1 THEOREM. (i) *Every* 1*-free group is abelian.*

(ii) *If* A *is a* 1*-free group, then the weak direct power* $A^{[\kappa]}$ *is* κ*-free.*

(iii) *If* X *is a nonempty pseudobasis for the abelian group* A, *then for each* $x \in X$ *there is a* 1*-free subgroup* A_x *with pseudobasis* $\{x\}$. *For each* $x, y \in X$, *there is an isomorphism between* A_x *and* A_y *taking* x *to* y. *The subgroup* $S = S_X = \langle \bigcup_{x \in X} A_x \rangle$ *is isomorphic to the weak direct power* $A_y^{[X]}$ *for any* $y \in X$; X *is a pseudobasis for* S, *and* S *is pure in* A (*i.e.,* $nS = S \cap nA$ *for all integers* n). *If* X *is finite, then* $S = A$.

(iv) *If* A *is a* 1*-free group with pseudobasis* $\{x\}$, *then its endomorphism ring* $E(A)$ *is an E-ring, hence a commutative unital ring. The map that takes* $a \in A$ *to the unique endomorphism* $[a]$ *taking* x *to* a *is an isomorphism from* A *onto the additive group of* $E(A)$.

(v) *If* A *has a pseudobasis with at least two elements, then* $E(A)$ *is noncommutative. Thus every nontrivial* 1*-free group is uniquely* 1*-free.*

Proof. (i) This is proved in [1], also in §0 above.

(ii) Use 1.3 and 1.5.

(iii) Let $X \neq \varnothing$ be a pseudobasis for the abelian group A, and let $A_x = A_{\{x\}}$ as in 1.8, 2.11. Then $\{x\}$ is a pseudobasis for A_x by 1.8(ii). A_x and A_y are isomorphic via an isomorphism taking x to y by (the proof of) 1.8(iii). By 1.8(i), $A_x \cap \langle \bigcup_{y \in X \setminus \{x\}} A_y \rangle \subseteq A_x \cap A_{X \setminus \{x\}} = \{0\}$, so S is isomorphic to the weak direct product of the groups A_x. Since these groups are all isomorphic, we have $S \cong A_y^{[X]}$ for any $y \in X$. X is then a pseudobasis for S by 1.3 and 1.5. If X is finite, $S = A$ by 2.11(ii). To show S is pure in A, suppose $a \in A$ and $na \in S$ for some integer n, say $na = k_1 a_1 + \cdots + k_l a_l$ where, for $1 \leq i \leq l$, k_i is an integer and $a_i \in A_{x_i}$. Set $\varphi = [f_{x_1}] + \cdots + [f_{x_l}]$. Then $b = \varphi(a) \in S$; moreover $nb = \varphi(na) = k_1 \varphi(a_1) + \cdots + k_l \varphi(a_l)$. Now $[f_{x_i}](a_i) = a_i$, and $[f_{x_j}](a_i) = [f_{x_j}]([f_{x_i}](a_i)) = 0$ if $j \neq i$, by 1.6. Thus $\varphi(na) = na$.

(iv) This is proved in [1], also, more generally, in 2.7.

(v) Use 1.2. □

3.2 REMARK. In 3.1(iii) there is a model-theoretic strengthening of the purity of $S = S_X$ in A that is also true. If Σ is a first order sentence of abelian group theory that is positive (i.e., built up from equations of terms via the logical operations of conjunction, disjunction, and universal and existential quantification) and contains extra constants that name elements of S, and if Σ is true in A, then Σ is true in S. This is due to the following fact: for each $a_1, \ldots, a_m \in S$, there is a homomorphism $\varphi : A \to S$ that fixes each a_i, $1 \leq i \leq m$.

Our first applications take the form of classification theorems. In the sequel, $\mathbb{Q}$ (resp. $\mathbb{Z}$, $\mathbb{Z}(n)$, $n = 1, 2, \ldots$) is the group or unital ring of rational numbers (resp.

integers, integers modulo n). Our intention as to which structure, group-theoretic or ring-theoretic, we have in mind will be made clear by context. (After all, $\mathbb{Q}$ is 1-free as a group, but 0-free as a unital ring.)

