ON BETWEENNESS AND EQUIDISTANCE IN METRIC SPACES

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ABSTRACT. The classic notions of *betweenness* and *equidistance* in Euclidean geometry readily generalize to the context of metric spaces. We view these notions from an axiomatic perspective, then analyze the role various axioms play when interpreted in a metric space−especially one where the metric is induced by a vector space norm.

The points lying between two given points constitute the *betweenness interval bracketed by* those points, and the points equidistant from two given points constitute the *equiset* with those points as *cocenters*. An equiset gives rise to a division of the underlying space into two *comparative nearness regions*; in the case of the Euclidean plane, each such region is a half-plane bounded by the line that is the equiset. Betweenness intervals naturally engender a notion of *convexity*, and one focus of this investigation is the issue of when equisets and their comparative nearness regions, as well as the betweenness intervals themselves, are convex. For normed vector spaces, betweenness intervals are always convex when the dimension is at most two, but this convexity property easily fails in higher dimensions. Equisets and comparative nearness regions in a normed vector space are convex precisely when the norm arises from an inner product. This is one of several characterizations we present of normed vector space properties purely in terms of abstract betweenness, equidistance and comparative nearness.

1. INTRODUCTION

The familiar notions of *betweenness* and *equidistance* in Euclidean geometry may easily be generalized to the metric context. Given a metric space $X = \langle X, \varrho \rangle$ and points $a, b \in X$, we make the following basic definitions.

- The **interval** $I(a, b)$, with **bracket points** a and b, is the set $\{x \in X : \varrho(a, b) = \varrho(a, x) + \varrho(x, b)\}$ of points **metrically between** a **and** b. Bracket points are not necessarily *end points* in the usual sense because it is entirely possible for $I(a, b)$ to equal $I(c, d)$ without $\{a, b\}$ and $\{c, d\}$ being the same set (see Examples 2.8, 5.1). We refer to $\{a, b\}$ as a **bracket set** for $I(a, b)$.
- The **equiset** $E(a, b)$, with **cocenters** a and b, is the set $\{x \in X : \varrho(x, a) = \varrho(x, b)\}$ of points **equidistant from** a **and** b.

When it makes sense to call attention to a specific metric, we use subscripts, $I_\varrho(a, b)$, etc.; otherwise we choose to keep the notation as simple as possible. Metric intervals were introduced in [15], and are also called *Menger intervals* in [1] and elsewhere. Metric equisets also have a distinguished history, being called *bisectors* in [9] and

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midsets in [5, 11, 12]. In this paper we prefer to use the term *bisector* in its traditional Euclidean sense. We also use the term *midset*, but in a more restrictive way.

- The **midset** $M(a, b)$, with **cocenters** a and b, is the set $I(a, b) \cap E(a, b)$ of **midpoints** of $I(a, b)$.

Each equiset $E(a, b)$ gives rise to a division of X into two subsets, called *comparative nearness regions*, which, in the case of the Euclidean plane, are the two half-planes whose common boundary is the line that constitutes the equiset. In general, we make the following definition.

- The **comparative nearness region** $R(a, b)$, with **center** a and **off-center** b, is the set $\{x \in X : \varrho(x, a) \leq \varrho(x, b)\}$ of points at least as near to a as to b.

Then $E(a, b) = R(a, b) \cap R(b, a)$. We frequently refer to comparative nearness regions more concisely as *nearness regions*. Following common usage, all mathematical structures are assumed to have nonempty underlying sets. With this in mind, we note that equisets and midsets (unlike intervals and nearness regions) can indeed be empty.

Note that when the set $\{a, b\}$ is *degenerate* (i.e, when $a = b$), $I(a, b)$ is degenerate too and $R(a, b) = X$. In the sequel, when $a \neq b$, we consider $E(a, b)$ (resp., $M(a, b)$) as a "wall separating a from b" in X (resp., in $I(a, b)$), and investigate ways in which we may view such a wall as satisfying what we informally refer to as *thinness*.

To make this idea precise, recall that a topological space is **connected** if it is not the disjoint union of two nonempty open subsets. A space that is not connected is called **disconnected**, and a cover consisting of two disjoint nonempty open sets is called a **disconnection** of the space. We say that a subset A of a space X **separates** X if $X \setminus A$ is disconnected; A **separates** a **from** b, for $a, b \in X$, if a and b lie in distinct members of a disconnection of $X \setminus A$.

In Proposition 3.1 below we show that all intervals and nearness regions are closed subsets of a metric space. Thus for each two distinct points $a, b \in X$, $E(a, b)$ (resp., $M(a, b)$) separates a from b in X (resp., in $I(a, b)$) in this sense.

Many of the metric spaces we consider are normed vector spaces $\langle X, \|\cdot\|\rangle$, over the real scalar field $\mathbb{R}$, where the metric is induced by the norm in the usual way: $\varrho(x, y) := \|x - y\|$. In such a space one may define, for points a and b, the **linear interval** bracketed by these points to be the line segment $[\![a, b]\!] := \{ta + (1 - t)b : 0 \leq t \leq 1\}$, with a and b as *end points* (and all other points as *interior points*). Clearly $[\![a, b]\!]$ is always contained in $I(a, b)$, but the metric interval may be much larger−even with nonempty topological interior. (See, e.g., the two-dimensional vector space $\mathbb{R}^2$, equipped with the *taxicab norm* in Example 5.1 below, where most intervals are solid rectangles.) The metric interval $I(a, b)$ is called **linear** if it equals $[\![a, b]\!]$. Linearity is a way of saying that an interval in a normed vector space is *thin* (in the informal sense referred to above).

1.1. **Remarks.** (i) Properly speaking, we should define linearity for *pairs of points*, not for the intervals they bracket. However, if $I(a, b) = I(c, d)$ and $I(a, b) = [\![a, b]\!]$, then we have $[\![c, d]\!] \subseteq [\![a, b]\!]$. If $x \in [\![a, b]\!] \setminus [\![c, d]\!]$, it is easy to check that $\|x - c\| + \|x - d\| > \|c - d\|$; so $x \notin I(c, d)$, a contradiction. Thus $\{c, d\} = \{a, b\}$; that is, linearity for $\langle a, b\rangle$ implies linearity for $\langle c, d\rangle$ whenever $I(c, d) = I(a, b)$.

(ii) It is worth noting—and trivial to prove—that if a and b are two points in any normed vector space, then $M(a,b) \cap [\![a,b]\!] = \{m\}$, where $m = \frac{1}{2}(a+b)$, the *halfway point* of $[\![a,b]\!]$.

We denote by B_X and S_X the unit ball $\{x \in X : \|x\| \leq 1\}$ and unit sphere $\{x \in X : \|x\| = 1\}$ of X, respectively; we will be interested in how the geometry of these sets relates to that of intervals, equisets and nearness regions in normed vector spaces.

The most classic example of a normed vector space is the Euclidean plane $\mathbb{R}^2_2$; i.e., $\mathbb{R}^2$ equipped with the Euclidean norm $\|\langle x,y\rangle\|_2 := (x^2+y^2)^{\frac{1}{2}}$. In this situation each interval is linear, and each midset is the singleton of the halfway point of the interval. Moreover, $R(a,b)$ is just the closed half-plane that contains a and whose boundary—in this case $E(a,b)$—is the usual perpendicular bisector of $[\![a,b]\!]$. In higher-dimensional Euclidean space the nondegenerate metric intervals are still the linear ones; but their corresponding nearness regions are closed half-spaces, and equisets are affine sets of codimension one. [1]

1.2. **Remark.** In a metric space X, with $S \subseteq X$ a finite subset, the *Voronoi region sited at* $a \in S$ is the set of points at least as near to a as to any $b \in S$ (see, e.g., [13]). The elements of S are called *sites*; and the collection of all Voronoi regions forms a cover of X, commonly known as the *Voronoi diagram with sites from* S. Clearly each site belongs to exactly one Voronoi region, and what we call *nearness regions* are just Voronoi regions where the site set S has at most two elements. The site belonging to a given Voronoi region is what we are referring to as the center of the region. Voronoi regions often overlap in nowhere dense sets, suggesting the usage *Voronoi tessellation* (and even *Voronoi partition*) found in the literature. But this terminology is misleading; as we see below in Example 5.1, it is entirely possible for the intersection of two Voronoi regions to have nonempty interior. In the present paper, all Voronoi diagrams have at most two sites.

In the next section we lay out several key features of betweenness, equidistance and comparative nearness, all expressed as abstract first-order axioms.

2. The IR-axioms

We single out a first-order predicate language whose atomic formulas are equalities and formulas of the form $I(y,x,z)$ and $R(x,y,z)$, where I and R are ternary relation symbols. We read $I(y,x,z)$ (resp, $R(x,y,z)$) as "x is between y and z" (resp., "x is at least as near to y as it is to z"). (For easier reading, we position the variables to reflect the geometric relationships among points.) If X is a metric space and $a,b,c \in X$, we interpret $I(b,a,c)$ as $a \in I(b,c)$ and $R(a,b,c)$ as $a \in R(b,c)$ (with the obvious abuse of notation). In the sequel it will be useful to include the atomic formulas $E(x,y,z)$ and $M(y,x,z)$ as abbreviations for the compound formulas, $R(x,y,z) \wedge R(x,z,y)$ and $I(y,x,z) \wedge E(x,y,z)$, respectively.

Let us define an **IR-structure** to be a triple $\langle X,I,R\rangle$, where I and R are arbitrary ternary relations on X. An IR-structure is **metric** if its I- and R-relations arise from a metric as described in the Introduction.

[1] See Example 5.1 below for more exotic normed planes (often referred to as *Minkowski* planes).

Many of the important features of betweenness and comparative nearness in metric spaces may be expressed as universally-quantified first-order sentences/axioms in this predicate language; there are three tranches of such *IR-axioms* that we consider. The first−labeled I1−I9, and involving I and equality only−emphasizes the elementary betweenness aspect of metric IR-structures. (The second tranche involves just R and equality, while the third involves all the relation symbols.) We drop the universal quantifiers, in the interests of notational simplicity.

(I1, Inclusivity) $I(x,x,y) \wedge I(x,y,y)$
(I2, Symmetry) $I(y,x,z) \to I(z,x,y)$
(I3, Uniqueness) $I(y,x,y) \to x = y$
(I4, Antisymmetry) $(I(y,x,z) \wedge I(y,z,x)) \to x = z$
(I5, I-Transitivity) $(I(y,w,x) \wedge I(y,x,z)) \to I(y,w,z)$
(I6, Concentration) $(I(y,x,z) \wedge I(y,w,x) \wedge I(x,w,z)) \to w = x$

Axioms I1−I3 are referred to as *basic betweenness axioms* in [1, 2, 3]. Antisymmetry features prominently in [1, 4] and elsewhere, and, when conjoined with inclusivity (I1), formally implies uniqueness (I3): just substitute y for z in I4 and invoke I1. Axioms I1−I4 clearly hold for metric IR-structures; I-transitivity−we consider other notions of *transitivity* in the sequel−also holds, but takes a few lines to prove (see [15]). Concentration (I6) is introduced in [3] as *slenderness*, and is closely related to antisymmetry (I4) (see [3, Theorem 5.0.6]). We changed its name to suggest that the intersection of certain pairs of intervals sharing a bracket point is *concentrated* at the point. We also wished to avoid confusion with what we are informally calling *thinness*, as it applies to intervals (as well as walls in Section 4).

To see how I6 holds in a metric space $\langle X, \varrho \rangle$, suppose that $w, x, y, z \in X$ are given so that $\varrho(y,x)+\varrho(x,z) = \varrho(y,z)$, $\varrho(y,w)+\varrho(w,x) = \varrho(y,x)$, and $\varrho(x,w)+\varrho(w,z) = \varrho(x,z)$ are all true. Substituting the second two equations into the first, we obtain $\varrho(y,w) + 2\varrho(w,x) + \varrho(w,z) = \varrho(y,z)$. But w is metrically between y and z, by I-transitivity (I5). Hence $\varrho(w,x) = 0$; i.e., $w = x$.

In any IR-structure $\langle X, I, R \rangle$, metric or not, intervals and regions are defined informally in the obvious way (but of course may not behave as we expect).

- $I(a,b) := \{x \in X : I(a,x,b) \text{ holds}\}$; and
- $R(a,b) := \{x \in X : R(x,a,b) \text{ holds}\}$.

Then the basic betweenness axioms I1−I3 respectively translate to the general assertions $\{a,b\} \subseteq I(a,b)$, $I(a,b) = I(b,a)$, and $I(a,a) = \{a\}$; antisymmetry (I4) says precisely that if $a,b,c \in X$ and both $c \in I(a,b)$ and $b \in I(a,c)$ hold, then $b = c$. As an immediate consequence of I1−I4 holding in an IR-structure, two distinct bracket sets for any interval in that structure must be disjoint.

I-transitivity (I5) is the statement that $I(a,c) \subseteq I(a,b)$ holds whenever $c \in I(a,b)$. And if we define a subset $S \subseteq X$ to be **star-shaped about** $a \in S$ if $I(a,b) \subseteq S$ for all $b \in S$, then I5 also is the assertion that intervals are star-shaped about each of their bracket points. Concentration (I6) amounts to saying that if $c \in I(a,b)$, then $I(a,c) \cap I(c,b) = \{c\}$. In any IR-structure satisfying I1−I3 and I-transitivity (I5), antisymmetry (I4) automatically follows from concentration (I6): For suppose $c \in I(a,b)$ and $b \in I(a,c)$ are both true. Then, by I-transitivity (I5), we have $I(a,b) = I(a,c)$. By concentration (I6) and symmetry (I2), we then have $\{c\} = I(a,c) \cap I(c,b) = I(a,b) \cap I(b,c) = \{b\}$.

