General Topology and its Applications 10 (1979) 231–232

NOTE

On "ULTRAPRODUCTS IN TOPOLOGY"

Paul BANKSTON

Department of Mathematics, Southern Illinois University at Carbondale, Cardondale, IL 62901, USA

Received 19 July 1977

We correct an error in the above-mentioned paper and provide a solution to an open problem contained therein.

AMS Subj. Class.: Primary 02H13, 54B10, 54G20; Secondary 02K25, 18A30, 18B99, 54F05.

ultraproducts   topological spaces

In this note we do two things: first we correct an error in the proof of Theorem 7.7 in [1], strengthening that theorem to avoid the use of P-points in $\beta(\omega)-\omega$; and second we provide E. van Douwen's solution (privately communicated to me) of Problem 10.2 (R. Button), also in [1] (to which the reader is referred for all notations and terminology).

First the correction (we are thankful to F. Galvin for helping to find the error).

Old Theorem 7.7. *Let $\mathscr{X}$ be regular, U a preselective ultrafilter* (*i.e. a* P*-point of $\beta(\omega)-\omega$*). *Then $\Delta(\mathscr{X})$ is closed in $\prod_U(\mathscr{X})$.*

In the proof we let $[f]\in\prod_U(\mathscr{X})-\Delta(\mathscr{X})$ and constructed an open ultrabox $\prod_U N_n$ about $[f]$ which was to miss $\Delta(\mathscr{X})$. Our claim was that the sets N_n are pairwise disjoint on a member of U. This is false. Nonetheless it is true that $\bigcup_{J\in U}\bigcap_{n\in J}N_n=\emptyset$; and this is all we needed. It is not at all difficult to adjust the proof so that it becomes correct. However, rather than do this, we present a proof of a strengthened version of 7.7.

New Theorem 7.7. *Let $\mathscr{X}$ be regular, U an ultrafilter on ω. Then $\Delta(\mathscr{X})$ is closed in $\prod_U(\mathscr{X})$.*

Proof. Let $[f]\in\prod_U(\mathscr{X})-\Delta(\mathscr{X})$. We show that there is an open ultrabox $\prod_U N_n$ containing $[f]$ and missing $\Delta(\mathscr{X})$. As in the old proof, we let

$$J=\{m<\omega:\lim([f])\neq f(m)\}.$$

Then $J \in U$; for if either $[f] \notin N(\mathscr{X})$ or $\lim([f]) \neq f(m)$ (all $m < \omega$), then $J = \omega$, and if $\lim([f]) = f(m_0)$ for some $m_0 < \omega$, then by Hausdorffness $J \supset \{m : f(m) \neq f(m_0)\} \in U$.

Now for each $m \in J$ let M_m be an open neighborhood of $f(m)$ such that $K_m = \{n \in J : f(n) \notin \overline{M_m}\} \in U$, and let $N_m = M_m - \bigcup\{\overline{M_n}: n < m, f(m) \notin \overline{M_n}\}$. Then N_m is an open neighborhood of $f(m)$.

For any $K \subset J$ with $K \in U$, we show $\bigcap_{m \in K} N_m = \emptyset$. Assuming otherwise, let $x \in \bigcap_{m \in K} N_m$, and let m_0 be the least member of K in the natural order on ω. Then $m_0 \notin K_{m_0}$, so let $m_1 \in K \cap K_{m_0}$. Then $m_0 < m_1$, $f(m_1) \notin \overline{M_{m_0}}$, and hence $\overline{M_{m_0}} \cap N_{m_1} = \emptyset$. But since $m_0, m_1 \in K$, we have $x \in N_{m_0} \cap N_{m_1} \subset \overline{M_{m_0}} \cap N_{m_1}$, a contradiction. □

Remarks. (i) With trivial modifications, Theorem 7.7 generalizes to the following: Let $\mathscr{X}$ be regular and κ-open (κ a cardinal), with U an ultrafilter on κ. Then $\Delta(\mathscr{X})$ is closed in $\prod_U(\mathscr{X})$.

(ii) A space is *Urysohn* if each pair of distinct points can be separated by open sets with disjoint closures. Clearly the Urysohn property lies between regularity and Hausdorffness. With minor changes in the above proof (i.e. in the definition of J), Theorem 7.7 holds for all Urysohn spaces $\mathscr{X}$.

(iii) With reference to [1], the reader may now eliminate preselectivity from Theorem 7.8 as well as MA from the first proof of Theorem 8.2.

Next we solve Problem 10.2 which asks whether ultraproducts of scattered spaces are scattered. Van Douwen demonstrated an example saying "no"; and a little extra work on his example leads to the following

Proposition. *Let U be a countably incomplete ultrafilter on a set I with $\mathscr{X}$ a topological space. Then the ultrapower $\prod_U(\mathscr{X})$ is scattered iff the Cantor–Bendixson rank of $\mathscr{X}$ is finite.*

Proof. We recall the notation $\mathscr{X}' = \mathscr{X}$—{isolated points}; and for α an ordinal, $\mathscr{X}^\alpha = \bigcap_{\beta<\alpha}(\mathscr{X}^\beta)'$. If $x \in \mathscr{X}$, then $\mathrm{rk}(x)$ = the least α such that $x \notin \mathscr{X}^\alpha$ if there is such an α, and ∞ otherwise. We set $\mathrm{rk}(\mathscr{X}) = \sup\{\mathrm{rk}(x) : x \in \mathscr{X}\}$. $\mathscr{X}$ is scattered iff $\mathrm{rk}(\mathscr{X}) < \infty$. It is easy to check that if $\mathrm{rk}(\mathscr{X}) < \omega$, then $\mathrm{rk}(\prod_U(\mathscr{X})) < \omega$ as well for any ultrafilter U. On the other hand, if U is countably incomplete, let $\langle J_n : n < \omega\rangle$ be a partition of I into sets whose complements are in U. Define $x : I \to \mathscr{X}$ so that if $i \in J_n$, then $\mathrm{rk}(x_i) \geq n$. Let

$$A = \left\{[y]_U \in \textstyle\prod_U(\mathscr{X}) : \text{for all } n < \omega\ \{i : \mathrm{rk}(y_i) \geq n\} \in U\right\}.$$

Clearly $[x]_U$ is in A, so $A \neq \emptyset$. And no matter how U behaves, A has no isolated points. Thus $\prod_U(\mathscr{X})$ is not scattered. □

References

[1] P. Bankston, Ultraproducts in topology, General Topology and Appl. 7 (1977) 283–308.