The following classification theorem, due to P. Schultz (see [5]), was originally phrased in terms of E-rings; we make the obvious translation to the context of groups.

3.3 THEOREM (P. Schultz). *The* 1*-free groups that are nonreduced* (*resp. torsion, finitely generated*) *are precisely those abelian groups of the form* $\mathbb{Q} \times \mathbb{Z}(n)$ (*resp.* $\mathbb{Z}(n)$, $\mathbb{Z}$ or $\mathbb{Z}(n)$). □

We quickly obtain the following generalization.

3.4 THEOREM. *Let* $1 \leq m < \aleph_0$. *Then the* m*-free abelian groups that are nonreduced* (*resp. torsion, finitely generated*) *are precisely those abelian groups of the form* $(\mathbb{Q} \times \mathbb{Z}(n))^m$ (*resp.* $\mathbb{Z}(n)^m$, $\mathbb{Z}^m$ *or* $\mathbb{Z}(n)^m$).

Proof. Suppose A is an m-free abelian group with pseudobasis X of cardinality m, and assume A is nonreduced. In the notation of 3.1(iii), set $B = A_x$ for any fixed $x \in X$ we like. Then $B^m \cong S_X = A$. Now B^m is nonreduced, so there is a nontrivial divisible subgroup $D \subseteq B^m$. Some projection of D onto B is nontrivial and divisible, hence B is nonreduced. By 3.3, B is of the form $\mathbb{Q} \times \mathbb{Z}(n)$. Of course every group of this form is m-free by 3.1(ii). The two other cases are proved similarly: when A is torsion, B is automatically torsion, so of the form $\mathbb{Z}(n)$; when A is finitely generated, so also is B, by the structure theorem for finitely generated abelian groups. □

We now set out to improve upon 3.4.

3.5 THEOREM. *No finitely generated abelian group is* κ*-free for infinite* κ.

Proof. Let A be finitely generated, with $X \subseteq A \backslash \{0\}$ an infinite set. It is enough to show X is not Marczewski independent. Marczewski independence implies usual independence in the context of abelian groups; i.e., if $n_1 x_1 + \cdots + n_k x_k = 0$ then $n_1 x_1 = \cdots = n_k x_k = 0$. (The two notions of independence are equivalent in the context of torsion-free groups.) Since A is finitely generated, the structure theorem allows us to write $A = \mathbb{Z}^l \times B$, where l is a positive integer and B is finite. Because X is infinite, there is an infinite subset of X, all of whose second coördinates are equal. Write this set as $Y \times \{b\} = \{\langle y_1, b\rangle, \langle y_2, b\rangle, \ldots\}$.

Then we have the inclusion $Y \subseteq \mathbb{Z}^l \subseteq \mathbb{Q}^l$. Since Y is infinite, Y is linearly dependent as a subset of the $\mathbb{Q}$-vector space $\mathbb{Q}^l$, so we have a linear equation

$q_1y_1 + \cdots + q_ky_k = 0$, where $q_1, \ldots, q_k \in \mathbb{Q}$ and, say, $q_1 \neq 0$. Let n be a positive integer such that $nq_1, \ldots, nq_k \in \mathbb{Z}$ and $nb = 0$. Then $nq_1\langle y_1, b\rangle + \cdots + nq_k\langle y_k, b\rangle = \langle 0, 0\rangle$ and $nq_1 \neq 0$. The order of $\langle y_1, b\rangle$ is infinite, so $nq_1\langle y_1, b\rangle \neq \langle 0, 0\rangle$. Thus $Y \times \{b\}$ is not independent, hence X is not a pseudobasis. □

3.6 COROLLARY. *Every finitely generated minimally free abelian group is of the form* $\mathbb{Z}^m$ *or* $\mathbb{Z}(n)^m$ *for some finite* m. □

As a further improvement of 3.4, we propose a complete classification of the minimally free torsion abelian groups.