If we paraphrase the ternary relation I as an "indexed family" of binary relations by writing $x \leq_y z$ for $I(y,x,z)$−so the first free variable is the "index"−then I4 looks like usual antisymmetry,

$$(x \leq_y z \wedge z \leq_y x) \to x = z,$$

and I5 looks like usual transitivity,

$$(w \leq_y x \wedge x \leq_y z) \to w \leq_y z.$$

Moreover, if $\langle X, I, R\rangle$ satisfies I1−I5, then each binary relation $\leq_a$, $a \in X$, is a partial ordering on X, with unique least element a. If $a, b \in X$, we let $\leq_{ab}$ denote the partial ordering $\leq_a$ restricted to $I(a, b)$. Then b is the unique $\leq_{ab}$-greatest element of $I(a, b)$, and $\leq_{ba}$ is the order-reversal of $\leq_{ab}$.

The next axiom, intuitively appealing and a natural strengthening of I-transitivity, is one of several *convexity* notions we consider, and holds only for some metric IR-structures.[2]

(I7, I-Convexity) $(I(u, w, v) \wedge I(x, u, y) \wedge I(x, v, y)) \to I(x, w, y)$

I-convexity (I7) says that $I(c, d) \subseteq I(a, b)$ for all $c, d \in I(a, b)$. If we define a subset $S \subseteq X$ to be **convex** if it is star-shaped about each of its points, we see I7 as the assertion that each interval is convex. In the context of vector spaces, we will use the terms *linearly star-shaped/convex* when linear intervals take the place of the−often much larger−metric ones. We investigate in later sections conditions under which I7 holds or fails in the metric−especially the normed vector space−context. (See, e.g., Corollary 2.12 and Theorem 5.14.)

The last two betweenness axioms are closely related to each other and also hold only for some metric IR-structures.

(I8, Weak Disjunctivity) $(I(x, u, y) \wedge I(x, v, y)) \to (I(x, u, v) \vee I(v, u, y))$
(I9, Strong Disjunctivity) $I(x, u, y) \to (I(x, u, v) \vee I(v, u, y))$

Weak disjunctivity says that every interval $I(a, b)$ is contained in the union $I(a, c) \cup I(c, b)$ whenever $c \in I(a, b)$; strong disjunctivity says the same, only without restriction on c. Suppose an IR-structure satisfies I8, as well as I1−I5. Then $I(a, b)$ actually equals $I(a, c) \cup I(c, b)$ for $c \in I(a, b)$. Concentration (I6) then holds too. For if $c \in I(a, b)$ and $d \in I(a, c) \cap I(c, b)$, then−by I8−either $c \in I(a, d)$ or $c \in I(d, b)$. In either case we infer $c = d$, by antisymmetry (I4). (See also [3, Theorem 5.0.6]. We do not know whether I6 formally follows from I1−I5.) Furthermore, in this situation each partial ordering $\leq_a$ is a tree order (i.e., the set of predecessors of each element is a total ordering) and each $\leq_{ab}$ is a total ordering [3, Propositions 5.0.4, 5.0.5]. Let us call a pair $\langle a, b\rangle$ of points **weakly disjunctive** if $I(a, b) \subseteq I(a, c) \cup I(c, b)$ for each $c \in I(a, b)$. Note that, in any IR-structure satisfying symmetry (I2), weak disjuctivity for $\langle a, b\rangle$ implies that for $\langle b, a\rangle$.

The next proposition demonstrates the simple fact that weak disjunctivity, like linearity, is a property of intervals, not just their bracket sets. As such it joins linearity as a way of saying intervals satisfy the informal notion of *thinness*.

2.1. **Proposition.** *If $\langle a, b\rangle$ is a weakly disjunctive pair of points in an IR-structure $\langle X, I, R\rangle$ satisfying I1−I4, and $\{c, d\}$ is a bracket set for $I(a, b)$, then $\{c, d\} = \{a, b\}$. Hence $\langle c, d\rangle$ is a weakly disjunctive pair as well.*

Proof. We have $c, d \in I(a, b)$; so since $\langle a, b\rangle$ is a weakly disjunctive pair, we also have $d \in I(a, c) \cup I(c, b)$. Assume $d \in I(c, b)$. Since $I(c, d) = I(a, b)$, we know $b \in I(c, d)$. By antisymmetry (I4), we infer $d = b$. But now $a \in I(c, b)$ and

[2]See [15] for a finite example, and [2] for some infinite ones, where the axiom fails.

$c \in I(a,b)$, so another application of I4 gives us $c = a$. In the event $d \in I(a,c)$, we obtain $d = a$ and $c = b$ using the same argument. □

Later on we show (see Corollary 5.6 and Proposition 5.7) that for normed vector spaces, weak disjunctivity (I8) holds precisely when the norm is *strictly convex* (commonly defined by the *rotundity* condition that whenever $a \neq 0 \neq b$ and $\|a + b\| = \|a\| + \|b\|$, then $a = tb$ for some $t > 0$), and strong disjunctivity (I9) holds precisely when the dimension is ≤ 1. [3]

2.2. **Proposition.** *In an IR-structure satisfying I1−I3 and I5, every weakly disjunctive interval is convex.*

Proof. Let a, b be two points of IR-structure $\langle X, I, R\rangle$, such that $I(a,b)$ is weakly disjunctive. Suppose $c, d \in I(a,b)$, with $e \in I(c,d)$. By weak disjunctivity, we have $d \in I(a,c)$ or $d \in I(c,b)$. In the first case, one application of I-transitivity (I5)−in the presence of symmetry (I2)−gives us $e \in I(a,c)$. A second application of I5 gives us $e \in I(a,b)$. The second case is handled exactly the same way; hence $I(a,b)$ is convex. □

The second tranche of IR-axioms−labeled R1−R5, and involving R and equality only−emphasizes the elementary comparative nearness aspect of metric IR-structures.

(R1, Self-nearness) $R(x,x,y)$
(R2, Degeneracy) $R(x,y,y)$
(R3, Positive-definiteness) $R(x,y,x) \to x = y$
(R4, Dichotomy) $R(x,y,z) \vee R(x,z,y)$
(R5, R-Transitivity) $(R(w,x,y) \wedge R(w,y,z))) \to R(w,x,z)$

If $\langle X, I, R\rangle$ is an IR-structure, R1 says that each nearness region $R(a,b)$ contains its center a, while R3 adds that $R(a,b)$ does not contain its off-center b when $b \neq a$. In the degenerate case where $a = b$, R2 says that $R(a,b) = X$. Axiom R4 tells us that each doubleton family $\{R(a,b), R(b,a)\}$ covers X, while R5 asserts that $R(a,b) \cap R(b,c)$ is always contained in $R(a,c)$. The reader may easily verify that axioms R1−R5 hold for all metric IR-structures.

If, as above with the ternary relation I, we paraphrase R by writing $y \preceq_x z$ for $R(x,y,z)$−where the first free variable is still the "index"−then R4 looks like usual dichotomy/totality,

$$y \preceq_x z \vee z \preceq_x y,$$

and R5 looks like usual transitivity,

$$(x \preceq_w y \wedge y \preceq_w z) \to x \preceq_w z.$$

If $\langle X, I, R\rangle$ satisfies R1−R5, then each binary relation $\preceq_a$, $a \in X$, is a total pre-ordering on X with unique least element a (antisymmetry being the only condition lacking). And if $\langle X, I, R\rangle$ also satisfies I1−I5, $a, b \in X$, and we let $\preceq_{ab}$ denote the pre-ordering $\preceq_a$ restricted to $I(a,b)$, then b is the unique $\preceq_{ab}$-greatest element of $I(a,b)$, and $\preceq_{ba}$ is the order-reversal of $\preceq_{ab}$. For $x, y \in I(a,b)$, define $x \sim_{ab} y$ if both $x \preceq_{ab} y$ and $y \preceq_{ab} x$ hold. Clearly $\sim_{ab} = \sim_{ba}$ is an equivalence relation on $I(a,b)$, and $\preceq_{ab}$ is a total ordering precisely when $\sim_{ab}$ is trivial.

2.3. **Remarks.**

[3] See [1, 2, 3, 4] for topological interpretations of betweenness in which strong disjunctivity is more the rule than the exception.

(i) The "R-version" of I4 that most retains the spirit of binary antisymmetry is the statement

$$(y \preceq_x z \wedge z \preceq_x y) \rightarrow y = z,$$

which amounts to saying that any equiset with two distinct cocenters is empty. (For example, the metric spaces described in Examples 2.5 (iv) and 3.2 (ii) below have this property.) In the present paper almost all metric spaces of interest are connected, in which case equisets are never empty (see Proposition 3.3 below).

(ii) The "I-version" of R4 that most retains the spirit of binary dichotomy is called *totality* in [4], and is the statement

$$y \leq_x z \vee z \leq_x y.$$

It is easy to check that a metric IR-structure satisfies this axiom if and only if its underlying set has at most two points, and so totality is of little interest here. On the other hand, with the continuum-theoretic interpretation of betweenness, where a *continuum* is a connected compact Hausdorff space and $I(y, x, z)$ holds precisely when x lies in every subcontinuum containing $\{y, z\}$, totality holds if and only if the continuum is *hereditarily indecomposable*; i.e., where any two subcontinua are either comparable or disjoint [4, Proposition 5.6].

The final tranche of IR-axioms, labeled IR1−IR5, IE1, IE2, and IM1, emphasizes how betweenness and comparative nearness interact in metric IR-structures.

(IR1, Weak Obstruction) $I(y, x, z) \rightarrow R(y, x, z)$
(IR2, Strong Obstruction) $(I(y, x, z) \wedge R(y, z, x)) \rightarrow x = z$
(IR3, IR-Transitivity) $(I(x, w, y) \wedge R(x, y, z)) \rightarrow R(w, y, z)$
(IR4, Complementarity) $(I(x, u, y) \wedge I(x, v, y) \wedge R(x, u, v)) \rightarrow R(y, v, u)$

Weak obstruction says of an IR-structure $\langle X, I, R\rangle$ that if $a \in I(b, c)$ (i.e., $a \leq_b c$), then $b \in R(a, c)$ (i.e., $a \preceq_b c$). Hence $\leq_a$ refines $\preceq_a$, for each $a \in X$. (It is also worth noting that if $a \in I(b, c)$ and symmetry (I2) holds in the IR-structure, then $c \in R(a, b)$ as well. It is easy to find metric counterexamples to the general assertion $R(y, x, z) \rightarrow R(z, x, y)$.) And if $a, b \in X$ are distinct, strong obstruction implies further that $b \notin R(c, a)$ ("a obstructs b from being nearer to c than to a"). Strong obstruction is easily shown to hold in metric IR-structures. It is a kind of "mixed antisymmetry" because, in indexed order terms, it becomes

$$(x \leq_y z \wedge z \preceq_y x) \rightarrow x = z,$$

involving two distinct order symbols.

It is easy to show—with the aid of R2 and R4—that IR1 formally follows from IR2. One then readily checks that both positive-definiteness (R3) and antisymmetry (I4) follow as well.

When expressed in indexed order terms, IR-transitivity becomes

$$(w \leq_x y \wedge y \preceq_x z) \rightarrow y \preceq_w z.$$

This is "mixed transitivity with a twist" because, in addition to two distinct order symbols, there are two distinct index variables. More revealingly, IR-transitivity says that each nearness region is star-shaped about its center.

2.4. **Proposition.** *Every metric IR-structure satisfies IR-transitivity.*

Proof. Let $\langle X, \varrho \rangle$ be a metric space, with $w, x, y, z \in X$ such that $w \in I(x, y)$ and $x \in R(y, z)$ both hold; i.e., we have $\varrho(w, x) + \varrho(w, y) = \varrho(x, y)$ and $\varrho(x, y) \leq \varrho(x, z)$. We wish to show that $w \in R(y, z)$ holds; so suppose otherwise. Then $\varrho(w, z) < \varrho(w, y)$; thus $\varrho(w, x) + \varrho(w, z) < \varrho(w, x) + \varrho(w, y) = \varrho(x, y) \leq \varrho(x, z)$, contradicting the triangle inequality for metrics. □

Complementarity (IR4) asserts that if c, d are both in $I(a, b)$ and a is at least as near to c as it is to d, then b is at least as near to d as it is to c. This axiom clearly holds in any metric IR-structure; and, in indexed order form, may be restated as

$$(u \leq_x y \wedge v \leq_x y) \rightarrow (u \preceq_x v \rightarrow v \preceq_y u).$$

I-convexity (I7) says that intervals are convex, and does not hold for all metric IR-structures. Recalling that $E(x, y, z)$ abbreviates $R(x, y, z) \wedge R(x, z, y)$ and $M(y, x, z)$ abbreviates $I(y, x, z) \wedge E(x, y, z)$, the following three axioms assert the convexity of nearness regions, equisets, and midsets, respectively. None of these axioms hold for all metric IR-structures, even when the metric arises from a vector space norm (see Examples 2.5 below).

(IR5, R-Convexity) $(I(u, w, v) \wedge R(u, x, y) \wedge R(v, x, y)) \rightarrow R(w, x, y)$
(IE1, E-Convexity) $(I(u, w, v) \wedge E(u, x, y) \wedge E(v, x, y)) \rightarrow E(w, x, y)$
(IM1, M-Convexity) $(I(u, w, v) \wedge M(x, u, y) \wedge M(x, v, y)) \rightarrow M(x, w, y)$

In the sequel, we will often say that a metric space (or IR-structure) *is I-convex* as shorthand for saying it *satisfies the I-convexity axiom*, etc.

By dint of the definitions of the predicates E and M, E-convexity logically follows from R-convexity, and M-convexity follows from the conjunction of I-convexity and E-convexity. We strongly suspect that M-convexity does not follow from either R-convexity or E-convexity alone, but have no examples to show this. The following shows that none of the other simple implications hold in general for metric IR-structures.

2.5. **Examples.**

(i) I-convexity does not imply any of the three other convexities. We have, by Theorem 5.14 below, the fact that all normed planes are I-convex. (This gives an affirmative answer to [2, Question 4.7]. In [2, Example 4.6] it is shown that a normed vector space of dimension three can fail to be I-convex.) In Example 5.1 below, however, we show that all three of the other convexities can indeed fail in dimension two.

