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## REDUCED COPRODUCTS OF COMPACT HAUSDORFF SPACES

PAUL BANKSTON

**Abstract.** By analyzing how one obtains the Stone space of the reduced product of an indexed collection of Boolean algebras from the Stone spaces of those algebras, we derive a topological construction, the “reduced coproduct”, which makes sense for indexed collections of arbitrary Tichonov spaces. When the filter in question is an ultrafilter, we show how the “ultracoproduct” can be obtained from the usual topological ultraproduct via a compactification process in the style of Wallman and Frink. We prove theorems dealing with the topological structure of reduced coproducts (especially ultracoproducts) and show in addition how one may use this construction to gain information about the category of compact Hausdorff spaces.

**§0. Introduction.** The study of reduced coproducts was initially motivated by the observation, made by several authors (see [9], [11] and [12] for more details and references), that the usual reduced product of finitary relational structures (in the sense of model theory [16]) can be viewed as a direct limit of cartesian products. This can now be easily translated into category-theoretic language, and we get the notion of “reduced product in a category”. The term “reduced coproduct”, then, simply refers to the reduced product in the dual (= opposite) of the category under consideration.

Ideal places to look for examples of reduced coproduct constructions are category dualities (in the sense of [25]) in which one of the participants is, say, an equational class (= variety) of finitary algebras (the duality theorems of Stone and of Pontryagin and van Kampen being particularly well known). We will be concerned in this paper with reduced coproducts of topological objects which have no additional distinguished structure. (The situation in the category of topological abelian groups is the topic of another report.)

With our viewpoint thus suitably restricted, there are two main lines of inquiry: the first asks how topological properties of a reduced coproduct are conditioned by topological properties of the factors and combinatorial properties of the filter used; the second line seeks to use reduced coproducts as a tool to answer questions of a

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category-theoretic or model-theoretic nature. By way of illustration, we show here that reduced coproducts of infinite spaces via a countably incomplete ultrafilter are never basically disconnected, although they are  $F$ -spaces whenever the factors are strongly 0-dimensional (see [19], [33] and [35] for terminology). On the other hand, B. Banaschewski [3] used a reduced coproduct argument to prove that the category  $KH$  of compact Hausdorff spaces and continuous maps cannot be category dual to any class of finitary relational structures (plus all homomorphisms) which is closed under cartesian powers and usual ultrapowers. (This answers a question raised in [9]. J. Rosický [31] independently solved the problem using different methods.) As a last illustration, the reduced coproduct construction in  $KH$  was used in [12] to answer questions concerning the preservation of the model-theoretic notion of elementary equivalence as we pass from one first-order representation of topological spaces to another; e.g. as we pass from the lattice of closed sets to the ring of continuous real-valued functions.

**§1. Basic notions.** Let  $\mathcal{A}$  be a category, let  $I$  be a set, let  $\langle A_i : i \in I \rangle$  be a family of  $\mathcal{A}$ -objects, and let  $\mathcal{F}$  be a filter of subsets of  $I$ . For each  $J \subseteq I$ , denote the  $\mathcal{A}$ -direct product by  $\prod_J^{\mathcal{A}} A_i$ ; and for each pair of subsets  $J, K \subseteq I$  with  $J \supseteq K$ , let  $\pi_{JK}$  be the canonical projection morphism from  $\prod_J^{\mathcal{A}} A_i$  to  $\prod_K^{\mathcal{A}} A_i$ . The set  $\mathcal{F}$  is directed under reverse inclusion; the resulting direct limit, when it exists, is the  $\mathcal{A}$ -reduced product and is denoted  $\prod_{\mathcal{F}}^{\mathcal{A}} A_i$ . When we wish to deal with the  $\mathcal{A}$ -reduced coproduct (i.e. the reduced product in the opposite category), we use the “inverse limit of coproducts” recipe and denote this object by  $\sum_{\mathcal{F}}^{\mathcal{A}} A_i$ .

When one takes a class  $\mathcal{K}$  of relational structures of the same finitary type and considers it as a category by throwing in all functions which preserve the atomic relations (i.e. the homomorphisms), then  $\mathcal{K}$ -reduced products will be the usual ones provided  $\mathcal{K}$  is closed under cartesian products (i.e. is a “ $P$ -class”) as well as the relevant usual reduced products. For example, if  $\mathcal{K}$  is an elementary  $P$ -class [16] then  $\mathcal{K}$ -ultraproducts are the usual ones. (If  $\mathcal{K}$  is a Horn class then all  $\mathcal{K}$ -reduced products are usual.)

By contrast,  $\mathcal{K}$ -reduced coproducts can be pathological, even when  $\mathcal{K}$  is a reasonable class of algebras or relational structures; quite often they are “trivial” (see [11]). However,  $\mathcal{A}$ -reduced coproducts turn out to have interesting properties when  $\mathcal{A}$  is a suitable “topological” category.

To motivate the approach we take, let us attempt to define the term “topological reduced coproduct”. Letting  $TOP$  denote the category of topological spaces and continuous maps, we note that coproducts in  $TOP$  are disjoint unions. It is not hard to see, then, that whenever  $\mathcal{F}$  is a “free” filter (i.e.  $\bigcap \mathcal{F} = \emptyset$ ),  $\sum_{\mathcal{F}}^{TOP} X_i$  is not very interesting because it is always empty. Being more judicious in our second attempt, we are guided by the assurance that, since reduced products of Boolean algebras are nontrivial, so too are reduced coproducts of Boolean (= totally disconnected compact Hausdorff) spaces (thanks to Stone duality). Thus let  $BS$  be the category of Boolean spaces. In this category, as well as in the larger category  $KH$  of compact Hausdorff spaces, one forms coproducts by taking the Stone-Ćech compactification of disjoint unions. Indeed,  $\sum_{\mathcal{F}}^{KH} X_i$  is  $\varprojlim \{ \beta(\bigcup_J X_i) : J \in \mathcal{F} \}$ , which is the closed subspace of  $\beta(\bigcup_I X_i)$  consisting of all ultrafilters  $p$  of zero-sets from

$\bigcup_I X_i (= \bigcup_I X_i \times \{i\})$  which “extend”  $\mathcal{F}$ , in the sense that  $\bigcup_I X_i \in p$  whenever  $J \in \mathcal{F}$ . (See [19] and [33] for information on compactification theory.)

We will adopt this as our definition of the topological reduced coproduct; the reader will no doubt observe that this definition makes sense when applied to any family of Tichonov spaces. The result is always compact Hausdorff (and is in fact Boolean whenever the factors are strongly 0-dimensional).

An alternative description of  $\sum_{\mathcal{F}} X_i$  (we drop the superscript) may be helpful. Assume the spaces  $X_i$  to be nonempty, and let  $\pi: \bigcup_I X_i \rightarrow I$  be projection onto the discrete space  $I$ . We then get the Stone-Čech lifting  $\bar{\pi} = \beta(\pi)$ . If we identify a filter  $\mathcal{F}$  on  $I$  with the closed subset of  $\beta(I)$  consisting of all ultrafilters on  $I$  which extend  $\mathcal{F}$ , then  $\sum_{\mathcal{F}} X_i$  is just the inverse image  $\bar{\pi}^{-1}[\mathcal{F}]$ . For any two filters  $\mathcal{F}$  and  $\mathcal{G}$ , if  $\mathcal{F} \subseteq \mathcal{G}$  then  $\sum_{\mathcal{F}} X_i \supseteq \sum_{\mathcal{G}} X_i$ ; on the other hand, if there is a  $J \subseteq I$  with  $J \in \mathcal{F}$  and  $I \setminus J \in \mathcal{G}$  then  $\sum_{\mathcal{F}} X_i \cap \sum_{\mathcal{G}} X_i = \emptyset$ . In particular the ultracoproducts (being the point-inverses of  $\bar{\pi}$ ) form a partition of  $\beta(\bigcup_I X_i)$  into  $|\beta(I)| = \exp^2(|I|)$  blocks. (Notation:  $|S|$  is the cardinality of the set  $S$ ,  $\mathbf{P}(S)$  is the power set of  $S$ ,  $\exp(\alpha) = |\mathbf{P}(\alpha)|$  for any cardinal  $\alpha$ , and  $\exp^n(\alpha)$  is the  $n$ -fold iterate of  $\exp(\alpha)$ .  $\alpha^+$  denotes the cardinal successor of  $\alpha$ ;  $\omega_1 = \omega^+$ , where  $\omega$  is the first infinite cardinal.)