3.7 LEMMA (L. Kulikov, Theorem 27.5 in [6]). *If* A *is an abelian group and* S *is a pure subgroup of* A *that is bounded (i.e.,* $nS = \{0\}$ *for some positive integer* n*), then* S *is a direct summand of* A. □

3.8 THEOREM. *Let* κ *be any cardinal number. Then the* κ*-free torsion abelian groups are precisely those abelian groups of the form* $\mathbb{Z}(n)^{[\kappa]}$.

Proof. Every weak direct power $\mathbb{Z}(n)^{[\kappa]}$ is a κ-free torsion abelian group by 3.1(ii). Let A be a torsion abelian group with pseudobasis X of cardinality κ. In the notation of 3.1(iii), set $B = A_x$ for any fixed $x \in X$. Then $B^{[\kappa]} \cong S = S_X$. Since B is a 1-free torsion group, there is some positive integer n such that $B \cong \mathbb{Z}(n)$, by 3.3. Thus $nS = \{0\}$. By 3.1(iii), S is pure in A. Thus, by 3.7, there is a subgroup C of A such that $S \cap C = \{0\}$ and $\langle S \cup C\rangle = A$. But $X \subseteq S$. Let $\psi \in End(A)$ take each element of x to 0 and each element of C to itself. (This is possible since $A \cong S \times C$.) Then, since X is a pseudobasis for A, ψ is the zero map. Consequently, $C = \{0\}$ and $S = A$. □

We next take up the issue of unique minimal freeness for abelian groups. The question of whether every nontrivial κ-free abelian group (or even every such group of the form $A^{[\kappa]}$, where A is 1-free) is uniquely κ-free is still open. We show below that every nontrivial countable minimally free abelian group is uniquely minimally free.

The approach we take is via the theory of modules. If A is any abelian group and X any set, we may view $A^{[X]}$ as an $E(A)$-module, where we define $\varphi \cdot \sum_{x \in X} a_x$ to be $\sum_{x \in X} \varphi(a_x)$. If A is 1-free, then A is isomorphic to the additive group of $E(A)$, and therefore $A^{[X]}$ is the free $E(A)$-module on $|X|$ generators. What is more, $E(A)$ is a commutative unital ring, as we showed in 2.7.

3.9 LEMMA (Corollary 2.12, p. 186, in [9]). *Let R be a commutative unital ring, with M a free R-module. Then any two free bases for M have the same cardinality.* □

3.10 REMARK. By 3.8, every minimally free torsion abelian group A is a free $\mathbb{Z}(n)$-module, where n is the maximum order of any element of A. [If $X \subseteq A$ is a pseudobasis, then all the elements of X have the same order, and the order of any element of A is a divisor of the order of an element of X.]

3.11 THEOREM. *Suppose A and B are 1-free groups, with A nontrivial, and let I and J be sets, with I nonempty. Then $A^{[I]} \cong B^{[J]}$ if and only if $A \cong B$ and $|I| = |J|$.*

Proof. Let $\eta : A^{[I]} \to B^{[J]}$ be an isomorphism, let $\{x\}$ and $\{y\}$ be pseudobases for A and B respectively, with $A \neq \{0\}$ and $I \neq \emptyset$. We denote by $\sigma_i, \sigma_j, \pi_i, \pi_j$ the respective canonical injections and projections. Then $I' = \{\sigma_i(x) : i \in I\}$, $J' = \{\sigma_j(y) : j \in J\}$ are pseudobases for $A^{[I]}$ and $B^{[J]}$ respectively, $|I'| = |I|$, $|J'| = |J|$. Fix $i \in I, j \in J$. Then there is a homomorphism $\varphi : A^{[I]} \to B^{[J]}$ taking $\sigma_i(x)$ to $\sigma_j(y)$, and a homomorphism $\psi : B^{[J]} \to A^{[I]}$ taking $\sigma_j(y)$ to $\sigma_i(x)$. [This is because $\eta(I')$ is a pseudobasis for $B^{[J]}$. So let φ be η followed by any endomorphism on $B^{[J]}$ taking $\eta(\sigma_i(x))$ to $\sigma_j(y)$.]