(ii) Neither R-convexity nor E-convexity is a consequence of M-convexity. Any normed vector space that is strictly convex but not an inner product space is M-convex, because midsets are all singletons, but it is not E-convex (or R-convex). (See Example 5.1, Corollary 5.2, and Remark 5.3 below.)

(iii) I-convexity is not a consequence of any of the three other convexities. The following finite example was used in [15] to demonstrate the failure of I-convexity; but−after a fair amount of tedious checking−we found that it satisfies all three other convexities. Let $X = \{a, b, c, d, e\}$, and define the metric as follows: $\varrho(a, c) = \varrho(b, d) = 2$; $\varrho(c, e) = \varrho(d, e) = 3$; $\varrho(b, e) = \varrho(a, e) = 5$; $\varrho(a, d) = \varrho(b, c) = \varrho(c, d) = 6$; and $\varrho(a, b) = 8$. (This amounts to a labeled complete undirected graph on five vertices.) Then $I(a, b) = \{a, b, c, d\}$, while $I(c, d) = \{c, d, e\} \not\subseteq I(a, b)$. Hence I-convexity (I7) fails. Checking that R-convexity holds is just a matter of trying all the possible cases. For example, $R(d,a) = \{b,d,e\}$, $I(b,d) = \{b,d\} \subseteq R(d,a)$, $I(b,e) = \{b,d,e\} \subseteq R(d,a)$, and $I(d,e) = \{d,e\} \subseteq R(d,a)$. As for M-convexity, one easily checks that each midset is empty, with the exception of $M(c,d) = \{e\}$.

(iv) A variation on the last example shows that R-convexity does not follow from E-convexity. Let X be the same five-point set, but reassign the metric values as follows: $\varrho(a,b) = 18$; $\varrho(a,c) = 13$; $\varrho(a,d) = 15$; $\varrho(a,e) = 11$; $\varrho(b,c) = 14$; $\varrho(b,d) = 8$; $\varrho(b,e) = 12$; $\varrho(c,d) = 10$; $\varrho(c,e) = 19$; and $\varrho(d,e) = 9$. (Of course, one checks the triangle inequality for each of the ten labeled triangles in this complete graph; hence the edge assignment gives a metric.) Since all ten nonzero distances are distinct, all equisets are empty; hence E-convexity vacuously holds. Finally we have $c, e \in R(a,b)$, $d \in I(c,e)$, but $d \in R(b,a) \setminus R(a,b)$; hence R-convexity does not hold.

The final axiom we consider is somewhat of a companion to linearity and weak disjunctivity (I8), in that it provides a third way of saying that intervals are *thin.*

(IE2, Narrowness) $(I(x,u,y) \wedge I(x,v,y) \wedge E(x,u,v)) \rightarrow u = v$

Informally, narrowness says that if two points of $I(a,b)$ are equidistant from a, then the points coincide. In the presence of axioms I1$-$I5, R1$-$R5, IE2 also says that, for each pair $\langle a,b\rangle$ of points, the binary relations $\preceq_{ab}$ is a total ordering on $I(a,b)$; i.e., that $\sim_{ab}$ is a trivial equivalence relation. The failure of syntactic symmetry in IE2 is remedied by noticing that: (i) it is a universally-quantified statement; and hence (ii) in the presence of symmetry (I2), the subformula $E(x,u,v)$ may be replaced with $E(x,u,v) \vee E(y,u,v)$.

2.6. **Remark.** Because each of the axioms considered above is a universal first-order sentence, the class of IR-structures satisfying any given set Σ of them is closed under ultraproducts and substructures. Being closed under ultraproducts means that we have a source of models of Σ that are not derived from metrics. As regards substructures: if $\langle Y, \varrho\rangle$ is a metric space and $X \subseteq Y$ is any subset, then the restriction of ϱ to X yields the same IR-structure on X as does the restriction to X of the metric-induced IR-structure on Y. Also a subset $S \subseteq X$ is star-shaped/convex in the substructure if and only if $S = T \cap X$ for some $T \subseteq Y$ which is star-shaped/convex in Y. (Just let T be the union of all Y-intervals with bracket sets contained in S.)

With the exception of I8, I9, and R4, all the axioms considered here are Horn sentences (see [7]). Hence any reduced product of IR-structures satisfying, say, R-convexity will still satisfy that axiom. However the direct product of two IR-structures satisfying dichotomy (R4), for example, will almost never satisfy that axiom.

Let $\mathsf{M}_\forall$ be the *universal theory of metric IR-structures*; i.e., the collection of universal IR-sentences that hold for all metric IR-structures. An IR-structure that models the axioms of $\mathsf{M}_\forall$ is called **metric-like**. We emphasize that its designated relations need not be induced by a metric on its underlyiing set; they just look like they do, from a first-order perspective. $\mathsf{M}_\forall$ includes I1$-$I6, R1$-$R5, and IR1$-$IR4 above (plus infinitely many others), but excludes all the convexity axioms.

Like weak disjunctivity (I8), narrowness (IE2) may be instantiated to individual pairs of points in an IR-structure $\langle X, I, R\rangle$. A pair $\langle a,b\rangle \in X^2$ is **narrow** if whenever $c, d \in I(a,b)$ are such that $a \in E(c,d)$, then $c = d$. Note that in any

IR-structure satisfying complementarity (IR4), narrowness for $\langle a, b\rangle$ implies that for $\langle b, a\rangle$.

2.7. **Theorem.** *Every weakly disjunctive pair of points in a metric-like IR-structure is narrow. Moreover, if E-convexity (IE1) holds in the structure, then so does narrowness (IE2).*

Proof. Let a, b be two points of the metric-like IR-structure $\langle X, I, R\rangle$, such that $\langle a, b\rangle$ is weakly disjunctive. Suppose $c, d \in I(a, b)$ are such that $E(a, c, d)$ holds. We need to show that $c = d$. By weak disjunctivity, we have $d \in I(a, c)$ or $d \in I(c, b)$. In the first instance, we have both $I(a, d, c)$ and $R(a, c, d)$; hence $c = d$ by strong obstruction (IR2). If $d \in I(c, b)$, then we have $I(b, d, c)$. Since $R(a, d, c)$ holds, we also have $R(b, c, d)$, by complementarity (IR4). Another application of IR2 gives us $c = d$. This shows that $\langle a, b\rangle$ is narrow.

Let $a, b, c, d \in X$ be such that $c, d \in I(a, b)$ and that $E(a, c, d)$ holds. By complementarity (IR4), $E(b, c, d)$ holds too; so a and b both belong to $E(c, d)$. If E-convexity holds, then we also have $c \in E(c, d)$. By self-nearness (R1) and positive-definiteness (R3), we infer that $c = d$. This shows that narrowness (IE2) holds. □

Narrowness does not imply weak disjunctivity; nor is it independent of choice of bracket set for an interval, as the following metric example shows.

2.8. **Example.** Let X be the five-element subset $\{a, b, c, d, e\}$ of $\mathbb{R}^2$, where $a = \langle 0, 0\rangle$, $b = \langle 3, 1\rangle$, $c = \langle 0, 1\rangle$, $d = \langle 3, 0\rangle$, and $e = \langle 1, 1\rangle$. For our metric, define $\varrho(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) := |x_1 - x_2| + |y_1 - y_2|$. This is the well-known *taxicab metric* [4] inherited from $\mathbb{R}^2$. Then $I(a, b) = X$, and no two distinct points of $I(a, b)$ are the same distance from a; so $\langle a, b\rangle$ is clearly narrow. On the other hand, $d \in I(a, b) \setminus (I(a, c) \cup I(c, b))$; hence $\langle a, b\rangle$ is not weakly disjunctive. Also we have $I(c, d) = I(a, b)$, with $a \sim_{cd} e$; hence $\langle c, d\rangle$ is not narrow.

In Theorem 5.5 (iii) below we show that for pairs of points in normed vector spaces, weak disjunctivity and narrowness are both equivalent to linearity. Leaving out linearity, weak disjunctivity and narrowness are in fact equivalent to each other for a much broader class of metric spaces, as we now show. First we take the usual definition of *arc* as being any topological space homeomorphic to a closed bounded interval in $\mathbb{R}$; equivalently it is a metrizable continuum with exactly two noncut points (i.e., points that do not separate the space). We will refer to the cut points of an arc as *interior* points; the noncut points will be referred to as the *end* points of−or points *joined by*−the arc.

An arc A contained in a metric space X and joining points a and b is a **metric segment** if A is isometric to a real closed bounded interval. (Such an isometry, being a homeomorphism, takes end points to end points, etc.) Clearly line segments in normed vector spaces are metric segments by this definition. The arc A is **taut** if it is contained in $I(a, b)$; metric segments are clearly taut arcs.

Define a metric space X to be **arc-connected** (resp., **segment-connected**) if for each two distinct points $a, b \in X$ there is an arc (resp., a metric segment) $A \subseteq X$ joining a and b. The space is **tautly arc-connected** if each two of its points may be joined by a taut arc. Segment-connected metric spaces are tautly arc-connected.

[4]This is also known as the *Manhattan*, or *rook's*, metric.

Arcs are the prototypical connected topological spaces; thereby making arc-connectedness a strong form of connectedness. Unlike arc-connectedness, taut arc-connectedness involves the metric in an essential way, and one sees readily that all open balls in a tautly arc-connected metric space are arc-connected. Hence taut arc-connectedness is a strong form of local connectedness (as well as connectedness).

2.9. **Remark.** If A is an arc and $x, y \in A$ are distinct, let $A[x, y]$ denote the unique subarc of A joining x and y. The tautness of A does not imply that of $A[x, y]$: in $\mathbb{R}^2$ with the taxicab metric and $a = \langle 0, 0\rangle$, $b = \langle 2, 2\rangle$, any arc A joining a and b is taut if it is contained in the square $I(a, b) = [0, 2]^2$. So if A contains, say, $c = \langle 0, 1\rangle$ and $d = \langle 2, 1\rangle$, then $I(c, d) = [\![c, d]\!]$; so $A[c, d]$ is not taut unless it equals $[\![c, d]\!]$. Call an arc A *hereditarily taut* if each $A[x, y]$ is taut. Metric segments are hereditarily taut, and may be parameterized as geodesics (see [2]). Note that, by a classical Koch "snowflake" construction in the plane with the taxicab metric, we may obtain taut arcs, no subarc of which is a metric segment.

It is well known [15] that any complete metric space is segment-connected if (and only if) each of its midsets is nonempty. Hence, for complete metric spaces, segment-connectedness and taut arc-connectedness are equivalent properties. All normed vector spaces are segment-connected, though, even the ones whose metrics are not complete (e.g., the space c_{00} of all eventually-zero real sequences, equipped with the supremum norm).

2.10. **Proposition.** *In a tautly arc-connected (resp., segment-connected) metric space, each metric interval $I(a, b)$ is the union of all taut arcs (resp., metric segments) joining a and b.*

Proof. By definition, any taut arc joining a and b lies in $I(a, b)$. For the reverse inclusion, let $I(a, b)$ be a nondegenerate interval, with $c \in I(a, b)$ arbitrary; we may safely assume $c \notin \{a, b\}$. Assuming taut arc-connectedness, we find an arc $A \subseteq I(a, c)$ joining a and c; likewise we find an arc $B \subseteq I(c, b)$ joining c and b. By I-transitivity (I5), $I(a, c) \cup I(c, b) \subseteq I(a, b)$, and by concentration (I6), we know $I(a, c) \cap I(c, b) = \{c\}$. Thus $A \cup B \subseteq I(a, b)$, being the union of two arcs sharing one end point and disjoint otherwise, is a taut arc joining a and b, and containing c.

If X is segment-connected, we mimic the argument above, replacing *arc* with *metric segment* throughout. Then, because the shared end point c is in $I(a, b)$, we conclude that $A \cup B$ is a metric segment joining a and b. □

2.11. **Theorem.** *Let $\langle X, \varrho\rangle$ be a segment-connected metric space. Then a pair of points in X is weakly disjunctive if and only if it is narrow.*

Proof. By Theorem 2.7, all weakly disjunctive pairs of points are narrow, so assume $\langle a, b\rangle$ is not weakly disjunctive. We pick $c, d \in I(a, b)$ such that $d \notin I(a, c) \cup I(c, b)$. If $\varrho(a, c) = \varrho(a, d)$, then c and d witness that $\langle a, b\rangle$ is not narrow. Assume $\varrho(a, c) > \varrho(a, d)$ and use segment-connectedness to fix a metric segment $A \subseteq I(a, c)$ joining a and c. Then there is an isometry φ between a real interval $[0, r]$ and A, such that $\varphi(0) = a$ and $\varphi(r) = c$. Clearly, since φ is an isometry, we have $r = \varrho(a, c)$. By assumption, then, we have $t = \varrho(a, d) \in [0, r]$. So let $e = \varphi(t)$. Then $e \in I(a, c) \subseteq I(a, b)$ (thanks again to I-transitivity (I5)), $e \neq d$ because $d \notin I(a, c)$, and $\varrho(a, e) = \varrho(a, d)$ because φ is an isometry. Thus we have a witness to the failure of $\langle a, b\rangle$ being narrow. Finally, we assume $\varrho(a, c) < \varrho(a, d)$. Then $\varrho(c, b) > \varrho(d, b)$. Thus we may find $e' \in I(c, b) \subseteq I(a, b)$ with $\varrho(e', b) = \varrho(d, b)$. This time we have e' and d witnessing that $\langle a, b\rangle$ is not narrow. □

The next assertion is a consequence of Theorems 2.7 and 2.11, Proposition 2.2, and the fact that M-convexity immediately follows from the conjunction of E-convexity and I-convexity.

2.12. **Corollary.** *In a segment-connected metric space, E-convexity (IE1) implies weak disjunctivity (I8), and hence both I-convexity (I7) and M-convexity (IM1).*

In light of Remark 2.6, we may add a little to Theorem 2.11 and Corollary 2.12.