Here are some items of specialized notation which will be of use later on. (i) If  $S_i \subseteq X_i$  for each  $i \in I$ , let  $\sigma_{\mathcal{F}} S_i$  denote  $\{p \in \sum_{\mathcal{F}} X_i: \bigcup_I S_i \text{ contains a member of } p\}$ . Then the topology on  $\sum_{\mathcal{F}} X_i$ , inherited from that of  $\beta(\bigcup_I X_i)$ , has a closed basis consisting of sets of the form  $\sigma_{\mathcal{F}} Z_i$ , where each  $Z_i \subseteq X_i$  is a zero-set. (ii) When  $X_i = X$  for all  $i \in I$ , the reduced copower is a subset of  $\beta(X \times I)$ , and is denoted  $\sum_{\mathcal{F}} X$ . If  $X$  is compact, there is a canonical “codiagonal” map  $V = V_{\mathcal{F}, X}: \sum_{\mathcal{F}} X \rightarrow X$ , given by the rule:  $V(p) = x$  iff  $U \times I$  contains a member of  $p$  for each open set  $U$  containing  $x$ .  $V$  is evidently a continuous surjection, and is the restriction to  $\sum_{\mathcal{F}} X$  of the Stone-Čech lifting of the projection  $X \times I \rightarrow X$ .

For the most part in the sequel, we will confine our attention to topological reduced products  $\sum_{\mathcal{D}} X_i$  in which  $\mathcal{D}$  is an ultrafilter. All spaces are assumed to be Tichonov. The resulting theory is greatly enriched because of the intimate connection with topological ultraproducts (as studied in [6], [7], [8] and [10]), as well as with classical model theory (see [12] and [13]). In particular,  $\sum_{\mathcal{D}} X_i$  is a Wallman-Frink style compactification of the corresponding ultraproduct, as we describe presently.

Let  $\langle X_i: i \in I \rangle$  be any family of topological spaces and let  $\mathcal{D}$  be an ultrafilter on  $I$ . The *topological ultraproduct*  $\prod_{\mathcal{D}} X_i$  is the space whose points are  $\mathcal{D}$ -equivalence classes of functions  $f \in \prod_I X_i$  (i.e.  $[f] = [f]_{\mathcal{D}} = \{g \in \prod_I X_i: \{i: f(i) = g(i)\} \in \mathcal{D}\}$ ), and whose open (closed) sets are basically generated by “open (closed) ultraboxes”  $\prod_{\mathcal{D}} M_i$ , where  $M_i$  is open (closed) in  $X_i$ . It is easy to verify that if  $\mathcal{B}_i$  is an open (closed) basis for  $X_i$ ,  $i \in I$ , then ultraboxes  $\prod_{\mathcal{D}} B_i$ ,  $B_i \in \mathcal{B}_i$ , generate the ultraproduct topology in the appropriate sense.

**1.1. REMARK.** The topological ultraproduct  $\prod_{\mathcal{D}} X_i$  is a quotient of the box product  $\prod_I X_i$ , not the usual (Tichonov) product [6]. There is no “reasonable” category  $\mathcal{A}$ , whose objects are topological spaces, which admits the topological ultraproduct as an  $\mathcal{A}$ -ultraproduct.

For a space  $X$ , we let  $F(X)$  (resp.  $Z(X), B(X)$ ) denote the lattice of closed (resp. zero-, clopen) subsets of  $X$ . When  $X$  is Tichonov,  $Z(X)$  is a “normal” basis of closed sets (in the sense of O. Frink [33]). If  $\langle X_i: i \in I \rangle$  is a family and  $\mathcal{D}$  is an ultrafilter, the

lattice ultraproduct  $\prod_{\mathcal{D}} Z(X_i)$  is, in an obvious sense, a normal basis for the topological ultraproduct. Hence  $\prod_{\mathcal{D}} X$  is Tichonov, and we can form the corresponding Wallman-Frink compactification  $\omega(\prod_{\mathcal{D}} Z(X_i))$  (i.e. points of  $\omega(\mathcal{N})$ , for any normal basis  $\mathcal{N}$  of  $X$ , are  $N$ -ultrafilters; and closed sets are generated by sets  $N^\#$ ,  $N \in \mathcal{N}$ , where, for any  $S \subseteq X$ ,  $S^\# = \{p \in \omega(\mathcal{N}): S \text{ contains a member of } p\}$ ).

**1.2. PROPOSITION** [12].  $\sum_{\mathcal{D}} X_i$  is naturally homeomorphic with  $\omega(\prod_{\mathcal{D}} Z(X_i))$ , sets of the form  $\sigma_{\mathcal{D}} M_i$  being identified with  $(\prod_{\mathcal{D}} M_i)^\#$  under this homeomorphism.

**1.3. REMARK.** The above proposition is false without the assumption that  $\mathcal{D}$  is maximal. For let  $\mathcal{F}$  be a nonmaximal filter and let each  $X_i$  be a singleton space. No matter how one defines the topological reduced product topology on the set  $\prod_{\mathcal{F}} X_i$ , the resulting space is again a singleton. But the reduced coproduct, being in one-one correspondence with the set of ultrafilters on  $I$  which extend  $\mathcal{F}$ , has a plurality of points. (This number will either be finite  $> 1$  or  $\exp^2(|I|)$ , since an infinite closed subset of  $\beta(I)$  is equinumerous with  $\beta(I)$  [19].)

**1.4. PROPOSITION.** The ultracoproduct  $\sum_{\mathcal{D}} X_i$  is infinite iff for each  $n < \omega$ ,  $\{i: |X_i| > n\} \in \mathcal{D}$ .

**PROOF.** If  $\sum_{\mathcal{D}} X_i$  is infinite then, by (1.2), so is  $\prod_{\mathcal{D}} X_i$ . By standard results about ultraproducts,  $\{i: |X_i| > n\} \in \mathcal{D}$  for each  $n < \omega$ . The converse is just as easy. ■

**1.5. PROPOSITION** [9, LEMMA 4.6]. The clopen algebra of the ultracoproduct is isomorphic with the ultraproduct of the clopen algebras (in symbols,  $B(\sum_{\mathcal{D}} X_i) \cong \prod_{\mathcal{D}} B(X_i)$ ). In particular,  $\sum_{\mathcal{D}} X_i$  is connected iff  $\{i: X_i \text{ is connected}\} \in \mathcal{D}$ .

Suppose, for each  $i \in I$ , that  $\theta_i: X_i \rightarrow Y_i$  is continuous. Then so are the ultraproduct map  $\prod_{\mathcal{D}} \theta_i$  (defined by  $[f]_{\mathcal{D}} \rightarrow [\langle \theta_i(f(i)): i \in I \rangle]_{\mathcal{D}}$ ) and the ultracoproduct map  $\sum_{\mathcal{D}} \theta_i$  (defined by  $\prod_{\mathcal{D}} Z_i \in (\sum_{\mathcal{D}} \theta_i)(p)$  iff  $p \in (\prod_{\mathcal{D}} \theta_i^{-1}[U_i])^\#$  whenever  $\prod_{\mathcal{D}} U_i$  is an ultrabox of cozero-sets containing  $\prod_{\mathcal{D}} Z_i$ ). (The definition of  $\sum_{\mathcal{D}} \theta_i$  in [12] is incorrect. However, it amounts to a typographic error in that neither the truth of Lemma 3.1 nor its proof are affected.) Moreover  $\sum_{\mathcal{D}} \theta_i$  extends  $\prod_{\mathcal{D}} \theta_i$  in the obvious sense. (All the standard properties of functions (e.g. injectivity, surjectivity, homeomorphism) pass from the factor maps to the ultraproduct and ultracoproduct maps, with the following exception:  $\sum_{\mathcal{D}} \theta_i$  need not be one-one just because the maps  $\theta_i$  are. One needs to assume in addition that the spaces are normal and the maps are closed.)

The next result tells us that ultracoproducts of noncompact spaces give us nothing new.

**1.6. PROPOSITION** [12, LEMMA 3.1]. If, for each  $i \in I$ ,  $\eta_i: X_i \rightarrow \beta(X_i)$  is the compactification embedding then  $\sum_{\mathcal{D}} \eta_i: \sum_{\mathcal{D}} X_i \rightarrow \sum_{\mathcal{D}} \beta(X_i)$  is a homeomorphism.

With the aid of (1.5) and (1.6), one can prove the following.

**1.7. PROPOSITION** [12, LEMMA 3.3].  $\sum_{\mathcal{D}} X_i$  is Boolean iff  $\{i: X_i \text{ is strongly 0-dimensional}\} \in \mathcal{D}$ . Moreover, if  $\mathcal{D}$  is countably incomplete (i.e.  $\bigcap \mathcal{S} = \emptyset$  for some countable  $\mathcal{S} \subseteq \mathcal{D}$ ) and  $\{i: |X_i| > n\} \in \mathcal{D}$  for each  $n < \omega$ , then  $\sum_{\mathcal{D}} X_i$  cannot be "basically disconnected" (i.e. where closures of cozero-sets are open).