Now let $\theta = \pi_j \circ \varphi \circ \sigma_i : A \to B$ and $\tau = \pi_i \circ \psi \circ \sigma_j : B \to A$. Then $\theta(x) = y$ and $\tau(y) = x$. Since A and B are 1-free, this implies θ and τ are mutually inverse isomorphisms.

Thus we are reduced to the question of whether $A^{[I]} \cong A^{[J]}$ implies $|I| = |J|$, where A is nontrivial 1-free and $I \neq \emptyset$. Suppose $\alpha : A^{[I]} \to A^{[J]}$ is any homomorphism. As mentioned in the paragraph before 3.9, $A^{[I]}$ is a free $E(A)$-module, where scalar multiplication is coördinatewise application. We claim α is a module homomorphism. Indeed, let $a \in A$, $\rho \in E(A)$. $\sigma_i(a)$ is a typical additive generator of $A^{[I]}$, so we need only check that $\alpha(\rho \cdot \sigma_i(a)) = \rho \cdot \alpha(\sigma_i(a))$. This is true just in case it is true under each projection. So let $j \in J$; then $\pi_j(\alpha(\rho \cdot \sigma_i(a))) = \pi_j(\alpha(\sigma_i(\rho(a))))$. Now $\pi_j \circ \alpha \circ \sigma_i \in E(A)$, a commutative ring by 2.7. Thus $\pi_j(\alpha(\sigma_i(\rho(a)))) = \rho(\pi_j(\alpha(\sigma_i(a)))) = \pi_j(\rho \cdot \alpha(\sigma_i(a)))$. Consequently, if $A^{[I]}$ and $A^{[J]}$ are isomorphic as abelian groups, they are also isomorphic as $E(A)$-modules. By 3.9, $|I| = |J|$. □

3.12 COROLLARY. *A nontrivial abelian group cannot be both m-free and n-free for different finite m and n.*

Proof. Use 3.1(iii) and 3.11. □

3.13 REMARK. In a private communication, R. Schutt announced the result 3.12 without proof.

3.14 COROLLARY. *Assume A is a nontrivial minimally free abelian group that is countable. Then A is uniquely minimally free.*

Proof. If A is finite, the result is immediate by 1.1. If A is countably infinite, A cannot have both finite and infinite pseudobases, again by 1.1. A cannot have two pseudobases of differing finite cardinalities, by 3.12. $\square$

The next result is only conditional, relying on the Generalized Continuum Hypothesis (G.C.H.).

3.15 COROLLARY (G.C.H.). *Let A be a nontrivial countable 1-free abelian group, $\kappa > 0$. Then $A^{[\kappa]}$ is uniquely κ-free.*

Proof. If κ is finite, then $A^{[\kappa]}$ is uniquely κ-free by 3.1(ii) and 3.14. If κ is infinite, then, by 1.1, $|End(A)| = |A^{[\kappa]}|^{\kappa} = \kappa^{\kappa} = 2^{\kappa}$. If $\lambda < \kappa$ and A has a pseudobasis of cardinality λ, λ must be infinite as well. But then $|End(A)| = 2^{\lambda}$. By the G.C.H., $2^{\lambda} < 2^{\kappa}$, a contradiction. $\square$

In special cases, it is unnecessary to resort to the G.C.H. in 3.15.

3.16 THEOREM. (i) *Every pseudobasis of $\mathbb{Q}^{[\kappa]}$, $\kappa > 0$, has cardinality κ; no pseudobasis is a basis.*

(ii) *Every pseudobasis for $(\mathbb{Q} \times \mathbb{Z}(n))^{[\kappa]}$, $0 < \kappa \leq \aleph_0$, has cardinality κ; no pseudobasis is a basis.*

(iii) *Every pseudobasis for $\mathbb{Z}^{[\kappa]}$ or for $\mathbb{Z}(n)^{[\kappa]}$, $\kappa > 0$, has cardinality κ and is also a basis.*

Proof. (i) If $X \subseteq \mathbb{Q}^{[\kappa]}$ is a pseudobasis, then X is a $\mathbb{Q}$-vector space basis by 2.10(i). Thus $|X| = \kappa$. Because of unique representation of elements of $\mathbb{Q}^{[\kappa]}$ in terms of linear combinations of elements of X, it is impossible to span $\mathbb{Q}^{[\kappa]}$ using only integer coefficients.