2.13. **Corollary.** *Let X be a metric space that isometrically embeds in a segment-connected metric space Y.*

(i) *If a pair of points of X is narrow as a pair in Y, then it is weakly disjunctive as a pair in X.*
(ii) *If Y is E-convex, then X is both weakly disjunctive (i.e., satisfies weak disjunctivity (I8)) and E-convex. Hence X is both I-convex and M-convex.*

2.14. **Example.** Let X be the *harmonic comb* in the Euclidean plane $Y = \mathbb{R}^2_2$, namely

$$X = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup \bigcup_{n=1}^{\infty} (\{1/n\} \times [0, 1]).$$

Then X is arc-connected, but−failing to be locally connected−it is not tautly arc-connected. (Indeed, no compatible metric on X can result in a tautly arc-connected space.) But Y is E-convex; so, by Corollary 2.13, X is E-convex (as well as weakly disjunctive, I-convex, and M-convex).

2.15. **Remark.** The ideas surrounding betweenness and comparative nearness may be naturally extended beyond the metric realm, but at a price: for example, we could replace metrics with *pseudometrics*, thereby allowing two distinct points to be zero distance from each other. A pseudometric on a set induces an IR-structure in the same way a metric does; and it is easy to see that any pseudometric IR-structure satisfies all the metric-like axioms above, except for those that involve equality (i.e., I3, I4, I6, R3, IR2). In particular, weak obstruction (IR1) still holds, even though strong obstruction (IR2) may not.

3. Topological IR-structures

An IR-structure $\langle X, I, R\rangle$ is **topological** if it is metric-like and there is an underlying topology on X such that all intervals and nearness regions are closed subsets. (This topology is not assumed to coincide with the open sets of a *possible* metric inducing the metric-like structure; but if a metric is specified at the outset, the topology is assumed to be induced by that metric.)

3.1. **Proposition.** *Every metric IR-structure $\langle X, I, R\rangle$ is topological, relative to the metric topology; indeed, both I and R are closed subsets of the product space X^3. Finally, metric intervals are bounded as well as closed: if $a, b \in X$ then $\varrho(x, y) \leq \varrho(a, b)$ for all $x, y \in I(a, b)$.*

Proof. Let $\langle X, \varrho\rangle$ be a metric space. Then, by the remarks in Section 2, $\langle X, I, R\rangle$ is a metric-like IR-structure. To show I and R are closed in X^3, suppose $a, b, c \in X$, with sequences $\langle a_n\rangle \to a$, $\langle b_n\rangle \to b$, $\langle c_n\rangle \to c$, such that $I(b_n, a_n, c_n)$ holds for each $n = 1, 2, \dots$. Then, for each n, we have $\varrho(b_n, a_n) + \varrho(a_n, c_n) = \varrho(b_n, c_n)$. By continuity of the metric function, we infer that $\varrho(b, a) + \varrho(a, c) = \varrho(b, c)$. Hence $I(b, a, c)$ holds. The argument for the closedness of R follows similar lines, and from this we clearly have all intervals and nearness regions closed in X.

If $x, y \in I(a, b)$, then $\varrho(a, x) + \varrho(x, b) = \varrho(a, b) = \varrho(a, y) + \varrho(y, b)$. We also have $\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y)$ and $\varrho(x, y) \leq \varrho(x, b) + \varrho(y, b)$. Adding these two inequalities, we see $2\varrho(x, y) \leq 2\varrho(a, b)$, showing the diameter of $I(a, b)$ to be $\varrho(a, b)$. □

Henceforth metric IR-structures are assumed to be topological IR-structures, where the underlying topology is that induced by the metric (as indicated above). Note that in any IR-structure, all intervals and nearness regions are guaranteed to be nonempty. However, the same cannot be said for equisets and midsets without adding some further conditions on the structure. The following example looks at three geometrically−but not topologically−distinct versions of Cantor space.

3.2. **Examples.**

(i) Consider the Cantor space $C_3 \subseteq \mathbb{R}$ resulting from the "middle thirds" deletion of intervals from $[0, 1]$. The metric is the one inherited from $\mathbb{R}$, and hence $E(a, b) = M(a, b)$ always. Midsets/equisets have at most one element; for example $M(0, \frac{2}{3}) = \{\frac{1}{3}\}$, but $M(0, 1) = \emptyset$.

(ii) Next consider the Cantor space $C_5 \subseteq \mathbb{R}$, similar to C_3 above except that we delete the "middle three-fifths" from intervals at each iteration. In this case midsets/equisets of distinct points are always empty. (See [14, Example 1.10].)

(iii) Finally consider the Cantor space 2^ω consisting of all $\{0, 1\}$-valued sequences $a = \langle a(0), a(1), \dots\rangle$, with the metric $\varrho(a, b) := 2^{-n}$, where $n = \min\{k : a(k) \neq b(k)\}$ for distinct $a, b \in 2^\omega$. Then plainly $M(a, b) = \emptyset$ for any $a \neq b$. Also if $a(0) \neq b(0)$ then $E(a, b) = \emptyset$ as well. However, if $a(0) = b(0)$, then any $c \in 2^\omega$ with $c(0) \neq a(0)$ is of unit distance from both a and b; hence $E(a, b)$ is infinite.

Of course Cantor spaces are anything but connected. They're *totally disconnected* in fact: all nondegenerate subsets are disconnected. In the sequel our focus is on connected spaces.

3.3. **Proposition.** *Let $\langle X, I, R\rangle$ be a topological IR-structure.*

(i) *If the underlying space is connected, then all equisets are nonempty.*

(ii) *If $a, b \in X$ and $I(a, b)$ is connected, then the midset $M(a, b)$ is nonempty.*

Proof. Ad (i): Note that, by degeneracy (R2), $E(a, a) = X$, nonempty by tacit assumption. So, assuming $a, b \in X$ are distinct, we use dichotomy (R4) to infer that $E(a, b)$ separates X, with $R(a, b) \setminus E(a, b)$ and $R(b, a) \setminus E(a, b)$ constituting a disconnection of $X \setminus E(a, b)$. Since X is connected, each $E(a, b)$ must be nonempty.

Ad (ii): Each interval $I(a, b)$ is a topological IR-structure in the obvious sense. Thus we may relativize the proof of (i) to $I(a, b)$. □

Helped by the argument above, we obtain the following.

3.4. **Theorem.** *In a segment-connected metric IR-structure, E-convexity (IE1) implies R-convexity (IR5), I-convexity (I7), and M-convexity (IM1).*

Proof. Let $\langle X, I, R\rangle$ be a segment-connected metric space satisfying E-convexity. In light of Corollary 2.12, all we need to prove is R-convexity. For the sake of contradiction, suppose X does not satisfy R-convexity. Then we have five points $a, b \in X$, $x, y \in R(a, b)$, and $z \in I(x, y) \setminus R(a, b)$ to witness the fact. Since $z \notin E(a, b)$, E-convexity implies that x and y cannot both be in $E(a, b)$. Without loss of generality, assume we have $x \in R(a, b) \setminus R(b, a)$. Because X is also tautly arc-connected, there is an arc $A \subseteq I(x, z)$ joining x and z. Using the argument in the proof of Proposition 3.3 (i)−and because arcs are connected−we know there exists $u \in I(x, z) \cap E(a, b)$. Because of I-transitivity (I5), $u \in I(x, y)$; and by Corollary 2.12, X satisfies weak disjunctivity (I8). Hence $I(x, y) = I(x, u) \cup I(u, y)$. By our assumption, plus dichotomy (R4), we know $z \in R(b, a) \setminus R(a, b)$; so $z \neq u$. By antisymmetry (I4), $z \notin I(x, u)$; hence $z \in I(u, y)$. Since $z \notin E(a, b)$, E-convexity tells us that $y \notin E(a, b)$ either. Thus $y \in R(a, b) \setminus R(b, a)$. Again using taut arc-connectedness, there is an arc $B \subseteq I(z, y)$ joining z and y; and we argue as above to find $v \in I(z, y) \cap E(a, b)$. Using weak disjunctivity (plus antisymmetry) we infer that $z \in I(u, v)$, running afoul of E-convexity. We therefore conclude that $I(x, y) \subseteq R(a, b)$, showing $R(a, b)$ to be convex. □

3.5. **Question.** Theorem 3.4 draws heavily upon the metric-dependent argument of Theorem 2.11. Does it still hold for any tautly arc-connected topological IR-structure?

We may use Remark 2.6 once again, and extract a little more from Theorem 3.4.

3.6. **Corollary.** *Let X be a metric space that isometrically embeds in a segment-connected metric space that is E-convex. Then X is R-convex.*

From this result, we may infer that any metric subspace of a Euclidean space (e.g., the harmonic comb from Example 2.14) is R-convex.

4. Divisions and walls

As mentioned above, we take the view that the sets $E(a, b)$ and $M(a, b)$ are "walls separating a from b." This brings us to the general notions of *division* and *wall.*

For any topological space X, a **division** of X is a cover $\mathcal{D} = \{A, B\}$ of X by two proper closed subsets; these sets are referred to as the **halves** of the division. The intersection $A \cap B$ of the two halves is the **wall** of the division, and−see the proof of Proposition 3.3 (i)−the doubleton $\{X \setminus A, X \setminus B\}$ is a disconnection of $X \setminus (A \cap B)$. Hence all divisions of connected spaces have nonempty walls. A division $\mathcal{D}$ **separates** two distinct points a, b if neither half contains both points; i.e., if the wall of the division separates a from b in the sense discussed in the Introduction. The half of the division containing a is denoted $\mathcal{D}(a)$. Points of $\mathcal{D}(a)$ may be thought of as being "nearer to a than to any point $b \in X \setminus \mathcal{D}(a)$," so we may regard $\mathcal{D}(a)$ as the nearness region $R(a, b)$, when $b \in X \setminus \mathcal{D}(a)$ is fixed. For a metric space $\langle X, \varrho\rangle$ and distinct points $a, b \in X$, the associated Voronoi diagram of nearness regions constitutes a division that separates a from b.

By way of notation: if S is a subset of a topological space X, then the *closure* and *interior* of S (relative to X) are written S^- and S°, respectively, and the *boundary* of S is just $S^- \cap (X \setminus S)^-$. A set is **nowhere dense** if its closure has empty interior.

A division $\mathcal{D} = \{A, B\}$ is **fine** if every neighborhood of a point in $A \cap B$ intersects both $A \setminus B$ and $B \setminus A$; i.e., if each half is the closure of the complement of the other. In a fine division the wall is nowhere dense (and may then be reasonably regarded as a "tessellation"), but the following example shows that divisions with nowhere dense walls need not be fine.

4.1. **Example.** Let X be the Euclidean plane $\mathbb{R}^2_2$, with $C \subseteq X$ the closed unit circle (i.e., $C = S_X$). Let U, V be the bounded and unbounded components, respectively, of $X \setminus C$. Then $\{U \cup C, V \cup C\}$ is a fine division of X, whose wall is C. On the other hand, suppose $D = C \cup ([-1, 1] \times \{0\})$, with U the union of the two bounded components of $X \setminus D$, and V the unbounded component. Then $\{U \cup D, V \cup D\}$ is a division whose wall is the nowhere dense set D. It is not fine, however, because there are points on the "crossbar" of D with neighborhoods that do not intersect V.

If $\mathcal{D} = \{A, B\}$ and $\mathcal{D}' = \{A', B'\}$ are two divisions of X, we say $\mathcal{D}'$ **refines** $\mathcal{D}$ if each half of $\mathcal{D}'$ is contained in a (necessarily unique) half of $\mathcal{D}$. In that case we write $\mathcal{D}' \leq \mathcal{D}$. If $\mathcal{D}$ separates a, b and $\mathcal{D}' \leq \mathcal{D}$, then $\mathcal{D}'$ clearly separates a, b too: For suppose $a \in A$, $b \in B$, $A' \subseteq A$, and $B' \subseteq B$. Then $a \notin B'$ because $a \notin B$. Hence $a \in A'$. Likewise for inferring $b \in B'$.

Let $\mathbb{D}$ denote the family of divisions of X. Then the relation of refinement is a partial order on $\mathbb{D}$; and a division is called **minimal** if it is minimal in this partial ordering.

4.2. **Theorem.** *If $\mathcal{D}$ is a division separating distinct points $a, b \in X$, then there is a minimal division separating a, b and refining $\mathcal{D}$.*

Proof. This is a simple application of Zorn's lemma, after we verify that $\mathbb{D}$ is closed under intersections of chains. Indeed, suppose $\mathcal{D} = \{A, B\}$, with $A = \mathcal{D}(a)$ and $B = \mathcal{D}(b)$. Let $\{\mathcal{D}_\lambda : \lambda \in \Lambda\}$ be an indexed chain of refinements of $\mathcal{D}$, where $\mathcal{D}_\lambda = \{A_\lambda, B_\lambda\}$, $\lambda \in \Lambda$, Λ is a totally ordered set, and $\mathcal{D}_\lambda \leq \mathcal{D}_\mu$ for $\lambda \leq \mu$ in Λ. Since each $\mathcal{D}_\lambda$ refines $\mathcal{D}$, it separates a from b. Thus we lose no generality in assuming $a \in A_\lambda \subseteq A_\mu$ (and hence $b \in B_\lambda \subseteq B_\mu$) for $\lambda \leq \mu$. Set $A = \bigcap\{A_\lambda : \lambda \in \Lambda\}$, $B = \bigcap\{B_\lambda : \lambda \in \Lambda\}$. Then A and B are both closed sets, $a \in A \setminus B$, and $b \in B \setminus A$. So it suffices to show that $A \cup B = X$. For any $x \in X$, if $x \notin A$, then there is some $\mu \in \Lambda$ with $x \notin A_\mu$. Hence, for each $\lambda \leq \mu$ we have $x \notin A_\lambda$. But then we have, for each $\lambda \leq \mu$, $x \in B_\lambda$; from which we infer that $x \in B$. □

4.3. **Lemma.** *Let $\{A', B'\} \leq \{A, B\}$ be divisions of X, say $A' \subseteq A$ and $B' \subseteq B$. Then $(A \setminus B)^- \subseteq A'$ and $(B \setminus A)^- \subseteq B'$.*

Proof. If $x \in A \setminus B$, then−since $x \notin B'$−we have $x \in A'$; so $A \setminus B \subseteq A'$. Since A' is closed, we have $(A \setminus B)^- \subseteq A'$. Likewise $(B \setminus A)^- \subseteq B'$. □

4.4. **Theorem.** *A division is minimal if and only if it is fine.*

Proof. Suppose $\mathcal{D} = \{A, B\}$ is fine, with $\mathcal{D}' = \{A', B'\} \leq \mathcal{D}$−say $A' \subseteq A$ and $B' \subseteq B$. By the fineness assumption, plus Lemma 4.3, we have $A = (A \setminus B)^- \subseteq A'$ and $B = (B \setminus A)^- \subseteq B'$, so $\mathcal{D}$ is minimal. Now suppose $\mathcal{D}$ is not fine. Then there is some $x \in A \cap B$ with an open neighborhood lying either in A or in B, say it is in B. Thus there is a nonempty open set $U \subseteq B$ such that $U \cap A \neq \emptyset$. Let

$A' = A \setminus U$. Then A' is closed and properly contained in A. Moreover, $A' \cup B = X$. Thus $\mathcal{D}' = \{A', B\}$ is a division that properly refines $\mathcal{D}$, showing that $\mathcal{D}$ is not minimal. □

From Theorems 4.2 and 4.4, we immediately have the following.