This last result can be used further to show that the topological ultracoproduct is "almost never" the Stone-Ćech compactification of the topological ultraproduct.

**1.8. PROPOSITION.** If  $\mathcal{D}$  is countably incomplete and  $\{i: |X_i| > n\} \in \mathcal{D}$  for each  $n < \omega$  then  $\sum_{\mathcal{D}} X_i$  and  $\beta(\prod_{\mathcal{D}} X_i)$  are not homeomorphic. In fact, their lattices of zero-sets are distinguishable via a first-order sentence in the language of partial orderings.

PROOF. By a fundamental result [6, Theorem 4.1],  $\prod_{\mathcal{D}} X_i$  is a “ $P$ -space” (i.e. intersections of countable families of open sets are open) and is hence basically disconnected [19, 4K.7]. Thus  $\beta(\prod_{\mathcal{D}} X_i)$  is also basically disconnected [19, 6M.1]. By (1.7), then,  $\beta(\sum_{\mathcal{D}} X_i) \neq \sum_{\mathcal{D}} X_i$ . The property of basic disconnectedness is easily shown to be expressible as a first-order sentence in the language of partial orderings (or lattices). ■

1.9. REMARK.  $\beta(\prod_{\mathcal{D}} X_i)$  and  $\sum_{\mathcal{D}} X_i$  need not even have the same cardinality. Let  $|X| = |I| = \alpha$  be infinite, and let  $X$  be discrete. If  $\mathcal{D}$  is an “ $\alpha$ -regular” ultrafilter on  $I$  (i.e. there is a subset  $\mathcal{S} \subseteq \mathcal{D}$  of cardinality  $\alpha$  such that  $\bigcap \mathcal{S}_0 = \emptyset$  for every infinite  $\mathcal{S}_0 \subseteq \mathcal{S}$ ), then

$$(\exp^2(|X|))^{|I|} = |\prod_{\mathcal{D}} \beta(X)| \leq |\sum_{\mathcal{D}} X| \leq |\beta(X \times I)| = \exp^2(|X|),$$

i.e.  $|\sum_{\mathcal{D}} X| = \exp^2(\alpha)$ . On the other hand,  $|\beta(\prod_{\mathcal{D}} X)| = \exp^2(|X|^{|I|}) = \exp^3(\alpha)$ .

An important (but easy) result which further links ultracoproducts with model theory is the following.

1.10. PROPOSITION [12, LEMMA 3.2]. *Suppose  $\theta: \prod_{\mathcal{D}} X_i \rightarrow \prod_{\mathcal{D}} Y_i$  is a homeomorphism which pairs up ultraboxes of zero-sets (i.e.  $\theta$  is a lattice isomorphism between  $\prod_{\mathcal{D}} Z(X_i)$  and  $\prod_{\mathcal{D}} Z(Y_i)$ ). Then  $\theta$  extends to a homeomorphism between  $\sum_{\mathcal{D}} X_i$  and  $\sum_{\mathcal{D}} Y_i$ . If the spaces in question are normal and  $\theta$  pairs up ultraboxes of closed sets, then the same conclusion obtains.*

1.11. REMARK. That  $\theta$  in (1.10) need be more than simply a homeomorphism is borne out by the following simple argument. By [6, A2.6], if  $X$  and  $Y$  are any two nonempty regular  $T_1$  spaces which are “self-dense” (i.e. with no isolated points) then there is a homeomorphism  $\theta: \prod_{\mathcal{D}} X \rightarrow \prod_{\mathcal{D}} Y$  for some ultrafilter  $\mathcal{D}$ . If we pick, say,  $X$  connected and  $Y$  disconnected then (1.5) tells us that  $\sum_{\mathcal{D}} X$  and  $\sum_{\mathcal{D}} Y$  can never be homeomorphic.

1.12. QUESTION. Suppose  $\sum_{\mathcal{D}} X \simeq \sum_{\mathcal{D}} Y$ . Does it follow that  $\prod_{\mathcal{D}} X \simeq \prod_{\mathcal{D}} Y$ ? Does it follow that  $\prod_{\mathcal{E}} X \simeq \prod_{\mathcal{F}} Y$  for some ultrafilters  $\mathcal{E}$  and  $\mathcal{F}$ ?

REMARK. If we assume the continuum hypothesis (C.H.) then the answer to the first question is no. For let  $X = \mathbf{2}^\omega = \{0, 1\}^\omega$  be the countable Tichonov power of the two-point discrete space, let  $Y = \mathbf{2}^{\omega_1}$ , and let  $\mathcal{D}$  be a free ultrafilter on a countable set. Then  $|\prod_{\mathcal{D}} X| = c = \exp(\omega)$ , and  $|\prod_{\mathcal{D}} Y| = \exp(\omega_1) = \exp(c) > c$ . However  $B(X)$  and  $B(Y)$ , being atomless Boolean algebras, are elementarily equivalent of cardinalities  $\omega$  and  $\omega_1$  respectively. Thus  $\prod_{\mathcal{D}} B(X)$  and  $\prod_{\mathcal{D}} B(Y)$  are elementarily equivalent  $\omega_1$ -saturated algebras of cardinality  $c = \omega_1$  (see [16]). Hence they are isomorphic, and  $\sum_{\mathcal{D}} X \simeq \sum_{\mathcal{D}} Y$ .

In the theory of topological ultrapowers [6], the diagonal map  $\Delta = \Delta_{\mathcal{D}, X}: X \rightarrow \prod_{\mathcal{D}} X$ , given by assigning to  $x \in X$  the  $\mathcal{D}$ -equivalence class of the “constantly  $x$ ” map, is generally closed and one-one but not continuous. (Model-theoretically,  $\Delta$  is an elementary embedding by the classic (Łoś) ultraproduct theorem. Continuity in this setting can fail because the image  $\Delta[X]$  in  $\prod_{\mathcal{D}} X$  will carry the discrete topology whenever  $X$  is first countable and  $\mathcal{D}$  is countably incomplete.) Since  $\Delta^{-1}[\prod_{\mathcal{D}} U_i] = \bigcup_{J \in \mathcal{D}} \bigcap_{i \in J} U_i$ , we can infer that  $\Delta$  will be an embedding provided  $X$  is a “ $P_\kappa$ -space” for  $\kappa > |I|$  (i.e. intersections of  $< \kappa$  open sets are open).

If  $X$  is compact Hausdorff, however,  $X$  cannot even be a  $P$ -space (=  $P_{\omega_1}$ -space) without being finite. But in this case the good news is that there is a continuous left-

inverse  $\Lambda = \Lambda_{\mathcal{D}, X}: \prod_{\mathcal{D}} X \rightarrow X$  for  $\Delta$ , given by the condition that  $x = \Lambda([f])$  iff  $[f] \in \prod_{\mathcal{D}} U$  for every open neighborhood  $U$  of  $x$ . ( $\Lambda$  is the “standard part” map, in the context of nonstandard topology [30]; the ultrapower topology is the “ $Q$ -topology” in that setting.) In light of (1.2), it is easy to prove the following.

**1.13. PROPOSITION.** *Let  $X$  be compact Hausdorff. Then  $V: \sum_{\mathcal{D}} X \rightarrow X$  extends  $\Lambda$ . Moreover,  $\Delta$  is a homeomorphism iff  $V$  is a homeomorphism.*

**1.14. COROLLARY.** *If  $X$  is a Tichonov space, of cardinality less than the first measurable cardinal, and  $\mathcal{D}$  is countably complete (i.e.  $\bigcap \mathcal{S} \in \mathcal{D}$  whenever  $\mathcal{S} \subseteq \mathcal{D}$  and  $|\mathcal{S}| \leq \omega$ ) then  $\sum_{\mathcal{D}} X \simeq \beta(X)$ .*

**PROOF.** Use (1.13) and (1.6), plus standard facts concerning ultraproducts via measures. ■

**§2. Topological properties of ultraproducts.**

**2.1. Iteration.** We next deal with iterated ultraproducts; these results will prove useful in later sections. By way of preliminary discussion, let  $X$  be any Tichonov space with normal bases  $\mathcal{M} \subseteq \mathcal{N}$ . We say that  $\mathcal{M}$  weakly separates  $\mathcal{N}$  if whenever we have two disjoint members of  $\mathcal{N}$ , there is a member of  $\mathcal{M}$  which contains one and is disjoint from the other. (For example,  $Z(X)$  weakly separates  $F(X)$  whenever  $X$  is normal.) Let  $p \in \omega(\mathcal{N})$ . It is easy to see that  $p \cap \mathcal{M} \in \omega(\mathcal{M})$ , and that the mapping  $\rho: \omega(\mathcal{N}) \rightarrow \omega(\mathcal{M})$ , taking  $p$  to  $p \cap \mathcal{M}$ , is a continuous surjection. If  $\mathcal{M}$  weakly separates  $\mathcal{N}$  then  $\rho$  is a homeomorphism (so  $\beta(X) = \omega(F(X))$  when  $X$  is normal). Now suppose  $\mathcal{N}_i$  is a normal basis for  $X_i$  extending  $Z(X_i)$ ,  $i \in I$ . Then disjoint members of  $\mathcal{N}_i$  are actually separable via disjoint zero-sets. Thus  $\rho: \omega(\prod_{\mathcal{D}} \mathcal{N}_i) \rightarrow \sum_{\mathcal{D}} X_i$  is a homeomorphism. In particular it will be convenient to speak of the points of  $\sum_{\mathcal{D}} X_i$ , for normal  $X_i$ , as ultrafilters of closed ultraboxes.