(ii) Every pseudobasis for $(\mathbb{Q} \times \mathbb{Z}(n))^{[\kappa]}$, $0 < \kappa \leq \aleph_0$, has cardinality κ by 3.14. No finite pseudobasis of $(\mathbb{Q} \times \mathbb{Z}(n))^{[\kappa]}$ is a generating set since the group is not finitely generated. Suppose $\kappa = \aleph_0$, and let $X \subseteq \mathbb{Q}^{[\kappa]} \times \mathbb{Z}(n)^{[\kappa]}$ be a pseudobasis. Then $|X| = \kappa$, and $nX \subseteq \mathbb{Q}^{[\kappa]} \times \{0\}$ is a vector space basis for $\mathbb{Q}^{[\kappa]} \times \{0\}$. By arguing as in the proof of (i) above, we infer that there is an element $a \in \mathbb{Q}^{[\kappa]} \times \{0\}$ that cannot be written as a linear combination of members of nX using only integer coefficients. Thus $a/n \in \mathbb{Q}^{[\kappa]} \times \{0\}$ cannot be written as a linear combination of members of X using only integer coefficients.

(iii) Let $X \subseteq \mathbb{Z}^{[\kappa]} \subseteq \mathbb{Q}^{[\kappa]}$ be a pseudobasis for $\mathbb{Z}^{[\kappa]}$. X is independent as a subset of $\mathbb{Z}^{[\kappa]}$, hence linearly independent as a subset of the $\mathbb{Q}$-vector space $\mathbb{Q}^{[\kappa]}$. Let Y be a vector space basis for $\mathbb{Q}^{[\kappa]}$ containing X. For each $y \in Y$, let $n(y)$ be the least positive integer k such that $ky \in \mathbb{Z}^{[\kappa]}$. (For $y \in X$, $n(y) = 1$.) Let $Y' = \{n(y)y : y \in Y\}$. Then $|Y'| = |Y| = \kappa$, $X \subseteq Y'$, and $Y' \subseteq \mathbb{Z}^{[\kappa]}$. Moreover Y' is linearly independent in $\mathbb{Q}^{[\kappa]}$. Thus Y' is a basis for $\mathbb{Q}^{[\kappa]}$, hence a free basis for $\mathbb{Z}^{[\kappa]}$. Since both X and Y' are pseudobases for $\mathbb{Z}^{[\kappa]}$ and $X \subseteq Y'$, we have $X = Y'$.

Now consider $\mathbb{Z}(n)^{[\kappa]}$. There is a pseudobasis of cardinality κ that is a basis, hence every pseudobasis of cardinality κ must be a basis. [Just move one onto the other via an automorphism.] If $\kappa > 0$ is countable, our assertion is proved by 3.14. Assume, by way of transfinite induction, that for each positive integer l and each positive cardinal $\lambda < \kappa$, where κ is infinite, that $\mathbb{Z}(l)^{[\lambda]}$ is uniquely λ-free. If $X \subseteq \mathbb{Z}(n)^{[\kappa]}$ is a pseudobasis, then $|X| \leq \kappa$. If $|X| = \lambda \leq \kappa$ then, by 3.8, $\mathbb{Z}(n)^{[\kappa]} \cong \mathbb{Z}(l)^{[\lambda]}$. By the inductive hypothesis, if $\lambda < \kappa$, $\mathbb{Z}(l)^{[\lambda]}$ is uniquely λ-free. But it has a pseudobasis of cardinality κ, a contradiction. Thus $\lambda = \kappa$. (Alternatively, by 3.11, $n = l$ and $\kappa = \lambda$.) □

3.17 REMARK. We do not know whether every κ-free abelian group that is not reduced is of the form $(\mathbb{Q} \times \mathbb{Z}(n))^{[\kappa]}$, for $\kappa \geq \aleph_0$. If the (rather dubious) hypothesis that every κ-free abelian group is of the form $A^{[\kappa]}$ for some 1-free group A were true, then we could conclude from 3.11 that every nontrivial minimally free abelian group is uniquely minimally free.