4.5. **Corollary.** *If $\mathcal{D}$ is a division separating distinct points $a, b \in X$, then there is a fine division, separating a and b, and refining $\mathcal{D}$.*

4.6. **Remark.** Without the refinement condition to worry about, obtaining a fine division separating two distinct points is relatively easy. For example, let X be a Hausdorff topological space, with $a, b \in X$ distinct. Then there is an open set V with $a \in V \subseteq V^- \subseteq X \setminus \{b\}$. Now let $U = V^{-\circ}$ and set $A = U^- = V^-$ and $B = X \setminus U$. Then clearly $\{A, B\}$ is a division separating a and b. To see that it is fine, we have first that $(X \setminus B)^- = U^- = A$, and we need to show that $(X \setminus A)^- = B$; i.e., that $(X \setminus U^-)^- = X \setminus U$. But $X \setminus U^- \subseteq X \setminus U$ and $X \setminus U$ is closed, so the containment $(X \setminus U^-)^- \subseteq X \setminus U$ is assured. Suppose $x \notin (X \setminus U^-)^-$. Then there is an open neighborhood W of x that is contained in U^-. But then $W = W^\circ \subseteq U^{-\circ} = V^{-\circ} = U$, so $x \notin X \setminus U$. This gives the reverse containment.

Strict convexity in a normed vector space is equivalent to the statement that each nondegenerate metric interval is linear [2, Proposition 4.1]; i.e., where the notions of *convex subset* and *linearly convex subset* coincide. It is also equivalent to saying that the unit sphere contains no nondegenerate line segments. This property plays an important role *vis à vis* the fineness of divisions. In Example 5.1 below, we see that when $\mathbb{R}^2$ is equipped with the taxicab norm, equisets can have nonempty topological interior, making their associated divisions far from fine. In the presence of strict convexity, this cannot happen.

4.7. **Theorem.** *Let $\langle X, \|\cdot\|\rangle$ be a strictly convex normed vector space. Then for each two distinct points $a, b \in X$, the division $\{R(a, b), R(b, a)\}$ is fine.*

Proof. Fix distinct $a, b \in X$, with $c \in E(a, b)$. Then the three points are also distinct. Let B be an open ball centered at c, and suppose−for the sake of contradiction−that B is entirely contained in one of the halves; say $B \subseteq R(b, a)$. (We also lose no generality in assuming $B \subseteq X \setminus \{a, b\}$.) Then we may fix a point $d \in ([\![a, c]\!] \setminus \{c\}) \cap B$. By assumption, $d \in R(b, a)$−i.e., $\|d - b\| \leq \|d - a\|$−and we have $\|c - d\| + \|d - b\| \leq \|c - d\| + \|d - a\| = \|c - a\|$ (because $d \in [\![a, c]\!] \subseteq I(a, c)$). But $\|c - a\| = \|c - b\|$, and therefore $d \in I(c, b)$. By strict convexity, however, we have $d \in [\![a, c]\!] \cap [\![c, b]\!]$; and $d \notin \{a, b, c\}$ by construction. By simple plane geometry, this means that the three points a, b, c are collinear, and therefore $c \in [\![a, b]\!]$. By concentration (I6), we have $c = d$, a contradiction. □

4.8. **Remark.** While we do not know if the converse of Theorem 4.7 is true for general normed vector spaces, it does hold in dimension 2: for let $\langle X, \|\cdot\|\rangle$ be a normed plane where strict convexity fails. Then the unit sphere S_X contains a nondegenerate line segment $[\![a, b]\!]$. By [9, Theorem 2.1], $E(a, b)$ has nonempty interior; hence $\{R(a, b), R(b, a)\}$ is not fine.

5. Convexity in normed vector spaces

We now consider in more detail the study of convexity in normed vector spaces, starting with the following well-known class of normed planes.

5.1. **Example.** Fix $1 \leq p < \infty$, and let $\mathbb{R}^2_p$ be the vector space $\mathbb{R}^2$, equipped with the p-norm

$$\|\langle x, y\rangle\|_p := (|x|^p + |y|^p)^{\frac{1}{p}}.$$

We also define the ∞-norm, given by

$$\|\langle x, y\rangle\|_\infty := \max\{|x|, |y|\} = \sup\{\|\langle x, y\rangle\|_p : 1 \leq p < \infty\}.$$

Note that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(x, y) = \langle x - y, x + y\rangle$, is an isomorphism such that $\|T(x, y)\|_\infty = \|\langle x, y\rangle\|_1$. Hence the two normed vector spaces at the opposite ends of the $[1, \infty]$-spectrum are isometrically isomorphic.

Because strict convexity in a normed vector space means that its unit sphere contains no nondegenerate line segments, we see that $\mathbb{R}^2_p$ is strictly convex if and only if $1 < p < \infty$.

A norm on $\mathbb{R}^2$ arises from an inner product if and only if its unit sphere−a simple closed curve in this case− describes an ellipse [8, Theorem 3.2]. Hence $\mathbb{R}^2_p$ is an inner product space if and only if $p = 2$. By [18, Corollary 1.3], a normed vector space is an inner product space if and only if each equiset is linearly convex. The main result of [9] is that the equisets in any normed plane have two basic forms: either they are unbounded curves, homeomorphic to $\mathbb{R}$; or they are the union of two disjoint cones whose apexes are joined by an arc. (Here, following [9, 13], a *cone* in two dimensions is the intersection of two closed half-planes whose boundary lines meet at a point, the *apex* of the cone.) All equisets in the planes $\mathbb{R}^2_p$, $1 < p < \infty$, are of the first form, but both forms occur if $p \in \{1, \infty\}$ (as we see below).

So, among the spaces $\mathbb{R}^2_p$, all equisets are straight lines precisely when $p = 2$. Since strict convexity holds for $\mathbb{R}^2_p$ if and only if $1 < p < \infty$, Theorem 4.7 and Remark 4.8 tell us that it is precisely in this range that all divisions $\{R(a, b), R(b, a)\}$, $a \neq b$, are fine. (We will see a concrete instance of this when we consider the case $p = 1$ below.)

Consider the case $p = 3$. If $a = \langle -1, -1\rangle$ and $b = \langle 1, 1\rangle$, then $E(a, b)$ actually is the perpendicular bisector of $[\![a, b]\!]$. However, if $a = \langle -2, -1\rangle$ and $b = \langle 2, 1\rangle$, then $E(a, b)$ is a curve, symmetric about the origin, and approaching the line $\sqrt{2}x+y = 0$ asymptotically. Outside the interval $-2 \leq x \leq 2$, the curve is a hyperbola defined by the equation $6x^2-3y^2+7 = 0$, and inside this interval the curve is defined by the cubic $x^3 + 12x + y^3 + 2y = 0$.

Now consider the case $p = 1$ (i.e., the *taxicab norm*). If $[\![a, b]\!]$ is parallel to either coordinate axis, then $E(a, b)$ is the perpendicular bisector of $[\![a, b]\!]$, as is easily verified. But when the slope of $[\![a, b]\!]$ is defined and nonzero, we get two quite different situations, depending on whether the slope has absolute value 1. For example, consider the case $a = \langle -1, -1\rangle$ and $b = \langle 1, 1\rangle$. Then $I(a, b) = [-1, 1]^2$, a solid rectangle. (Note that, in this case, $I(a, b) = I(c, d)$, where $c = \langle -1, 1\rangle$ and $d = \langle 1, -1\rangle$. This gives another illustration of how intervals do not necessarily determine their bracket sets.) Furthermore, $E(a, b) = ((-\infty, -1]\times[1, \infty))\cup\{\langle t, -t\rangle : -1 \leq t \leq 1\}\cup([1, \infty)\times(-\infty, -1])$, the union of two disjoint quarter-plane cones and the line segment−namely $M(a, b)$−joining the apexes of those cones. Note that $E(a, b)$ is not convex because it is not linearly convex. $M(a, b)$, on the other hand is linearly convex but not convex: while $c, d \in M(a,b)$, we have $I(c,d) = I(a,b) \not\subseteq M(a,b)$.

Because $E(a,b)$ has nonempty interior, the division $\{R(a,b), R(b,a)\}$ is decidedly not fine. We recover fineness if the slope of $[\![a,b]\!]$ differs from ± 1: for if we take $a = \langle -2,-1\rangle$ and $b = \langle 2,1\rangle$, then $I(a,b) = [-2,2] \times [-1,1]$ and $E(a,b) = (\{-1\} \times [1,\infty)) \cup \{\langle t,-t\rangle : -1 \le t \le 1\} \cup (\{1\} \times (-\infty,-1])$, a broken line. Hence $\{R(a,b), R(b,a)\}$ is fine. (The situation when $p = \infty$ is the same, except that everything is rotated counterclockwise through an angle of $\pi/4$ under the norm-preserving isomorphism T described above.)

[18, Corollary 1.3] says that a normed vector space is an inner product space if and only if all of its equisets are linearly convex. With just a little more work, one can show the following.

5.2. **Corollary.** *Let X be a normed vector space. The following three conditions are equivalent.*

(a) *The norm of X arises from an inner product.*
(b) *X is E-convex.*
(c) *X is R-convex.*

Proof. Assume (a). Then X is strictly convex; hence all metric intervals are linear [2, Proposition 4.1]. By [18, Corollary 1.3], all equisets are linearly convex; by linearity of metric intervals, we know (b) holds.

Now assume (b). Then each equiset is linearly convex. Another application of [18, Corollary 1.3] gives us that X is an inner product space, so (a) holds.

That (b) follows from (c) is trivial; that (c) follows from (b) is a consequence of Theorem 3.4, since normed vector spaces are segment-connected metric IR-structures. □

5.3. **Remark.** We do not know how to characterize I-convexity or M-convexity for normed vector spaces, along the lines of Corollary 5.2. Strict convexity is too strong a condition: it clearly implies both I-convexity (because intervals are line segments) and M-convexity (because midsets are singletons); but $\mathbb{R}^2_1$ is I-convex without being strictly convex, and we exhibit in Example 5.4 a normed plane where M-convexity holds but strict convexity does not. (Among the spaces $\mathbb{R}^2_p$ from Example 5.1, though, M-convexity does coincide with strict convexity.)

In the two-dimensional case the situation is fairly well understood: Theorem 5.12 gives a simple characterization of M-convexity in a normed plane in terms of the geometry of its unit ball; and Theorem 5.14 asserts that *every* normed plane is I-convex. However, even in dimension three the characterization problems for I-convexity and M-convexity appear to be quite challenging.

For geometric analysis of nondegenerate metric intervals in a normed vector space X, it suffices to focus on those of the form $I(0,a)$, where $a \in S_X$. This is because of the invertible affine transformations $x \mapsto \frac{x-a}{\|b-a\|}$, for each $a \neq b$, taking $I(a,b)$ to $I(0, \frac{b-a}{\|b-a\|})$. Each interval is a rescaled translate of the other; in particular, one is convex if and only if the other is. Furthermore, if $a, b \in X$ are distinct and one of $I(a,b)$, $I(0, \frac{b-a}{\|b-a\|})$ is linear, then the other is also linear, and parallel to the first.

We will refer to this invariance of qualitative geometric properties under invertible affine transformations as the *Rescaled Translation Principle* (RTP for short).

Recall that a point e in a linearly convex set $K \subseteq X$ is an **extreme point** of K if no line segment containing e as an interior point lies entirely in K. An immediate consequence of Lemma 5.8 below is that, for $c \in S_X$, $I(0,c)$ is linear if c is an extreme point of B_X. For any $x \in X$, define the vector subspace $L(x) := \{tx : t \in \mathbb{R}\}$. Then for any two distinct points $a, b \in X$ we may conclude that $I(a,b)$ is linear if $L(b-a) \cap S_X$ consists of extreme points of B_X. From this discussion, we can show that, while strictly convex normed vector spaces are M-convex (because midsets are singletons, see Remark 1.1 (ii) above), the converse is false.

5.4. **Example.** In $\mathbb{R}^2$, let $B := \{\langle x,y\rangle : \max\{|x+y|, x^2+y^2\} \leq 1\}$. Then B is linearly convex and symmetric about the origin. Relative to the Euclidean norm, B is also closed and bounded, with the origin in its interior. Let $\|\cdot\|_B$ be its *Minkowski functional*; i.e., $\|a\|_B := \inf\{t > 0 : t^{-1}a \in B\}$. Then $\|\cdot\|_B$ is the norm on $\mathbb{R}^2$ whose unit ball is B. (It is a "hybrid" norm: in the first and third quadrants it agrees with $\|\cdot\|_1$; in the second and fourth quadrants it agrees with $\|\cdot\|_2$.) Let $X = \langle \mathbb{R}^2, \|\cdot\|_B\rangle$. Then $B = B_X$, and we may write S_X as the union $A_1 \cup A_2 \cup A_3 \cup A_4$, where A_1 (resp., A_3) is the intersection of the line $y = 1 - x$ (resp., $y = -1 - x$) and the first (resp., third) quadrant, and A_2 (resp., A_4) is the intersection of the circle $x^2 + y^2 = 1$ and the second (resp., fourth) quadrant.