Back to iteration, let  $\mathcal{D}$  be an ultrafilter on  $I$ ; and for each  $i \in I$ , let  $\mathcal{E}_i$  be an ultrafilter on  $J_i$ . Let  $K = \bigcup_{i \in I} (\{i\} \times J_i)$  and let  $\mathcal{F} = \bigoplus_{\mathcal{D}} \mathcal{E}_i = \{R \subseteq K: \{i: \langle i, j \rangle \in R\} \in \mathcal{E}_i\} \in \mathcal{D}$ . It is well known [16] that  $\mathcal{F}$  is an ultrafilter on  $K$ ; moreover if  $\langle A_{ij}: \langle i, j \rangle \in K \rangle$  is any family of relational structures of the same type then the mapping  $\eta: \prod_{\mathcal{F}} A_{ij} \rightarrow \prod_{\mathcal{D}} \prod_{\mathcal{E}_i} A_{ij}$ , given by the obvious assignment from  $\prod_K A_{ij}$  to  $\prod_I \prod_{J_i} A_{ij}$  (i.e.  $[f]_{\mathcal{F}} \rightarrow [g]_{\mathcal{D}}$ , where  $g(i) = [h_i]_{\mathcal{E}_i}$  and  $h_i(j) = f(i, j)$ ), is an isomorphism. If  $J_i = J$  and  $\mathcal{E}_i = \mathcal{E}$  for each  $i \in I$  then  $\mathcal{F}$  is written  $\mathcal{D} \cdot \mathcal{E}$ . In the case of iterated ultrapowers, we also get that the isomorphism  $\eta$  commutes with diagonal maps:  $\eta \circ \Delta_{\mathcal{D} \cdot \mathcal{E}, X} = \Delta_{\mathcal{D}, \prod_{\mathcal{E}} X} \circ \Delta_{\mathcal{E}, X}$ . The dual version of all this clearly works for Boolean spaces because of Stone duality. It also works in general.

**2.1.1. THEOREM.** (i) *With the above notation, there is a canonical homeomorphism  $\gamma: \sum_{\mathcal{D}} \sum_{\mathcal{E}_i} X_{ij} \rightarrow \sum_{\mathcal{F}} X_{ij}$  for arbitrary Tichonov spaces  $\langle X_{ij}: \langle i, j \rangle \in K \rangle$ .*

(ii) *If  $X \in KH$  then the canonical homeomorphism  $\gamma: \sum_{\mathcal{D}} \sum_{\mathcal{E}} X \rightarrow \sum_{\mathcal{D} \cdot \mathcal{E}} X$  commutes with codiagonal maps:  $V_{\mathcal{D} \cdot \mathcal{E}, X} \circ \gamma = V_{\mathcal{E}, X} \circ V_{\mathcal{D}, \sum_{\mathcal{E}} X}$ .*

(iii) *Let  $X \in KH$  and let  $\mathcal{D}, \mathcal{E}$  be two ultrafilters. Then there is an ultrafilter  $\mathcal{F}$  and a homeomorphism  $\delta: \sum_{\mathcal{F} \cdot \mathcal{D}} X \rightarrow \sum_{\mathcal{F} \cdot \mathcal{E}} X$  such that  $V_{\mathcal{F} \cdot \mathcal{E}, X} \circ \delta = V_{\mathcal{F} \cdot \mathcal{D}, X}$ .*

**PROOF.** Thanks to (1.6) we can assume all spaces to be compact Hausdorff. To prove (i), let  $\eta: \prod_{\mathcal{D}} \prod_{\mathcal{E}_i} X_{ij} \rightarrow \prod_{\mathcal{F}} X_{ij}$  be the (inverse of the) natural homeomorphism defined above. (Properly speaking,  $\eta$  arises from an isomorphism between lattices.) We will extend  $\eta$  to a homeomorphism  $\gamma: \sum_{\mathcal{D}} \sum_{\mathcal{E}_i} X_{ij} \rightarrow \sum_{\mathcal{F}} X_{ij}$ . Now by our preliminary remarks,  $\sum_{\mathcal{F}} X_{ij} = \omega(\prod_{\mathcal{F}} F(X_{ij}))$ . Identify a closed ultrabox  $\prod_{\mathcal{F}} C_{ij}$  with

$\prod_{\mathcal{Q}} \prod_{\mathcal{E}_i} C_{ij}$ ; and for each  $i \in I$ , let  $\mathcal{N}_i = \{(\prod_{\mathcal{E}_i} C_{ij})^\# : C_{ij} \in F(X_{ij})\}$ . Then  $\mathcal{N}_i \subseteq F(\sum_{\mathcal{E}_i} X_{ij})$ , both families are normal bases, and disjoint closed subsets of  $\sum_{\mathcal{E}_i} X_{ij}$  are separable via disjoint members of  $\mathcal{N}_i$ . Therefore  $\prod_{\mathcal{Q}} \mathcal{N}_i$  weakly separates  $\prod_{\mathcal{Q}} F(\sum_{\mathcal{E}_i} X_{ij})$ . This means that the natural map  $\rho: \omega(\prod_{\mathcal{Q}} F(\sum_{\mathcal{E}_i} X_{ij})) \rightarrow \omega(\prod_{\mathcal{Q}} \mathcal{N}_i)$  is a homeomorphism. Now the domain of  $\rho$  is just  $\sum_{\mathcal{Q}} \sum_{\mathcal{E}_i} X_{ij}$ , so we need to find a natural homeomorphism connecting  $\omega(\prod_{\mathcal{Q}} \mathcal{N}_i)$  and  $\sum_{\mathcal{F}} X_{ij}$ . Indeed, for  $p \in \omega(\prod_{\mathcal{Q}} \mathcal{N}_i)$  let  $v(p) = \{\prod_{\mathcal{F}} C_{ij} : \prod_{\mathcal{Q}} (\prod_{\mathcal{E}_i} C_{ij})^\# \in p\}$ . This is easily seen to be the homeomorphism we want, and we set  $\gamma = v \circ \rho$ . That  $\gamma$  extends  $\eta$  is also an easy exercise.

(ii) This follows easily from the definition of  $\gamma$ .

(iii) This is essentially a model-theoretic proof. Suppose  $A, B$ , and  $C$  are arbitrary relational structures of the same type, and suppose  $\alpha: A \rightarrow B$  and  $\beta: A \rightarrow C$  are elementary embeddings. By expanding the structures  $B$  and  $C$  with constants from  $A$ , we infer that the new structures  $(B, \alpha(a))_{a \in A}$  and  $(C, \beta(a))_{a \in A}$  are elementarily equivalent. By the (Keisler-Shelah) ultrapower theorem [16], we obtain an ultrafilter  $\mathcal{F}$  and an isomorphism  $\varepsilon: \prod_{\mathcal{F}} (B, \alpha(a))_{a \in A} \rightarrow \prod_{\mathcal{F}} (C, \beta(a))_{a \in A}$ . But this amounts to an isomorphism between  $\prod_{\mathcal{F}} B$  and  $\prod_{\mathcal{F}} C$  which commutes with the diagonal maps composed with  $\alpha$  and  $\beta$  (i.e.  $\varepsilon \circ \Delta_{\mathcal{F}, B} \circ \alpha = \Delta_{\mathcal{F}, C} \circ \beta$ ).

Now apply this observation to the case in which  $A = F(X)$ ,  $B = \prod_{\mathcal{Q}} F(X)$ ,  $C = \prod_{\mathcal{E}} F(X)$ ,  $\alpha = \Delta_{\mathcal{Q}, F(X)}$  and  $\beta = \Delta_{\mathcal{E}, F(X)}$ . Using the above remarks plus what we know about iteration, we can establish an isomorphism  $\varepsilon: \prod_{\mathcal{F}} F(X) \rightarrow \prod_{\mathcal{F}, \mathcal{E}} F(X)$  so that  $\varepsilon \circ \Delta_{\mathcal{F}, \mathcal{Q}, F(X)} = \Delta_{\mathcal{F}, \mathcal{E}, F(X)}$ . The way in which ultracoproducts are constructed immediately gives the result: just apply (1.10) and (i), (ii) above. ■

**2.2. Dimension.** We now turn to the issue of dimension of ultrapowers of compact Hausdorff spaces. The three most important dimension functions (see [28]) are *weak inductive* (or *small inductive* or *Urysohn-Menger*), denoted by  $\text{ind}$ ; *strong inductive* (or *large inductive* or *Čech*), denoted by  $\text{Ind}$ ; and *covering* (or *Lebesgue*), denoted by  $\text{dim}$ . All three functions agree on separable metric spaces, only the second two agree on arbitrary metric spaces, and there seems to be no general relationship (aside from inequalities) among the three for any other reasonable class of spaces.