3.18 COROLLARY. (i) *Let* $1 \leq m < \aleph_0$. *Then every* m*-free abelian group that is not reduced is uniquely* m*-free.*

(ii) *Every nontrivial minimally free torsion or finitely generated abelian group is uniquely minimally free.*

Proof. (i) Use 3.4 and 3.16(ii).

(ii) Use 3.8 and 3.16(iii) for torsion; 1.1 and 3.6 for finitely generated. □

3.19 REMARK. There are many open problems in the area of minimally free abelian groups. Some we have already alluded to, but the reader will no doubt think of several other interesting ones. We did not pay any attention to the torsion-free divisible case, as that is already subsumed by the trivial situation in vector space theory.

4. Minimally free idempotent algebras

In this section we take Ω to be a type with no constants. An Ω-algebra A is an *idempotent algebra* if every element of A is an idempotent. (Equivalently: (i) every singleton subset is a subalgebra; or (ii) every constant self-map is an endomorphism.) The variety of idempotent Ω-algebras is denoted $\mathbf{I}_\Omega$.

4.1 PROPOSITION. *If $A \in \mathbf{I}_\Omega$ is 0-free or 1-free, then A is trivial.*

Proof. If A is 0-free then A is trivial because the constant maps are endomorphisms. If A is 1-free with pseudobasis $\{x\}$, then both id_A and the constantly x map are endomorphisms on A that agree on x. They are thus equal. □

Recall that a *left zero* of a semigroup S is any element $z \in S$ such that $z \cdot s = z$ for all $s \in S$.

4.2 PROPOSITION. *Let $A \in \mathbf{I}_\Omega$. Then the left zeros of $End^\circ(A)$ are the constant maps on A.*

Proof. Clearly every constant endomorphism is a left zero of $End^\circ(A)$. Suppose $\varphi \in End^\circ(A)$ is nonconstant, say $\varphi(a_1) = b_1$, $\varphi(a_2) = b_2$, and $b_1 \neq b_2$. Let ψ be constantly a_2. Then $(\varphi \circ \psi)(a_1) = b_2 \neq b_1 = \varphi(a_1)$; hence φ is not a left zero of $End^\circ(A)$. □

Let $A \in \mathbf{I}_\Omega$. Define $\gamma : A \to End(A)$ to be the map that takes $a \in A$ to the constantly a endomorphism.

4.3 PROPOSITION. *γ is an Ω-embedding of A into A^A; moreover $\gamma(A)$ is a subsemigroup of $End^\circ(A)$.*

Proof. Trivial. □

For idempotent Ω-algebras, the notion of "n-free" for finite n may be viewed as first order, from the standpoint of model theory.

4.4 PROPOSITION. *For each finite cardinal n, there is a first order sentence σ_n over the language of monoids such that if $A \in \mathbf{I}_\Omega$, then A is n-free if and only if σ_n is true in $End^\circ(A)$.*

Proof. If $n = 0$, the assertion is always true, regardless of type: just express the statement that there is one element. For $n > 0$, use 4.2 and the facts that the set of left zeros of a semigroup is first order definable and for $a \in A$, $\varphi \in End(A)$, $\gamma(\varphi(a)) = \varphi \circ (\gamma(a))$. □

The class of operationally commutative idempotent algebras is denoted $\mathbf{IOC}_\Omega = \mathbf{I}_\Omega \cap \mathbf{OC}_\Omega$. Semilattices (i.e., commutative idempotent semigroups), and, more generally, normal bands (i.e., idempotent semigroups satisfying the medial law) are well known special cases of this kind of algebra (see, e.g., [11]).