So if $a \in A_2 \cup A_4$ is an extreme point of B, then—by Lemma 5.8—$I(0,a) = [\![0,a]\!]$. Thus $M(0,a) = \{\frac{1}{2}a\}$ is trivially convex. If a is in, say, A_1, let $a = \langle r, 1-r\rangle$. Then $I(0,a) = [0,r] \times [0,1-r]$—witnessing that X is not strictly convex—and $M(0,a) = [\![b,c]\!]$, a line segment parallel to the line segment A_1. (For example, if $r \geq \frac{1}{2}$, then we may take $b = \langle \frac{1}{2}, 0\rangle$ and $c = \langle r - \frac{1}{2}, 1-r\rangle$.) Then $I(b,c) = [\![b,c]\!] = M(0,a)$ because $L(b-c) \cap S_X \subseteq A_2 \cup A_4$, consisting of extreme points of B. If x and y are distinct points of $M(0,a)$, then $L(x-y) = L(b-c)$. Hence $I(x,y) = [\![x,y]\!] \subseteq [\![b,c]\!] = M(0,a)$. This shows that every midset of X is convex.

The following result lists some basic geometric features of intervals, equisets, and midsets in normed vector spaces. By way of notation: If a and b are distinct vectors, let $b[\![a,\infty) := \{b + t(a-b) : t \geq 1\}$ denote the closed half-line with end point a, and pointing away from b. (This notation is intended to be consistent with that for line segments. For example, the closed half-line with end point a and proceeding through $b \neq a$ is $[\![a,b]\!] \cup a[\![b,\infty)$.) As per common usage, a *plane* in a vector space is a parallel translation of a two-dimensional subspace.

5.5. **Theorem.** *Let $\langle X, \|\cdot\|\rangle$ be a normed vector space, with $a, b \in X$ distinct.*

(i) *If $P \subseteq X$ is any plane containing $[\![a,b]\!]$, then $P \cap E(a,b)$ is unbounded in P.*
(ii) *Both the metric interval $I(a,b)$ and the midset $M(a,b)$ are linearly convex; $M(a,b)$ is degenerate if and only if $I(a,b)$ is linear.*
(iii) *The pair $\langle a,b\rangle$ is weakly disjunctive if and only if it is narrow, if and only if $I(a,b)$ is linear.*
(iv) *If $\langle X, \|\cdot\|\rangle$ is a normed plane and $I(a,b)$ is not linear, then $M(a,b)$ is a nondegenerate line segment whose end points lie on the boundary of $I(a,b)$.*

Proof. Ad (i): If $z = b + t(a - b)$ for some $t \geq 1$, then $\|z - a\| = (t - 1)\|a - b\| < t\|a-b\| = \|z-b\|$; so $b[\![a, \infty) \subseteq P \setminus E(a, b)$. Likewise we have $a[\![b, \infty) \subseteq P \setminus E(a, b)$. Suppose $P \cap E(a, b)$ is bounded in P. Then we may choose $C \subseteq P$, a circle that properly encloses $\{a, b\} \cup (P \cap E(a, b))$. But then $C \cup b[\![a, \infty) \cup a[\![b, \infty)$ is a connected subset of P containing a and b, and disjoint from $P \cap E(a, b)$. This contradicts the fact that $P \cap E(a, b)$ separates P (Proposition 3.3 (i)), and we conclude that $P \cap E(a, b)$ is unbounded.

Ad (ii): We first show that $I(a, b)$ is linearly convex (see also [2, Proposition 4.3]). So let $x, y \in I(a, b)$, with $z \in [\![x, y]\!]$, say $z = sx + ty$, where $s, t \geq 0$ and $s + t = 1$. To show $z \in I(a, b)$, it suffices to show $\|a - z\| + \|z - b\| \leq \|a - b\|$. Indeed, the left-hand side is $\|a - (sx + ty)\| + \|(sx + ty) - b\|$, which equals $|(s+t)a-(sx+ty)\|+\|(sx+ty)-(s+t)b\| = \|s(a-x)+t(a-y)\|+\|s(x-b)+t(y-b)\| \leq s\|a-x\|+t\|a-y\|+s\|x-b\|+t\|y-b\| = s(\|a-x\|+\|x-b)\|)+t(\|a-y\|+\|y-b\|) = (s + t)\|a - b\| = \|a - b\|$, since $x, y \in I(a, b)$.

The linear convexity of $M(a, b)$ would follow instantly from that of $E(a, b)$ and $I(a, b)$, but Example 5.1 demonstrates that $E(a, b)$ need not be so graced. So let us assume $x, y \in M(a, b)$, with $z \in [\![x, y]\!]$ as above. Then $\|a - z\| = \|s(a - x) + t(a - y)\| \leq s\|a - x\| + t\|a - y\|$. But $\|a - x\| = \|a - y\| = \frac{1}{2}\|a - b\|$; so we have $\|a - z\| \leq \frac{1}{2}\|a - b\|$. Similarly, $\|z - b\| \leq \frac{1}{2}\|a - b\|$. But, because $I(a, b)$ is linearly convex, we also have $\|a-z\|+\|z-b\| = \|a-b\|$. Hence $\|a-z\| = \|z-b\| = \frac{1}{2}\|a-b\|$, and we infer that $z \in M(a, b)$. Hence $M(a, b)$ is linearly convex.

Let $m = \frac{1}{2}(a+b)$, the halfway point of $[\![a, b]\!]$. Then $m \in M(a, b)$. (See Remark 1.1 (ii).) Clearly if $I(a, b) = [\![a, b]\!]$, then $M(a, b) = \{m\}$. For the converse, suppose $x \in I(a, b) \setminus [\![a, b]\!]$. If $x \in M(a, b)$, then—because $x \neq m$—we have $M(a, b)$ nondegenerate and we are done. Suppose $x \notin M(a, b)$. Then, by Proposition 3.3 (ii), we know that $M(a, b)$ separates $I(a, b)$; i.e.,

$$\{(R(a, b) \cap I(a, b)) \setminus M(a, b), (R(b, a) \cap I(a, b)) \setminus M(a, b)\}$$

is a disconnection of $I(a, b) \setminus M(a, b)$. Without loss of generality, assume x is in the member of the disconnection not containing b. Then, since $I(a, b)$ is linearly convex, it must be the case that $[\![x, b]\!]$ intersects $M(a, b)$ in some point y. But $[\![x, b]\!] \cap [\![a, b]\!] = \{b\}$; so $y \in M(a, b) \setminus \{m\}$. Hence $M(a, b)$ is nondegenerate.

Ad (iii): If $I(a, b) = [\![a, b]\!]$, then disjunctivity and narrowness for $\langle a, b\rangle$ follow immediately. By Theorem 2.7, narrowness follows from weak disjunctivity; so it suffices to show that linearity follows from narrowness. But if $I(a, b)$ is not linear, then $M(a, b)$ is nondegenerate, by (ii) above. Any two points in $M(a, b)$ witness the fact that $\langle a, b\rangle$ is not narrow.

Ad (iv): Without loss of generality, we may assume $b = -a$. Let $M = M(a, -a)$, which we assume to be nondegenerate because of (ii) above. Since M is linearly convex, by (ii), as well as closed and bounded, by Proposition 3.1, we will know it is a line segment once we show it to be contained in a straight line.

For any nonzero vector x, the subspace $L(x)$ generated by x is the line containing $[\![0, x]\!]$. Since $M \cap [\![a, -a]\!] = M \cap L(a) = \{0\}$, we may fix $c \in M \setminus [\![a, -a]\!]$. We show $M \subseteq L(c)$. Indeed, let $y \in M$ be arbitrary; we may as well assume $y \neq 0$. Since M is clearly symmetric about the origin and $y \in L(c)$ if and only if $-y \in L(c)$, we lose no generality in assuming y is not on the same side of $L(a)$ as is c. But then $[\![c, y]\!]$ intersects $L(a)$ in some point z. Since M is linearly convex, we have $z = 0$. Thus $y \in L(c)$, as desired; hence M is a line segment. The end points of M lie on the boundary of $I = I(a,-a)$ because−see the proof of (i) above−I is connected and $I \setminus M$ is not. □

Regarding Theorem 5.5 (iv): In Remark 5.17 (i) below we cite an example of a three-dimensional normed vector space where some midsets are singletons, some are nondegenerate line segments, but none have dimension more than one. With $\mathbb{R}^3_1$, on the other hand, there are linearly convex polygons among the midsets, as well as line segments and singletons.

Corollary 5.2 provides a characterization of a normed vector space property−namely being an inner product space−in terms of universal first-order IR-axioms. A second normed vector space property, characterizable in similar terms, is strict convexity.

5.6. **Corollary.** *Let X be a normed vector space. The following three conditions are equivalent.*

(a) *X is strictly convex.*
(b) *The associated IR-structure of X satisfies weak disjunctivity (I8).*
(c) *The associated IR-structure of X satisfies narrowness (IE2).*

Proof. From [2, Proposition 4.1], strict convexity is equivalent to all intervals being linear. Now invoke Theorem 5.5 (iii). □

We note that the equivalence of items (a) and (b) above is proved in [1, Theorem 4.9] using an argument different from that afforded by Theorem 5.5 (iii).

A third property of normed vector spaces serves to highlight the restrictiveness of strong disjunctivity in that context.

5.7. **Proposition.** *The associated IR-structure of a normed vector space satisfies strong disjunctivity (I9) if and only if the space has dimension ≤ 1.*

Proof. There is nothing to prove in dimension zero, so suppose the dimension of $\langle X, \|\cdot\|\rangle$ is one; so we may take X to be $\mathbb{R}$. Every one-dimensional normed vector space is strictly convex; hence, for $a, b \in \mathbb{R}$, say $a \leq b$, we have $I(a,b) = [\![a,b]\!] = [a,b]$. Strong disjunctivity immediately follows from elementary facts about total orderings.

If X has dimension ≥ 2 and strict convexity fails, then so does weak disjunctivity, by Corollary 5.6. If strict convexity holds, then all metric intervals are line segments. So if a, b, c are any three noncollinear points, then $I(a,b) \cap (I(a,c) \cup I(c,b)) = \{a,b\}$; hence $I(a,b) \not\subseteq I(a,c) \cup I(c,b)$, and strong disjunctivity fails. □

We next set about investigating more deeply the geometry of metric intervals in normed vector spaces. If X is any vector space and $a, b \in X$ are distinct, let $b(\!(a,\infty) := b[\![a,\infty) \setminus \{a\}$ be the open half-line with end point a, pointing away from b. If $K \subseteq X$ is linearly convex and $a \in K$, then the K**-face** of a, denoted $F_K(a)$, consists of a, together with those $b \in K \setminus \{a\}$ such that $b(\!(a,\infty)$ intersects K. A **face** of K is simply the K-face of some $a \in K$. This terminology is equivalent to the one that is most widely used in classical convexity theory. Our notion *K-face of a* appears in [6, TVS II.87] as *K-facet of a*; it is designed to classify the points of a convex set. For example, suppose K is the unit ball B from Example 5.4 above. If a lies in the interior of K, then $F_K(a) = K$. If a is an interior point of the line segment A_1 (resp., A_3), then $F_K(a)$ is A_1 (resp., A_3). Finally, if $a \in A_2 \cup A_4$, then $F_K(a) = \{a\}$.

Faces of linearly convex sets are themselves linearly convex; the degenerate faces of K constitute the extreme points of K. Note that if $b \in F_K(a) \setminus \{a\}$ and $t > 1$ is such that $c = b + t(a - b) \in K$, then $b = c + s(a - c)$, where $s = \frac{t}{t-1} > 1$. Hence $c \in F_K(a)$ too.

Given a nonempty $B \subseteq X$ and $a \in X$, the B**-fan** with **apex** a, denoted $V_B(a)$, is the union $\bigcup\{[\![a, b]\!] : b \in B\}$ of all line segments with a as one endpoint and a point of B as the other. Simple plane geometry tells us that if B is linearly convex, then so too is any B-fan.

By the RTP discussed above, we need only focus on intervals of the form $I(0, a)$, where $a \in S_X$. When $a \in S_X$, we let $F(a)$ denote the face $F_{B_X}(a)$ and $V(a)$ denote the $F(a)$-fan $V_{F(a)}(0)$ with apex 0. The following is basically half of [2, Theorem 4.10].

5.8. **Lemma.** *Let $\langle X, \|\cdot\|\rangle$ be a normed vector space, $a \in S_X$. Then $I(0, a) \subseteq V(a)$.*

Proof. Let $x \in I(0, a)$. Then $\|x\| + \|a - x\| = \|a\| = 1$. If $x \in \{0, a\}$, then clearly $x \in V(a)$; so we may assume that $0 < \|x\| < 1$. Let $b = \frac{x}{\|x\|}$ (automatically in S_X). We are done once we show that $b \in F(a)$. Indeed, with $t = \frac{1}{1-\|x\|}$, we have $t > 1$ and

$$b + t(a - b) = \frac{(1 - \|x\|)b + (a - b)}{1 - \|x\|} = \frac{a - \|x\|b}{1 - \|x\|} = \frac{a - x}{\|a - x\|}.$$

Hence $\|b + t(a - b)\| = 1$; so $b + t(a - b) \in B_X$ and thus $b \in F(a)$, as desired. □

If $a, b \in X$, then $[\![a, b]\!] \subseteq I(a, b)$. Since, by Proposition 3.1, $\|a - b\|$ is the diameter of $I(a, b)$, we refer to $[\![a, b]\!]$ as a **spine** of $I(a, b)$. (There are as many spines for an interval as there are bracket sets.) If $m = \frac{1}{2}(a+b)$ is the halfway point of $[\![a, b]\!]$ then the translation $x \mapsto x - m$ moves $I(a, b)$ to the interval $I(-c, c)$, where $c = \frac{1}{2}(a - b)$. As $I(-c, c)$ is symmetric about the origin, the RTP implies that $I(a, b)$ is symmetric about the halfway point of any of its spines.