We will deal with finite dimension; the empty space will have dimension  $-1$ . Given a space  $X$ , we let  $\text{Cl}(S) = \text{Cl}_X(S)$  be the closure of  $S$  in  $X$ . Let  $n$  be a natural number. We say “ $\text{ind}(X) \leq n$ ” if, for each  $x \in X$  and open  $U$  containing  $x$ , there is an open  $V \subseteq U$  containing  $x$  such that  $\text{ind}(\text{Fr}(V)) (= \text{ind}(\text{Cl}(V) \setminus V)) \leq n - 1$ . We say “ $\text{Ind}(X) \leq n$ ” if in the above definition we can replace the point  $x$  with an arbitrary closed set. We say “ $\text{dim}(X) \leq n$ ” if for any finite open covering  $\mathcal{U}$ , there is an open covering  $\mathcal{V}$  such that  $\mathcal{V}$  “refines”  $\mathcal{U}$  (i.e. each  $V \in \mathcal{V}$  is contained in some  $U \in \mathcal{U}$ ) and  $\mathcal{V}$  has “order”  $\leq n + 1$  (i.e. for each  $x \in X$ ,  $|\{V \in \mathcal{V} : x \in V\}| \leq n + 1$ ). If  $d$  is any dimension function, we say  $d(X) = n$  if  $d(X) \leq n$  and it is not the case that  $d(X) \leq n - 1$ . If  $d(X) \not\leq n$  for all  $n < \omega$ , we put  $d(X) = \infty$ .

**2.2.1. REMARKS.** (i) A slightly modified definition of  $\text{dim}$  is given in [19] and concerns only covers by cozero sets. It agrees with the usual definition for normal spaces, and has the attractive feature that  $\text{dim}(X) = \text{dim}(\beta(X))$  for any Tichonov space  $X$ . (The equality for normal  $X$  is an old theorem of Vedenisov [28].)

(ii) If  $\mathcal{U} = \{U_1, \dots, U_k\}$  is an open cover refining to an open cover  $\mathcal{V} = \{V_\xi : \xi < \kappa\}$  of order  $\leq n + 1$ , let  $\phi: \kappa \rightarrow \{1, \dots, k\}$  be any function such that  $V_\xi \subseteq U_{\phi(\xi)}$  for each

$\xi < \kappa$ , and define  $W_i = \bigcup \{V_\xi; \phi(\xi) = i\}$ . Then  $\mathscr{W} = \{W_1, \dots, W_k\}$  is an open cover which is a “precise” refinement of  $\mathscr{U}$  (i.e.  $W_i \subseteq U_i$ ; however there may be some repetitions). Moreover  $\mathscr{W}$  has order  $\leq n + 1$ . We will need this standard fact in the following.

**2.2.2. THEOREM.** *Let  $X \in KH$  and let  $\mathscr{D}$  be an ultrafilter. Then  $\dim(\sum_{\mathscr{D}} X) = \dim(X)$ .*

**PROOF.** Assume  $\dim(X) \leq n$ , and let  $\mathscr{U}$  be any finite open cover of  $\sum_{\mathscr{D}} X$ . By compactness, we can take  $\mathscr{U}$  to be a basic open cover, say  $\mathscr{U} = \{(\prod_{\mathscr{D}} U_{i,1})^\#, \dots, (\prod_{\mathscr{D}} U_{i,k})^\#\}$ . Then, letting  $\mathscr{U}_i = \{U_{i,1}, \dots, U_{i,k}\}$ ,  $i \in I$ , we have that  $J = \{i: \mathscr{U}_i \text{ is an open cover of } X\} \in \mathscr{D}$ . By assumption, plus (2.2.1(ii)) above, let  $\mathscr{V}_i = \{V_{i,1}, \dots, V_{i,k}\}$  be an open cover precisely refining  $\mathscr{U}_i$ , and of order  $\leq n + 1$  for each  $i \in J$ . We now let  $\mathscr{V} = \{(\prod_{\mathscr{D}} V_{i,1})^\#, \dots, (\prod_{\mathscr{D}} V_{i,k})^\#\}$ .  $\mathscr{V}$  is a precise refinement of  $\mathscr{U}$ . To see that  $\mathscr{V}$  is a cover, it is easy to check the steps in the following calculation:

$$\begin{aligned} (\prod_{\mathscr{D}} V_{i,1})^\# \cup \dots \cup (\prod_{\mathscr{D}} V_{i,k})^\# &= (\prod_{\mathscr{D}} V_{i,1} \cup \dots \cup \prod_{\mathscr{D}} V_{i,k})^\# \\ &= (\prod_{\mathscr{D}} (V_{i,1} \cup \dots \cup V_{i,k}))^\# = (\prod_{\mathscr{D}} X)^\# = \sum_{\mathscr{D}} X. \end{aligned}$$

To see that  $\mathscr{V}$  has order  $\leq n + 1$ , suppose not. Then there are  $m > n + 1$  distinct sets  $(\prod_{\mathscr{D}} V_{i,1})^\#, \dots, (\prod_{\mathscr{D}} V_{i,m})^\#$  containing some  $p$ . But then, for almost all  $i \pmod{\mathscr{D}}$ ,  $V_{i,1}, \dots, V_{i,m}$  are all distinct and have empty intersection. This says that  $\emptyset = \prod_{\mathscr{D}} (V_{i,1} \cap \dots \cap V_{i,m}) \in p$ , an impossibility.

Now assume  $\dim(\sum_{\mathscr{D}} X) \leq n$ , and let  $\mathscr{U} = \{U_1, \dots, U_k\}$  be an open cover of  $X$ . Then  $\{(\prod_{\mathscr{D}} U_1)^\#, \dots, (\prod_{\mathscr{D}} U_k)^\#\}$  is an open cover of  $\sum_{\mathscr{D}} X$ ; so there is a precise refinement of order  $\leq n + 1$ , which we can take to be basic (because of compactness and the fact that both  $\prod_{\mathscr{D}}(\cdot)$  and  $(\cdot)^\#$  commute with finite set operations), denoted by  $\{(\prod_{\mathscr{D}} V_{i,1})^\#, \dots, (\prod_{\mathscr{D}} V_{i,k})^\#\}$ . Thus for almost all indices  $i \pmod{\mathscr{D}}$ ,  $\{V_{i,1}, \dots, V_{i,k}\}$  is an open cover of  $X$ , refining  $\mathscr{U}$ , and of order  $\leq n + 1$ . ■

**2.2.3. THEOREM.** *Let  $X \in KH$ , and let  $\mathscr{D}$  be an ultrafilter. Then  $\text{Ind}(\sum_{\mathscr{D}} X) \leq \text{Ind}(X)$ .*

**PROOF.** This is done inductively. What we really need to show is that if  $\langle X_i; i \in I \rangle$  is any family of compact Hausdorff spaces such that  $\{i: \text{Ind}(X_i) \leq n\} \in \mathscr{D}$ , then  $\text{Ind}(\sum_{\mathscr{D}} X_i) \leq n$ . (1.7) establishes the assertion for  $n = 0$ ; so we work on the inductive step, and assume the more general statement true for dimensions  $\leq n$ . Assume  $\text{Ind}(X) \leq n + 1$ , and let  $F \subseteq U \subseteq \sum_{\mathscr{D}} X_i$ ,  $F$  closed and  $U$  open. Since  $F$  is compact and the operations  $\prod_{\mathscr{D}}(\cdot)$  and  $(\cdot)^\#$  commute with finite set operations, there are open sets  $U_i \subseteq X_i$  such that  $F \subseteq (\prod_{\mathscr{D}} U_i)^\# \subseteq U$ . Moreover, since  $F$  is closed and  $\sum_{\mathscr{D}} X_i$  is compact, there are closed sets  $F_i \subseteq X_i$  such that  $F \subseteq (\prod_{\mathscr{D}} F_i)^\# \subseteq (\prod_{\mathscr{D}} U_i)^\# \subseteq U$ . Thus  $J = \{i: F_i \subseteq U_i\} \in \mathscr{D}$ ; and for each  $i \in J$  we have open sets  $V_i$  with  $F_i \subseteq V_i \subseteq U_i$  and  $\text{Ind}(\text{Fr}(V_i)) \leq n$ . Thus  $F \subseteq (\prod_{\mathscr{D}} V_i)^\# \subseteq U$ , and we claim that  $\text{Ind}(\text{Fr}((\prod_{\mathscr{D}} V_i)^\#)) \leq n$ . This follows immediately from the inductive hypothesis, plus the following (easily verified) facts: (i) both  $\prod_{\mathscr{D}}(\cdot)$  and  $(\cdot)^\#$  commute with finite set operations (as mentioned before); and (ii) whenever  $C_i \subseteq X_i$  is closed for each  $i \in I$  then  $(\prod_{\mathscr{D}} C_i)^\#$ , as a subspace of  $\sum_{\mathscr{D}} X_i$ , is naturally homeomorphic with  $\sum_{\mathscr{D}} C_i$ . Thus