Let $A \in \mathbf{IOC}_\Omega$ have pseudobasis X. For each $Y \subseteq X$, let $\Phi(Y) = \{\varphi \in End(A) : \varphi(x) = x \text{ for all } x \in X \backslash Y\}$.

4.5 THEOREM. *Suppose $A \in \mathbf{IOC}_\Omega$ is nontrivial with pseudobasis X. Let $\mathbf{P}(X)$ be the complete bounded meet-semilattice of subsets of X. Then*:

(i) *Φ is a complete embedding of $\mathbf{P}(X)$ into the complete bounded meet-semilattice of subalgebras of $End^\circ_\Omega(A)$.*

(ii) *For each $Y \subseteq X$, let $\rho_Y : A^Y \to A^X$ be the natural embedding, i.e.,*

$$\rho_Y(f)(x) = \begin{cases} f(x) & \text{if } x \in Y \\ x & \text{if } x \in X \backslash Y. \end{cases}$$

Then $[\cdot]_Y = [\cdot] \circ \rho_Y$ is an isomorphism between A^Y and $\Phi(Y) \mid \Omega$.

(iii) *For $Y, Z \subseteq X$, if $|Y \backslash Z| = |Z \backslash Y|$, then there is an involutionary automorphism on $End^\circ_\Omega(A)$ taking $\Phi(Y)$ onto $\Phi(Z)$.*

(iv) *If $|X| = 2$, then, for $Y \subseteq X$, $\Phi(Y)$ is a commutative submonoid if and only if $|Y| \leq 1$.*

(v) *If $|X| > 2$, then, for $Y \subseteq X$, $\Phi(Y)$ is a commutative submonoid if and only if $Y = \varnothing$.*

Proof. (i) The verifications are routine. First check that $\{id_A\}$ is a subalgebra of $End^\circ_\Omega(A)$, so the subalgebra lattice is indeed bounded. For each $Y \subseteq X$, $\Phi(Y)$ is closed under function composition; if $\mu \in \Omega_m$ and $\varphi_1, \ldots, \varphi_m \in \Phi(Y)$, let $x \in X \backslash Y$. Then $\mu(\varphi_1, \ldots, \varphi_m)(x) = \mu(x, \ldots, x) = x$ since $A \in \mathbf{IOC}_\Omega$. Thus $\mu(\varphi_1, \ldots, \varphi_m) \in \Phi(Y)$, so $\Phi(Y)$ is a subalgebra of $End^\circ_\Omega(A)$. Next we have the equalities $\Phi(X) = End(A)$, $\Phi(\varnothing) = \{id_A\}$, and for any family $\langle Y_i : i \in I \rangle$ of subsets of X, $\Phi(\bigcap_{i \in I} Y_i) = \bigcap_{i \in I} \Phi(Y_i)$. Finally, if $Y, Z \subseteq X$ are distinct, say $y \in Y \backslash Z$, let $\varphi \in End(A)$ move y and fix all $x \in X \backslash Y$ (since A is nontrivial). Then $\varphi \in \Phi(Y) \backslash \Phi(Z)$; whence Φ is one-one.

(ii) $[\cdot]_Y = [\cdot] \circ \rho_Y$ is an embedding of A^Y into $End_\Omega(A)$ by 2.5; its image is $\Phi(Y)$.

(iii) Let $h_0 : Y \backslash Z \to Z \backslash Y$ be a bijection, and define $h : X \to X$ as follows:

$$h(x) = \begin{cases} h_0(x) & \text{if } x \in Y \backslash Z \\ h_0^{-1}(x) & \text{if } x \in Z \backslash Y \\ x & \text{otherwise.} \end{cases}$$

Then $[h]$ is an involutionary automorphism on A. For any $\varphi \in End(A)$, define $\eta(\varphi) = [h] \circ \varphi \circ [h]$. Then η is easily seen to be an involutionary automorphism on $End^\circ_\Omega(A)$, taking $\Phi(Y)$ onto $\Phi(Z)$.