In dimension two we can be quite explicit about the shape of intervals. Note that, for normed planes, either all points of the unit sphere are extreme points of the unit ball (the strictly convex case) or the unit sphere contains a nondegenerate line segment. Intervals are line segments in the first case; the following addresses the second.

5.9. **Theorem.** *Let $X = \langle\mathbb{R}^2, \|\cdot\|\rangle$ be a normed plane that is not strictly convex, with p, q distinct extreme points of B_X, such that $[\![p, q]\!] \subseteq S_X$. Fix $a \in [\![p, q]\!]$, and fix unique $\alpha, \beta \in [0,\infty)$ so that $\alpha + \beta = 1$ and $a = \alpha p + \beta q$. Let P be the parallelogram $\{\alpha' p + \beta' q : 0 \le \alpha' \le \alpha, 0 \le \beta' \le \beta\}$ (a line segment if and only if $a \in \{p, q\}$). Then $I(0, a) = P$; in particular, when $a \notin \{p, q\}$ then $I(0, a)$ is a parallelogram with $[\![0, a]\!]$ as one of its two diagonals. Furthermore, if $a \in [\![p, q]\!] \setminus \{p, q\}$ then $M(0, a)$ is a nondegenerate line segment parallel to $[\![p, q]\!]$.*

Proof. First, to show $P \subseteq I(0, a)$, it suffices to show all four corners lie in $I(0, a)$ and invoke linear convexity (Theorem 5.5 (ii)). The bracket points are in $I(0, a)$ by definition, so consider αp. We need to show $\|\alpha p\| + \|a - \alpha p\| = \|a\|$. But the left-hand side is $\|\alpha p\| + \|\beta q\| = \alpha\|p\| + \beta\|q\| = \alpha + \beta = 1 = \|a\|$, and we are done. Similarly, we show $\beta q \in I(0, a)$.

Next, if $a \in \{p, q\}$, say $a = p$, then $F(a) = \{p\}$−because p is an extreme point of B_X−and $V(a) = [\![0, a]\!]$. By Lemma 5.8 (and because $[\![0, a]\!] \subseteq I(0, a)$ generally), we have $I(0,a) = [\![0,a]\!]$. Note that in this case $\alpha = 1$ and $\beta = 0$, so P also is $[\![0,a]\!]$. (We handle the case $a = q$ similarly.)

Now assume $a \in [\![p,q]\!] \setminus \{p,q\}$ (so both α and β are less than one). Then $F(a) = [\![p,q]\!]$, and $V(a)$ is the triangle $T = \triangle(0,p,q)$ with corners $0,p,q$. Both T and P are *nontrivial*, in that all stated corners are distinct and no three of them are collinear. By Lemma 5.8, we have $I(0,a) \subseteq T$. Suppose $c \in X \setminus P$ is arbitrary. We wish to show $c \in X \setminus I(0,a)$. Since $I(0,a) \subseteq T$, we may assume $c \in T$. Let $m = \frac{1}{2}a$, the halfway point of $[\![0,a]\!]$. Then P is symmetric about m, and c is distinct from m. Let L be the line through $\{c,m\}$. The triangle T is the union of P plus the nontrivial triangles $\triangle(\alpha p,p,a)$ and $\triangle(\beta q,q,a)$. Without loss of generality, assume $c \in \triangle(\beta q,q,a)$. Then, by basic Euclidean geometry, L intersects the line segment $[\![\beta q,a]\!]$ in a unique point d; and it intersects the line segment $[\![0,\alpha p]\!]$, parallel to $[\![\beta q,a]\!]$, in a unique point e. Because P is symmetric about m, and $c \notin P$, we have $d \in [\![m,c]\!] \setminus \{m,c\}$, and therefore $e \in [\![m-c,m]\!] \setminus \{m-c,m\}$. But e is on the boundary of T and $m \in T$; hence $m - c \notin T$. By another application of Lemma 5.8, we have $m - c \notin I(0,a)$. But $I(0,a)$ is symmetric about m; hence we have $c \notin I(0,a)$. This completes the proof that $I(0,a) = P$. By the description of P, it is plain that $[\![0,a]\!]$ is one of the two diagonals of P.

In the case a is an interior point of $[\![p,q]\!]$, we saw above that $I(0,a)$ is nonlinear. Hence, by Theorem 5.5 (iv), $M(0,a)$ is a nondegenerate line segment. Let $c,d \in M(0,a)$ be two distinct points. Then $\|c\| = \|d\| = \frac{1}{2}$, so $2c$ and $2d$ are distinct points in S_X. Since c and d are points in $I(0,a) \subseteq T$, we know $2c$ and $2d$ lie on $[\![p,q]\!]$. Since $[\![c,d]\!]$ and $[\![2c,2d]\!]$ are parallel to each other, we infer the same fact about $M(0,a)$ and $[\![p,q]\!]$. □

If X is a normed vector space, $a \in S_X$, and P is any plane containing $[\![0,a]\!]$, then $P \cap B_X$ is the unit ball in the induced normed plane P, $P \cap F(a)$ is the face of a in $P \cap B_X$, and $P \cap I(0,a)$ is the interval in P bracketed by $\{0,a\}$. Thus, by Theorem 5.9 and the RTP, we have the following.

5.10. **Corollary.** *In any normed vector space, the intersection of a nondegenerate interval $I(a,b)$ with a plane containing $[\![a,b]\!]$ is either $[\![a,b]\!]$ itself or a nontrivial parallelogram with $[\![a,b]\!]$ as a spine.*

5.11. **Remark.** Parallelograms in $\mathbb{R}^2$ are definable using vector space notions only; whether a parallelogram has zero, one, or two spines is dictated by a given norm. Consider the real plane $\mathbb{R}^2$, with $a = \langle \frac{1}{2}, \frac{1}{2} \rangle$, $b = \langle \frac{1}{2}, 0 \rangle$, and $c = \langle 0, \frac{1}{2} \rangle$. Under the taxicab norm (see Example 5.1), $I(0,a)$ is the parallelogram $[0, \frac{1}{2}]^2$ and both diagonals are spines. If we now consider the norm from Example 5.4, $I(0,a)$ is the same parallelogram. But $I(b,c) = [\![b,c]\!]$ is linear and therefore not equal to I(0,a). Hence $[\![0,a]\!]$ is the only spine of $I(0,a)$. With either of these norms−the Euclidean norm too−any nontrivial parallelogram not a rectangle is utterly spineless.

The argument in Example 5.4 may be souped up, using Theorem 5.9, to obtain a characterization result.

5.12. **Theorem.** *Let X be a normed plane. The following two conditions are equivalent.*

(a) *X is M-convex.*

(b) *If* $b, c \in X$ *are distinct and* $[\![b, c]\!] \subseteq S_X$*, then* $L(b-c) \cap S_X$ *consists of extreme points of* B_X*.*

Proof. Assume (a) holds, and let $b, c \in X$ be distinct such that $[\![b, c]\!] \subseteq S_X$. Fix $a \in [\![b, c]\!] \setminus \{b, c\}$. Then, by Theorem 5.9, $M(0, a)$ is a nondegenerate line segment, say $M(0, a) = [\![d, e]\!]$. Since X is M-convex, we have $I(d, e) \subseteq M(0, a)$; i.e., $I(d, e)$ is linear. Using 5.9 again, we infer that $L(d-e) \cap S_X$ consists of extreme points of B_X. By a third appeal to 5.9, we know $[\![d, e]\!]$ and $[\![b, c]\!]$ are parallel to each other. Hence $L(d-e) = L(b-c)$.

Assume (b) holds. By the RTP, we need only consider midsets of the form $M(0, a)$, with $a \in S_X$. If $I(0, a)$ is linear, then $M(0, a)$ is a singleton and therefore convex. Alternatively, there exist distinct $b, c \in X$ with $a \in [\![b, c]\!] \setminus \{b, c\}$ and $[\![b, c]\!] \subseteq S_X$. By condition (b), we know $L(b-c) \cap S_X$ consists of extreme points of B_X. By Theorem 5.9, $M(0, a)$ is a nondegenerate line segment $[\![d, e]\!]$, which is parallel to $[\![b, c]\!]$. Hence $L(d-e)$ intersects S_X in extreme points of B_X; and we infer that $I(d, e)$ is linear, as is $I(x, y)$ for any $x, y \in I(d, e)$. Therefore $M(0, a)$ is convex. □

5.13. **Remark.** Let $n \geq 3$. Then radial symmetry holds for a regular n-sided polygon $B \subseteq \mathbb{R}^2$ centered at the origin if and only if n is even. In that case the Minkowski functional for B defines a norm $\|\cdot\|_B$ for which B is the unit ball (see Example 5.4). Then, by Theorem 5.12, $X = \langle \mathbb{R}^2, \|\cdot\|_B \rangle$ is M-convex if and only if $n/2$ is odd.

We now provide an affirmative answer to [2, Question 4.7].

5.14. **Theorem.** *Every normed plane is I-convex.*

Proof. Let $X = \langle \mathbb{R}^2, \|\cdot\| \rangle$ be a normed plane. To prove our assertion, we need only check convexity for nonlinear intervals; and by Theorem 5.9 (and the RTP), we need only consider intervals of the form $I(0, a)$, such that there exist extreme points p, q of B_X with $[\![p, q]\!] \subseteq S_X$ and $a \in [\![p, q]\!] \setminus \{p, q\}$.

Since a is not an end point of $[\![p, q]\!]$, we may fix $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, and $a = \alpha p + \beta q$. Fix distinct $b, c \in I(0, a)$. We need to show that $I(b, c) \subseteq I(0, a)$.

By way of notation: if $u, v, x, y \in X$, we let $\triangle(u, v, x)$ (resp., $\Diamond(u, v, x, y)$) be the triangle (resp., quadrilateral) with the indicated vectors as corners (Recall that these figures are said to be *nontrivial* if all indicated corners are distinct, and no three of them are collinear.) Then, by Theorem 5.9, $I(0, a) = \Diamond(0, \alpha p, \beta q, a)$ is a nontrivial parallelogram. As another bit of notation, we let $S + x$ be the translate $\{y + x : y \in S\}$ for $S \subseteq X$ and $x \in X$. And, as before, $L(x)$ denotes the vector subspace generated by nonzero vector x.

We first note that if $e \in I(0, a)$ then there are $0 \leq \alpha_e \leq \alpha$ and $0 \leq \beta_e \leq \beta$ such that $e = \alpha_e p + \beta_e q$. Then the quadrilateral $\Diamond(0, \alpha_e p, \beta_e q, e)$ is a (possibly trivial) parallelogram contained within the parallelogram $\Diamond(0, \alpha p, \beta q, a)$. Moreover, again by Theorem 5.9, $I(0, e) = \Diamond(0, \alpha_e p, \beta_e q, e)$.

To show $I(b, c) \subseteq I(0, a)$ for all $b, c \in I(0, a)$, we have two cases, depending on the orientation of $b - c$; i.e., whether or not $L(b-c)$ intersects $[\![p, q]\!]$. We observe that S_X, being radially symmetric and homeomorphic to a circle, may be written as $[\![p, q]\!] \cup A \cup [\![-q, -p]\!] \cup -A$, where A is an arc joining p and $-q$, and $-A := \{-x : x \in A\}$, the radial reflection of A, is an arc joining $-p$ and q. Furthermore, no two of these four arcs share any points other than a common end point, and any half-line emanating from the origin intersects S_X in exactly one point.

Since $b, c \in I(0,a)$, there exist $0 \le \alpha_b, \alpha_c \le \alpha$, $0 \le \beta_b, \beta_c \le \beta$ such that $b = \alpha_b p + \beta_b q$ and $c = \alpha_c p + \beta_c q$. Note that, by Proposition 3.1, we know $\|b-c\| \le \|a\| = 1$; so $b - c \in B_X$. We lose no generality in assuming that either $b - c \in \triangle(0,p,q)$ or $L(b-c) \cap (A \setminus \{p,q\}) \neq \emptyset$.

Case 1 [$b-c \in \triangle(0,p,q)$; i.e., $\alpha_c \le \alpha_b$ and $\beta_c \le \beta_b$]: Then $b-c = (\alpha_b-\alpha_c)p+(\beta_b-\beta_c)q \in I(0,a)$, and therefore $I(0,b-c) = \Diamond(0,(\alpha_b-\alpha_c)p,(\beta_b-\beta_c)q,b-c)$ is contained in $I(0,a)$. Now, $I(b,c) = I(0,b-c)+c$. So if $x \in I(b,c)$ then $x-c \in I(0,b-c)$. Thus there are $0 \le \alpha' \le \alpha_b - \alpha_c$ and $0 \le \beta' \le \beta_b - \beta_c$ such that $x - c = \alpha' p + \beta' q$. Hence $x = (\alpha'+\alpha_c)p + (\beta'+\beta_c)q \in I(0,a)$, since $0 \le \alpha'+\alpha_c \le \alpha_b \le \alpha$ and $0 \le \beta'+\beta_c \le \beta_b \le \beta$. Thus we infer $I(b,c) \subseteq I(0,a)$.

Case 2 [$L(b-c) \cap (A \setminus \{p,q\}) \neq \emptyset$; i.e., $\alpha_c \le \alpha_b$ and $\beta_c \ge \beta_b$]: Then $b-c = (\alpha_b-\alpha_c)p+(\beta_c-\beta_b)(-q)$. Let $P = \Diamond(0,(\alpha_b-\alpha_c)p,(\beta_c-\beta_b)(-q),b-c)$. Then P is a nontrivial parallelogram. We are done once we show: (1) $P + c \subseteq I(0,a)$; and (2) $I(0,b-c) \subseteq P$. For then we will have $I(b,c) = I(0,b-c)+c \subseteq P+c \subseteq I(0,a)$. Item (1) is treated much as above. If $x \in P + c$, then $x - c \in P$ and we have $0 \le \alpha' \le \alpha_b - \alpha_c$, $0 \le \beta' \le \beta_c - \beta_b$, so that $x - c = \alpha' p + \beta'(-q)$. Hence $x = (\alpha'+\alpha_c)p + (\beta_c - \beta')q \in I(0,a)$, since $0 \le \alpha'+\alpha_c \le \alpha_b \le \alpha$ and $0 \le \beta_c - \beta' \le \beta_c \le \beta$. This establishes (1).