$$\text{Ind}(\text{Fr}((\prod_{\mathscr{D}} V_i)^\#)) = \text{Ind}(\sum_{\mathscr{D}} \text{Fr}(V_i)) \leq n;$$

hence  $\text{Ind}(\sum_{\mathscr{D}} X_i) \leq n + 1$ . ■

**2.2.4. QUESTION AND REMARKS.** (i) Does equality hold in (2.2.3)? The equality in (2.2.2) is true for all normal spaces  $X$  because of (1.6) and Vedenisov’s theorem (i.e.

$\dim(X) = \dim(\beta(X))$ ). With the modified definition of  $\dim$  given in [19], a version of (2.2.2) can be made to hold for arbitrary Tichonov spaces. Likewise the inequality in (2.2.3) holds for all normal  $X$  because of Isbell’s theorem [28] (i.e.  $\text{Ind}(X) = \text{Ind}(\beta(X))$ ).

(ii) The inequality in (2.2.3), as well as the corresponding one in (2.2.2), actually holds for all reduced copowers. A proof using standard dimension-theoretic results goes as follows. Let  $d$  be either  $\text{Ind}$  or  $\dim$ , and suppose  $d(X) \leq n$ . If  $I$  is given the discrete topology (as usual) then  $d(X \times I) \leq n$ . Moreover  $X \times I$  is normal, so  $d(\beta(X \times I)) \leq n$  (see (i) above). Since  $\sum_{\mathcal{F}} X$  is a closed subset of  $\beta(X \times I)$ , we have  $d(\sum_{\mathcal{F}} X) \leq n$ .

**2.3. Cardinal invariants.** In this section we are concerned with cardinality issues in connection with ultracopowers of compact spaces. The following is a very simple consequence of what we have developed so far.

**2.3.1. PROPOSITION.** *Let  $X \in KH$  be infinite, and assume  $\mathcal{D}$  is countably incomplete. Then  $\sum_{\mathcal{D}} X$  is neither metrizable, nor does it satisfy the countable chain condition (c.c.c.).*

**PROOF.**  $X$  is infinite and compact, hence nondiscrete. Thus  $\prod_{\mathcal{D}} X$  is a nondiscrete  $P$ -space [6], hence nonmetrizable. By (1.2),  $\sum_{\mathcal{D}} X$  is nonmetrizable.

Let  $\mathcal{U}$  be any infinite collection of pairwise disjoint open subsets of  $X$ . Then  $|\prod_{\mathcal{D}} \mathcal{U}| \geq c = \exp(\omega)$  (see [16]), so  $\{(\prod_{\mathcal{D}} U_i)^{\#} : U_i \in \mathcal{U}\}$  is a collection of  $\geq c$  pairwise disjoint open subsets of  $\sum_{\mathcal{D}} X$ . ■

Recall the definition of “ $\kappa$ -regular” ultrafilter from (1.9). If  $I = \kappa$  then there exist many  $\kappa$ -regular ultrafilters, but no  $\lambda$ -regular ones for  $\lambda > \kappa$ .  $\mathcal{D}$  is *regular* if  $\mathcal{D}$  is  $|I|$ -regular. The classical result concerning regular ultrafilters in model theory is the following.

**2.3.2. LEMMA [16].** *Let  $\mathcal{D}$  be regular on  $I$ , and let  $S$  be any infinite set. Then  $|\prod_{\mathcal{D}} S| = |S|^{|I|}$ .*

Recall that for any space  $X$ , the weight  $w(X)$  of  $X$  is the least infinite cardinal of a basis for  $X$ . If  $X$  is infinite Boolean, then it is easily shown [17] that  $w(X) = |\mathcal{B}(X)|$ . We thus immediately get the fact that if  $X$  is an infinite Boolean space and  $\mathcal{D}$  is a regular ultrafilter on  $I$  then  $w(\sum_{\mathcal{D}} X) = w(X)^{|I|}$ . This actually holds for general infinite  $X \in KH$ , and we are grateful to K. Kunen for suggesting the proof of this result.

**2.3.3. THEOREM.** *Let  $\mathcal{D}$  be a regular ultrafilter on  $I$ , and let  $X$  be an infinite compact Hausdorff space. Then  $w(\sum_{\mathcal{D}} X) = w(X)^{|I|}$ .*

**PROOF.** Fix  $X \in KH$  infinite and let  $C(X)$  denote the set of continuous real-valued functions on  $X$  endowed with the “sup-norm” metric (i.e.  $\|\phi - \psi\| = \sup\{|\phi(x) - \psi(x)| : x \in X\}$ ). If  $d(Y)$  denotes the “density” of a space  $Y$ , that is the smallest cardinality of a dense subset of  $Y$ , then it is known that  $w(X) = d(C(X))$ . To see this, let  $D \subseteq C(X)$  be dense. Then the inverse images of the half-open interval  $[0, 1/2)$  under members of  $D$  form an open basis for  $X$  of cardinality  $\leq |D|$ . Thus  $w(X) \leq d(C(X))$ . The reverse inequality is a straightforward application of the Stone-Weierstrass theorem (see [35]).

The next simple fact we need is that there is a family  $\Phi$  of continuous functions from  $X$  into the closed unit interval  $[0, 1]$  such that  $|\Phi| = w(X)$  and  $\|\phi - \psi\| \geq 1/2$  for all distinct  $\phi, \psi \in \Phi$ . Such a family can be obtained using a Zorn’s lemma

argument: if  $\Phi$  is a family of continuous maps from  $X$  to  $[0, 1]$  which is maximal with respect to the property that  $\|\phi - \psi\| \geq 1/2$  for distinct  $\phi, \psi \in \Phi$ , then the inverse images of  $[0, 1/2)$  under members of  $\Phi$  form an open basis for  $X$  of cardinality  $\leq |\Phi|$ . Thus  $w(X) \leq |\Phi|$ . That  $|\Phi| \leq d(C(X)) = w(X)$  is clear.

To establish the  $\leq$ -inequality, first note that since  $\sum_{\mathcal{F}} X \subseteq \beta(X \times I)$  for any filter  $\mathcal{F}$ , we always have  $w(\sum_{\mathcal{F}} X) \leq \exp(w(X) \cdot |I|)$  (see [19]). Since  $w(X)^{|I|}$  is a truly better upper bound, a sharper argument is required. So let  $\mathcal{B}$  be an open basis for  $X$  of cardinality  $w(X)$ , and assume  $\mathcal{B}$  is closed under finite unions. Let  $\mathcal{B}' = \{(\prod_{\mathcal{Q}} B_i)^{\#} : B_i \in \mathcal{B}\}$ . Then  $|\mathcal{B}'| = |\prod_{\mathcal{Q}} \mathcal{B}| \leq w(X)^{|I|}$ .  $\mathcal{B}'$  is easily seen to be an open basis for  $\sum_{\mathcal{Q}} X$ ; for if  $p \in (\prod_{\mathcal{Q}} U_i)^{\#}$ , where each  $U_i$  is open in  $X$ , then there is a closed ultrabox  $\prod_{\mathcal{Q}} F_i \subseteq p$  contained in  $\prod_{\mathcal{Q}} U_i$ . For each  $i$  such that  $F_i \subseteq U_i$ , use compactness of  $X$ , plus the fact the  $\mathcal{B}$  is closed under finite unions, to find a  $B_i \in \mathcal{B}$  with  $F_i \subseteq B_i \subseteq U_i$ . Thus  $p \in (\prod_{\mathcal{Q}} B_i)^{\#} \subseteq (\prod_{\mathcal{Q}} U_i)^{\#}$ , establishing the inequality.