(iv) For each $Y \subseteq X$, let $\langle A, Y \rangle$ be the $(\Omega \cup Y)$-algebra in which the elements of Y are regarded as distinguished constants. Then it is easy to see that Y is always a pseudobasis for $\langle A, X \setminus Y \rangle$, and that the submonoid $\Phi(Y) \mid \{\circ, id\}$ is isomorphic to $End^{\circ}(\langle A, X \setminus Y \rangle)$. Suppose $|X| = 2$, $Y \subseteq X$. If $Y = \varnothing$ then $\Phi(Y) = \{id_A\}$. If $|Y| = 1$, then $\Phi(Y)$ is a commutative submonoid by 2.7. If $Y = X$, then $\Phi(Y)$ is noncommutative because of 1.2.

(v) Suppose $|X| > 2$. If $Y \neq \varnothing$, then $\Phi(Y)$ is noncommutative, again by 1.2. (If $|Y| > 1$, then there are too many pseudobasis elements; if $|Y| = 1$, then there are too many distinguished constants.) □

If $A \in \mathbf{IOC}_{\Omega}$ is 2-free, there is a first order statement that holds for $End^{\circ}(A)$, but which appears to fail for $End^{\circ}(B)$ when B is κ-free, $\kappa > 2$. Define a semigroup S to be *somewhere commutative* if there is a left zero z such that: (i) every left zero $z' \in S$ is a multiple $s \cdot z$ for some $s \in S$; and (ii) whenever $s, t \in S$ both commute with z, then $s \cdot t = t \cdot s$.

4.6 PROPOSITION. *Let* $A \in \mathbf{IOC}_{\Omega}$ *be 2-free. Then* $End^{\circ}(A)$ *is somewhere commutative.*

Proof. This is immediate from 4.2, (the proof of) 4.4, and 4.5(iv). □

We believe that every 2-free $A \in \mathbf{IOC}_{\Omega}$ is uniquely 2-free, but have so far been unable to prove it. Also we are singularly lacking in interesting examples of minimally free operationally commutative idempotent algebras (i.e., that are not free). We leave the topic with the following small result.

4.7 PROPOSITION. *Suppose* $A \in \mathbf{IOC}_{\Omega}$ *has two pseudobases* X *and* U, *with* $|X| = 2 \neq |U|$. *Then* $X \cap U = \varnothing$.

Proof. By 4.1, $|U| > 2$. If $x \in X \cap U$, then because $|X| = 2$, we know that any two endomorphisms fixing x must commute, by 4.5(iv). But since $|U \setminus \{x\}| \geq 2$, we know this cannot be true by 4.5(v). □

Afterword. After this paper was prepared, I discovered in my notes of a telephone conversation I held with R. Schutt in 1987, that Schutt had announced to me the results 3.1(iii) (i.e., the statement that every m-free abelian group is a weak direct power of a 1-free group, $1 \leq m < \aleph_0$) as well as 3.8. I do not know to what extent my proofs resemble his.

On a different topic, R. Villemaire has recently communicated to me an example of an ω-free abelian group that is not a weak direct power of any 1-free group, thus laying to rest the "rather dubious" hypothesis in Remark 3.17. The example is simple, given the machinery of [6], and Villemaire has kindly consented to let it be sketched here.

For a fixed prime number p, let J_p be the additive group of p-adic integers. J_p is easily shown to be 1-free, so by 3.1(ii), $J_p^{[\omega]}$ is ω-free with pseudobasis X consisting of all ω-sequences that are 0 almost everywhere and 1 everywhere else. Let A now be the p-adic completion of $J_p^{[\omega]}$. Then every function from X into A extends uniquely to a homomorphism from $J_p^{[\omega]}$ to A. Since homomorphic images of Cauchy sequences are again Cauchy sequences, we have that every homomorphism from $J_p^{[\omega]}$ to A extends uniquely to an endomorphism on A. Thus X is a pseudobasis for A, making A ω-free.

Now suppose $A = B^{[\kappa]}$ for some 1-free group B. By Corollary 39.10 in [6], B is a bounded group. But then A is bounded, a contradiction.

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