To prove (2), we have $b-c = (\alpha_b-\alpha_c)p+(\beta_c-\beta_b)(-q)$. Let d be the unique point on $L(b-c) \cap A$. Then $d = \frac{b-c}{\|b-c\|}$. If d is an extreme point of B_X, then (Theorem 5.9) $I(b,c) = [\![b,c]\!]$; so, since $I(0,a)$ is linearly convex, we have $I(b,c) \subseteq I(0,a)$. Assume now that d is not an extreme point of B_X. Because we are in the plane, the face $F(d)$ of d in B_X is a nondegenerate line segment $[\![r,s]\!] \subseteq S_X$, where r and s are extreme points of B_X. Since no two such faces can have more than an end point in common, we know that $[\![r,s]\!] \subseteq A$.

For convenience, we linearly order the arc A so that $-q \le r < d < s \le p$. This ordering naturally induces a linear ordering of the line segments with the origin as one end point and a point of A as the other; i.e., the inequalities $[\![0,-q]\!] \le [\![0,r]\!] < [\![0,d]\!] < [\![0,s]\!] \le [\![0,p]\!]$ make sense.

Fix $0 \le \gamma, \delta \le 1$ such that $b - c = \gamma r + \delta s$. Then, by Theorem 5.9 and the RTP, $I(0,b-c)$ is the parallelogram $\Diamond(0,\gamma r,\delta s,b-c)$.

The rest of the argument is basic plane geometry: Since P is nontrivial, each of its sides is nondegenerate. Since the side $[\![(\beta_c-\beta_b)(-q),b-c]\!]$ is parallel to $[\![0,p]\!]$ and $[\![0,-q]\!] \le [\![0,r]\!] < [\![0,d]\!]$, there is a unique point in the side's intersection with $[\![0,r]\!]$. Fix $0 \le \gamma' \le 1$ such that $\gamma' r \in [\![0,r]\!] \cap [\![(\beta_c-\beta_b)(-q),b-c]\!]$. Similarly we may fix $0 \le \delta' \le 1$ such that $\delta' s \in [\![0,s]\!] \cap [\![(\alpha_b-\alpha_c)p,b-c]\!]$. Then, because $[\![0,r]\!] < [\![0,d]\!]$, and so $[\![\gamma r,b-c]\!]$ is a nondegenerate line segment parallel to $[\![0,s]\!]$, we have $\gamma \le \gamma'$; similarly we argue that $\delta \le \delta'$. Hence $I(0,b-c) = \Diamond(0,\gamma r,\delta s,b-c) \subseteq \Diamond(0,\gamma' r,\delta' s,b-c) \subseteq P$. This completes the proof. □

5.15. **Problem.** Characterize I-convexity and M-convexity in a normed vector space, in terms of the geometry of its unit ball.

The following gives one necessary and one sufficient "unit ball condition" for I-convexity, but one of these conditions is clearly weaker than the other. By way of notation: if A is a nondegenerate line segment that does not contain the origin of X, denote by P_A the unique plane containing $A \cup \{0\}$.

5.16. **Corollary.** *A normed vector space X is I-convex if all the non-extreme points of its unit ball B_X are coplanar. The space is not I-convex if there exist two nondegenerate line segments $A, B \subseteq S_X$ such that A is a face of B_X and $B \cap P_A$ consists of a single interior point of B.*

Proof. The first statement follows immediately from Theorem 5.14, plus the oft-noted fact that intervals $I(0, a)$, for $a \in S_X$, are linear if and only if a is an extreme point of B_X.

As for the second statement, fix an interior point $a \in A$, so that A is the face $F(a)$ of a in B_X. Then, by Theorem 5.9, $I(0, a)$ is a nontrivial parallelogram contained in P_A. Now suppose $B \cap P_A = \{d\}$, an interior point of B. Then, since $I(0, a)$ has nonempty interior in P_A, there exist distinct $b, c \in I(0, a)$ with $d \in L(b - c)$; i.e., $d = \frac{b-c}{\|b-c\|}$. Applying Theorem 5.9 to the plane P_B (with norm inherited from that of X), and using the RTP, we infer that $I(0, d) \cap P_B$ has nonempty interior in P_B. Hence $I(b, c)$ cannot lie in P_A, let alone in $I(0, a)$. □

5.17. **Remarks.**

(i) An example of a space illustrating the first assertion of Corollary 5.16 comes from [2, Example 4.12]: in $\mathbb{R}^3$, let $\|\cdot\|$ be the Minkowski functional (as in Example 5.4) for the "ball"

$$\{\langle x, y, z\rangle : \langle x, y\rangle \in [-1, 1]^2 \text{ and } z^2 \leq (1 - x^2)(1 - y^2)\}.$$

The non-extreme points of the unit ball lie on the xy-plane, and constitute the interiors of the sides of the square $[-1, 1]^2$.

(ii) In [2, Example 4.6] we give a rather unenlightening example of an interval in $\mathbb{R}^3_\infty$ that is not convex. From the second assertion of Corollary 5.16, we gain a better understanding of *why* this space is not I-convex.

6. Connectedness in topological IR-structures

A topological IR-structure is **I-connected** if each of its intervals is connected in the subspace topology. Similarly we define *R-connected*, *E-connected*, and *M-connected*, except that in the latter two cases we also require all equisets and midsets to be nonempty. We also define *I-arc-connected*, etc., using the same pattern.

6.1. **Proposition.** *A topological IR-structure is I-arc-connected if and only if it is tautly arc-connected.*

Proof. Taut arc-connectedness is a trivial consequence of I-arc-connectedness, so assume the underlying topological space X is tautly arc-connected, with $I(a, b)$ a nondegenerate interval in X. Fix distinct points $x, y \in I(a, b)$. Since IR-structures satisfy I-transitivity (I5), we have $I(a, x) \cup I(a, y) \subseteq I(a, b)$. Let $A \subseteq I(a, x)$ (resp., $B \subseteq I(a, y)$) be an arc joining a and x (resp., a and y). Then $A \cup B$ is a Peano continuum, and is hence arc-connected (see [16, Chapter VIII]). Since $x, y \in A \cup B$, this shows $I(a, b)$ to be arc-connected. □

It is immediate from the definition of topological connectedness that a space is connected (resp., arc-connected) if and only if each two of its points are contained in a connected (resp., arc-connected) subset. Hence connectedness (resp., arc-connectedness) in a topological IR-structure immediately follows from I-connectedness (resp., I-arc-connectedness). On the other hand, R-arc-connectedness does not imply connectedness, as the two-point discrete space attests. But this is the only counterexample.

6.2. **Theorem.** *An R-connected (resp., R-arc-connected) topological IR-structure is either connected (resp., arc-connected) or it is a two-point discrete space.*

Proof. Assume $\langle X, I, R\rangle$ is R-connected (resp., R-arc-connected). If X is not connected (resp., not arc-connected), then it has two distinct components (resp., arc components), A and B. Let $a \in A$ and $b \in B$. Then $R(a, b)$, being connected (resp., arc-connected) and containing a, must be contained in A. Likewise, $R(b, a) \subseteq B$. But, by dichotomy (R4), we have $R(a, b) = A$ and $R(b, a) = B$. Thus X has exactly two components (resp., arc components). If there were a third point $c \in X \setminus \{a, b\}$, say $c \in A$, then both $R(a, c)$ and $R(c, a)$ would have to be contained in A, contradicting dichotomy. Hence $X = \{a, b\}$. Since $R(a, b) = \{a\}$ and $R(b, a) = \{b\}$ are closed sets, the topology on X is discrete. □

6.3. **Remark.** E-connectedness and M-connectedness come up short in implying connectedness for topological−even metric−IR-structures. In the rational line $\mathbb{Q} \subseteq \mathbb{R}$, equisets are midsets, and all of them are singletons, but the space is totally disconnected. (The metric is far from complete, though, see Proposition 6.1.) There is another simple example that appears in [17]: Let X be the metric subspace of $\mathbb{R}^2_2$, whose underlying set is $\{\langle -1, 1\rangle\} \cup \{\langle x, x^2\rangle : x \geq 0\}$. Then X is a disconnected complete subspace, in which each equiset is a singleton. The IR-structure of X is *I-discrete*; i.e., each of its nondegenerate intervals is a doubleton. In particular, all midsets are empty.

E-connectedness in a metric space does imply connectedness if one further assumes that each equiset is nondegenerate [12, Theorem 2.1].

6.4. **Theorem.** *If a topological IR-structure is I-connected (resp., I-arc-connected), then it is also R-connected (resp., R-arc-connected), and all equisets are nonempty.*

Proof. Fix $a, b \in X$ and suppose $x, y \in R(a, b)$. Then, by I-connectedness and IR-transitivity (IR3), $I(a, x) \cup I(a, y)$ is a connected set that contains x and y, and is contained in $R(a, b)$. Thus the structure is R-connected. It is also connected, by virtue of being I-connected; therefore all equisets are nonempty, by Proposition 3.3. If we assume the structure to be I-arc-connected, we have arcs $A \subseteq I(a, x)$ and $B \subseteq I(a, y)$, as in the proof of Proposition 6.1. Then it is possible to find an arc in $R(a, b)$ that joins x and y. This shows the structure to be R-arc-connected. □

6.5. **Corollary.** *Every tautly arc-connected topological IR-structure is R-arc-connected.*

I-discreteness stands in stark contrast to taut arc-connectedness. The standard unit circle in the Euclidean plane, with its inherited metric, is an example of a metric IR-structure where: (1) the betweenness structure is I-discrete; and (2) each nearness region−being a closed semicircle− is arc-connected. Hence the hypothesis of Theorem 6.4 is hardly necessary. Indeed, we can make a general statement in this regard to show that R-connectedness/R-arc-connectedness does not imply I-connectedness/I-arc-connectedness.

6.6. **Proposition.** *On a given set X, every metric ϱ is topologically equivalent to a metric ϱ' such that: (1) the ϱ'-betweenness structure is I-discrete; and (2) for each $a, b \in X$, $R_\varrho(a,b) = R_{\varrho'}(a,b)$.*

Proof. Given $\langle X, \varrho\rangle$, define $\varrho'(x,y) := \sqrt{\varrho(x,y)})$. Then−see [1, Example 4.3 (i)] for details−ϱ' is topologically equivalent to ϱ and has I-discrete betweenness structure. Furthermore, since the square root function is increasing, $\varrho(x,a) \leq \varrho(x,b)$ holds if and only if $\varrho'(x,a) \leq \varrho'(x,b)$ does too. Hence both metrics give rise to the same comparative nearness structure. □

6.7. **Theorem.** *Normed vector spaces are I-, R-, E-, and M-arc-connected.*

Proof. Normed vector spaces are tautly arc-connected; so I-arc-connectedness and R-arc-connectedness follow from Proposition 6.1 and Corollary 6.5. Normed vector spaces are also M-arc-connected because their midsets are linearly convex, by Theorem 5.5 (ii). To prove E-arc-connectedness, fix normed vector space X, with $a, b \in X$ distinct, and $c = \frac{1}{2}(a+b)$ the halfway point of $[\![a,b]\!]$. In dimension one, all equisets are singletons; so we may assume X is at least two-dimensional. Then $E(a,b)$ is the union of the slices $E(a,b) \cap P$, as P ranges over all planes in X that contain $[\![a,b]\!]$. Since c is common to all such slices, we need only prove each slice is arc-connected. But by the main result of [9]−see the last paragraph of Example 5.1−equisets in normed planes (such as P) are either homeomorphic to $\mathbb{R}$ or are the union of two disjoint linearly convex sets and an arc intersecting each. In either case the equiset is arc-connected. □

6.8. **Remark.** An alternate proof of E-connectedness in Theorem 6.7 does not require the geometric intricacy of the argument in [9], and proceeds as follows to show each slice $E(a,b) \cap P$ is connected: For $n = 1, 2, \ldots$, let D_n be the closed disk in P, of radius n, and centered at c. Fix m large enough so that $a, b \in D_m$, and use the fact that D_m is linearly convex, plus the fact that $R(a,b)$ is linearly star-shaped about a, to infer that $R(a,b) \cap D_m$ is linearly star-shaped about a too. Hence $\{R(a,b) \cap D_m, R(b,a) \cap D_m\}$ is a cover of D_m by closed connected subsets. Finite-dimensional Euclidean balls are well-known to be *unicoherent*; i.e., continua such that whenever they are covered by two subcontinua, the intersection of those subcontinua is connected (see [10, §57,III]). Hence $E(a,b) \cap D_m = (R(a,b) \cap D_m) \cap (R(b,a)) \cap D_m)$ is connected. Since $E(a,b) \cap P = \bigcup_{n=m}^{\infty}(E(a,b) \cap D_n)$, an increasing union of connected sets, we conclude that $E(a,b) \cap P$ is connected, as desired.

One cannot readily adapt this proof to show $E(a,b)$ is arc-connected, even though we know each $R(a,b)$ is arc-connected. For example, let $X = [0,1] \times [-2,2] \subseteq \mathbb{R}^2_2$. Let $M = (\{0\} \times [-2,1]) \cup \{\langle x,y\rangle \in X : x > 0 \text{ and } y \leq \sin\frac{1}{x}\}$, $N = (\{0\} \times [-1,2]) \cup \{\langle x,y\rangle \in X : x > 0 \text{ and } y \geq \sin\frac{1}{x}\}$. Then X is a 2-cell, $X = M \cup N$, each of M, N is arc-connected, but $M \cap N$ is−while connected−not arc-connected. Perhaps one can prove the following: if a normed plane is the union of two star-shaped sets, then the intersection of those sets is arc-connected.

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