We now establish that  $w(\sum_{\mathcal{Q}} X) \geq w(X)^{|I|}$ . Since  $w(\sum_{\mathcal{Q}} X) = d(C(\sum_{\mathcal{Q}} X))$ , we construct a family  $\Phi$  of maps from  $\sum_{\mathcal{Q}} X$  into  $[0, 1]$  such that  $|\Phi| \geq w(X)^{|I|}$  and  $\|\phi - \psi\| \geq 1/2$  for all distinct  $\phi, \psi \in \Phi$ . Let  $\Phi_0$  be a family of  $w(X)$  continuous maps from  $X$  into  $[0, 1]$  with this "spread out" property, and let  $\langle \theta_i : i \in I \rangle$  be an  $I$ -indexed family of maps from  $\Phi_0$ . Then  $\nabla \circ \sum_{\mathcal{Q}} \theta_i$  is a continuous map from  $\sum_{\mathcal{Q}} X$  to  $[0, 1]$ , where  $\nabla = \nabla_{\mathcal{Q}, [0, 1]}$ . Let  $\Phi$  be the set of all continuous maps constructed in this way. It is straightforward to show that  $\Phi$  is "spread out" and that  $|\Phi| = |\prod_{\mathcal{Q}} \Phi_0|$ . We now use the regularity of  $\mathcal{Q}$  to conclude that this cardinal is  $w(X)^{|I|}$ . ■

The cardinality of  $\sum_{\mathcal{Q}} X$  is much harder to pin down than the weight. Since  $\sum_{\mathcal{F}} X \subseteq \beta(X \times I)$  for any filter  $\mathcal{F}$ , we always have  $|\sum_{\mathcal{F}} X| \leq \exp^2(|X| \cdot |I|)$  [19]. (Equality can hold: set  $\mathcal{F} = \{I\}$  and  $X = \{\text{point}\}$ .) On the other hand, if  $\mathcal{Q}$  is an ultrafilter, then  $|\sum_{\mathcal{Q}} X| \leq \exp(w(\sum_{\mathcal{Q}} X)) \leq \exp(w(X)^{|I|})$ . The first inequality is true because weight is bounded by cardinality for compact Hausdorff spaces (an old Alexandrov-Urysohn result); the second follows from (2.3.3). Also, since cardinality does bound weight, this new upper bound is sharper than the old one, and strict inequality can definitely occur. Now suppose  $\mathcal{Q}$  is regular. Then  $|\sum_{\mathcal{Q}} X| \geq |\prod_{\mathcal{Q}} X| = X^{|I|}$ . Thus we can specify  $|\sum_{\mathcal{Q}} X|$  under certain circumstances (say, when  $w(X) = \exp(|I|)$  and  $|X| = \exp(w(X))$ ).

**2.3.4. EXAMPLE.** Let  $\mathcal{Q}$  be a free ultrafilter on  $\omega$ , and let  $X$  be the Tichonov power  $[0, 1]^c$  of a continuous number of intervals. Then  $w(\sum_{\mathcal{Q}} X) = w(X)^\omega = w(X) = c$ , and  $|\sum_{\mathcal{Q}} X| = |X|^\omega = |X| = \exp(c)$ . (There are numerous ways in which we can see that  $X$  and  $\sum_{\mathcal{Q}} X$  are topologically distinct, however. A very simple test is to note that, since  $[0, 1]$  satisfies the c.c.c., so too does any cube, say  $X$ . However, by (2.3.1),  $\sum_{\mathcal{Q}} X$  fails miserably in this regard. A more exotic test is to use (point-) homogeneity.  $X$  is a Tichonov power of the Hilbert cube, well known to be homogeneous. Thus  $X$  is homogeneous.  $\sum_{\mathcal{Q}} X$ , on the other hand, contains a dense set (i.e.  $\prod_{\mathcal{Q}} X$ ) of "P-points" (i.e. points lying in the interiors of all intersections of countable families of neighborhoods). This makes  $\sum_{\mathcal{Q}} X$  an "almost P-space": every nonempty  $G_\delta$ -set has nonempty interior. Since  $\sum_{\mathcal{Q}} X$ , a compact Hausdorff space, is infinite, it cannot be a P-space. Thus it cannot be homogeneous.)

To get better lower bounds on  $|\sum_{\mathcal{Q}} X|$  we bring in another well-known combinatorial property of ultrafilters, namely that of "goodness" (see [16] and [17]). Given an infinite cardinal  $\kappa$ , an ultrafilter  $\mathcal{Q}$  on  $I$  is  $\kappa$ -good if, for all  $\lambda < \kappa$ , any

“monotone”  $f: \mathbf{P}_\omega(\lambda) \rightarrow \mathcal{D}$  from the finite subsets of  $\lambda$  to  $\mathcal{D}$  (i.e.  $s \subseteq t \Rightarrow f(t) \subseteq f(s)$ ) “dominates” a “multiplicative”  $g: \mathbf{P}_\omega(\lambda) \rightarrow \mathcal{D}$  (i.e.  $g(s) \subseteq f(s)$  for all  $s \in \mathbf{P}_\omega(\lambda)$  and  $g(s \cup t) = g(s) \cap g(t)$  holds for  $s, t \in \mathbf{P}_\omega(\lambda)$ ). Every countably incomplete ultrafilter is  $\omega_1$ -good, and every  $\kappa$ -good countably incomplete ultrafilter is  $\lambda$ -regular for all  $\lambda < \kappa$ . There exist  $|I|^+$ -good countably incomplete ultrafilters on  $I$  (in fact  $\exp^2(|I|)$  of them [17]) and this is the maximal degree of goodness possible. The major lemma which will be of use to us here is the following.

**2.3.5. LEMMA** [16]. *Let  $\mathcal{D}$  be a  $\kappa$ -good countably incomplete ultrafilter on  $I$ , and let  $\langle A_i; i \in I \rangle$  be relational structures of the same type. Then the ultraproduct  $\prod_{\mathcal{D}} A_i$  is “ $\kappa$ -saturated” (i.e. if  $S$  is any subset of  $\prod_{\mathcal{D}} A_i$  of cardinality  $< \kappa$  and  $\Phi = \Phi(v_1, \dots, v_n)$  is any set of formulas in the variables  $v_1, \dots, v_n$  with constants from  $S$ , then  $\prod_{\mathcal{D}} A_i$  realizes  $\Phi$  (with particular elements substituting for the variables), provided the same is true for each finite subset of  $\Phi$ ).*

The following is an easy consequence of (2.3.5).

**2.3.6. PROPOSITION.** *Let  $\mathcal{D}$  be a  $\kappa$ -good countably incomplete ultrafilter on  $I$ , and let  $\langle X_i; i \in I \rangle$  be arbitrary topological spaces. Let  $\mathcal{B}$  be the basis of open ultraboxes  $\prod_{\mathcal{D}} U_i$  for the topological ultraproduct. Then  $\mathcal{B}$  satisfies the following three conditions.*

(i) (“ $\kappa$ -intersection”) *If  $\mathcal{U} \subseteq \mathcal{B}$  has cardinality  $< \kappa$  and  $\bigcap \mathcal{U} = \emptyset$ , then  $\bigcap \mathcal{U}_0 = \emptyset$  for some finite  $\mathcal{U}_0 \subseteq \mathcal{U}$ .*

(ii) (“ $\kappa$ -cover”) *If  $\mathcal{U} \subseteq \mathcal{B}$  has cardinality  $< \kappa$  and  $\bigcup \mathcal{U} = \prod_{\mathcal{D}} X_i$ , then  $\bigcup \mathcal{U}_0 = \prod_{\mathcal{D}} X_i$  for some finite  $\mathcal{U}_0 \subseteq \mathcal{U}$ .*

(iii) (“ $\kappa$ -additivity”) *If  $\mathcal{U} \subseteq \mathcal{B}$  has cardinality  $< \kappa$  then  $\bigcap \mathcal{U}$  is an open set. (Actually,  $\kappa$ -additivity (Sikorski’s terminology) follows from the fact that  $\mathcal{D}$  is  $\lambda$ -regular for all  $\lambda < \kappa$ , i.e. the topological ultraproduct is a  $P_\kappa$ -space [6].)*

Topological ultracoproducts of Boolean spaces via good ultrafilters enjoy a property which ensures the existence of many “ $C^*$ -embedded” subsets (i.e. every bounded continuous real-valued function on such a subset extends to the whole space). This will come in handy when we try to get lower bounds on cardinality. Given an infinite cardinal  $\kappa$  and  $X \in KH$ , we say  $X$  is an  $F_\kappa$ -space (we take the most convenient definition which works for compact spaces; see [32]) if every open set which is the union of  $< \kappa$  cozero-sets is  $C^*$ -embedded. Equivalently,  $X$  is an  $F_\kappa$ -space if any two disjoint open sets which are unions of  $< \kappa$  cozero-sets have disjoint closures. If  $X \in BS$  we can replace “cozero” by “clopen” in the above characterizations. Since unions of countably many cozero-sets are cozero,  $X$  is an  $F_{\omega_1}$ -space iff  $X$  is an “ $F$ -space” (i.e. disjoint cozero-sets have disjoint closures). (See [17], [19], [32] and [33] for more details.)

**2.3.7. THEOREM.** *Let  $\mathcal{D}$  be an ultrafilter on  $I$  and let  $\langle X_i; i \in I \rangle$  be a family in  $KH$ .*

(i) *If  $\mathcal{D}$  is  $\kappa$ -regular then  $\sum_{\mathcal{D}} X_i$  is an “almost- $P_\kappa^+$ -space” (i.e. nonempty intersections of  $< \kappa^+$  open sets have nonempty interiors).*

(ii) *If  $\mathcal{D}$  is  $\kappa$ -good countably incomplete and each  $X_i$  is Boolean then  $\sum_{\mathcal{D}} X_i$  is an  $F_\kappa$ -space.*

**PROOF.** (i) As mentioned before, a basic result of [6] is that  $\prod_{\mathcal{D}} X_i$  is a  $P_\kappa^+$ -space. Since  $\prod_{\mathcal{D}} X_i$  is dense in  $\sum_{\mathcal{D}} X_i$ , it is immediate that  $\sum_{\mathcal{D}} X_i$  is an almost- $P_\kappa^+$ -space.

(ii)  $B(\sum_{\mathcal{D}} X_i) \cong \prod_{\mathcal{D}} B(X_i)$  is a  $\kappa$ -saturated Boolean algebra, by (1.5) and (2.3.5). Suppose  $\langle B_\xi; \xi < \lambda \rangle$  and  $\langle C_\zeta; \zeta < \delta \rangle$  are two collections of  $< \kappa$  clopen sets, whose unions are disjoint. Every finite subcollection of  $B_\xi$ ’s and  $C_\zeta$ ’s can be separated by a

clopen set. By  $\kappa$ -saturatedness, then, there is a clopen  $D$  containing each  $B_\xi$  and disjoint from each  $C_\xi$ . Thus  $\sum_{\mathcal{D}} X_i$  is an  $F_\kappa$ -space. ■

**2.3.8. QUESTION.** Is the “Boolean” hypothesis necessary in (2.3.7(ii))?

**2.3.9. REMARK.** Combining (2.3.7(ii)) and (1.7), we see that the ultracoproduct construction provides a machine for generating  $F$ -spaces which are not basically disconnected.

The following is a well-known result.

**2.3.10. LEMMA [33].** *Let  $X \in KH$  be an  $F$ -space. Then every countable subset of  $X$  is  $C^*$ -embedded. If  $X$  is infinite as well then  $X$  contains a countably infinite discrete subset; hence an embedded copy of  $\beta(\omega)$ . Thus  $|X| \geq \exp(c)$ .*

**2.3.11. REMARK.** The obvious higher cardinal analogue to (2.3.10) is false: in [15] it is shown that for any infinite cardinal  $\kappa$ , the space of uniform ultrafilters on a set of cardinality  $\kappa$  is an  $F_\kappa$ -space which contains a discrete subset of cardinality  $\omega_1$  which is not  $C^*$ -embedded. (This result is attributed to E. K. van Douwen.)

Another step toward solving the cardinality problem for ultracopowers is the following result concerning ultracoproducts of finite sets.

**2.3.12. LEMMA.** *Let  $\mathcal{D}$  be a countably incomplete ultrafilter on a set  $I$  of cardinality  $\kappa$ , and let  $\langle X_i; i \in I \rangle$  be a family of finite discrete spaces such that  $\sum_{\mathcal{D}} X_i$  is infinite (i.e. such that for each  $n < \omega$ ,  $\{i: |X_i| \geq n\} \in \mathcal{D}$ ). Then:*

(i)  $\sum_{\mathcal{D}} X_i$  contains an embedded copy of  $\beta(\omega)$ , and hence has cardinality  $\geq \exp(c)$ .

(ii) If  $\mathcal{D}$  is  $\lambda$ -good,  $\lambda$  an infinite cardinal, then  $|\sum_{\mathcal{D}} X_i| \geq \exp(\lambda)$ .

(iii) If  $\mathcal{D}$  is  $\kappa^+$ -good then  $\exp(\kappa^+) \leq |\sum_{\mathcal{D}} X_i| \leq \exp^2(\kappa)$ . Hence equality holds if either  $\kappa = \omega$  or  $\kappa^+ = \exp(\kappa)$ .

PROOF: (i) This is a direct application of (2.3.7(ii)) and (2.3.10).

(ii) (This is similar to the proof of Theorem 2.4(i) in [10].) Inductively build a  $\lambda$ -level binary tree  $T$  consisting of infinite sets of the form  $(\prod_{\mathcal{D}} S_i)^\#$ ,  $S_i \subseteq X_i$ , and ordered by reverse inclusion. For each  $\xi < \lambda$ , denote the  $\xi$ th level by  $T_\xi$ , and let  $T|(\xi) = \bigcup_{\gamma < \xi} T_\gamma$ . Let  $T_0 = \{(\prod_{\mathcal{D}} X_i)^\#\}$ . Assuming that  $T|(\xi + 1)$  has been defined, let  $T_{\xi+1}$  be formed by taking each  $(\prod_{\mathcal{D}} S_i)^\#$  in  $T_\xi$  and letting  $S_i$  be a disjoint union  $S_i^1 \cup S_i^2$  in such a way that both  $\prod_{\mathcal{D}} S_i^1$  and  $\prod_{\mathcal{D}} S_i^2$  are infinite.  $T_{\xi+1} = \bigcup \{ \{(\prod_{\mathcal{D}} S_i^1)^\#, (\prod_{\mathcal{D}} S_i^2)^\#\} : (\prod_{\mathcal{D}} S_i)^\# \in T_\xi \}$ . In the limit case, assume  $T|(\xi)$  has been defined, and let  $\mathcal{B} = \langle (\prod_{\mathcal{D}} S_i^\gamma)^\# : \gamma < \xi \rangle$  be a branch in  $T|(\xi)$ . Since  $|\xi| < \lambda$  and  $\prod_{\mathcal{D}} F(X_i)$  is a  $\lambda$ -saturated lattice, the decreasing sequence  $\langle \prod_{\mathcal{D}} S_i^\gamma : \gamma < \xi \rangle$  is eventually constant. Thus  $\bigcap \mathcal{B} = (\prod_{\mathcal{D}} S_i^\beta)^\#$  for some  $\beta < \xi$ . Define  $T_\xi$  to be the collection of all such intersections.

Letting  $T = T|(\lambda)$ , we note that each member of  $T$  is a closed subset of a compact Hausdorff space. Thus each branch has nonempty intersection and we immediately have  $|\sum_{\mathcal{D}} X_i| \geq \exp(\lambda)$ .

(iii) This is immediate from (i) and (ii). ■

**2.3.13. QUESTION.** If  $\mathcal{D}$  is a  $\kappa^+$ -good countably incomplete ultrafilter on a set of cardinality  $\kappa$  and  $\sum_{\mathcal{D}} X_i$  is an infinite ultracoproduct of finite discrete spaces, is it always true that  $|\sum_{\mathcal{D}} X_i| = \exp^2(\kappa)$ ?

**2.3.14. THEOREM.** *Let  $\mathcal{D}$  be a  $\kappa^+$ -good countably incomplete ultrafilter on a set of cardinality  $\kappa$ , and let  $X \in KH$  be infinite. Then  $|\sum_{\mathcal{D}} X| \geq \exp(c \cdot \kappa^+) \cdot |X|^\kappa$ .*

PROOF. Since  $\mathcal{D}$  is  $\kappa^+$ -good countably incomplete on a set of cardinality  $\kappa$ ,  $\mathcal{D}$  is regular. Thus  $|\sum_{\mathcal{D}} X| \geq |\prod_{\mathcal{D}} X| = |X|^\kappa$  by (2.3.2). Let  $\langle F_i; i \in I \rangle$  be a family of finite

subsets of  $X$ , chosen so that  $\{i: |F_i| \geq n\} \in \mathcal{D}$  for each  $n < \omega$ , and let  $\theta_i: F_i \rightarrow X$  be the inclusion map for each  $i$ . Since all spaces under consideration are compact (hence normal), and each  $\theta_i$  is a closed map, the ultracopower map  $\sum_{\mathcal{D}} \theta_i$  is an embedding. By (2.3.12),  $|\sum_{\mathcal{D}} X| \geq |\sum_{\mathcal{D}} F_i| \geq \exp(c \cdot \kappa^+)$ . ■

**2.3.15. COROLLARY.** *Let  $\mathcal{D}$  be a free ultrafilter on  $\omega$ , and let  $X \in KH$  be infinite of cardinality and weight  $\leq c$ . Then  $|\sum_{\mathcal{D}} X| = \exp(c)$  and  $w(\sum_{\mathcal{D}} X) = c$ .*

PROOF. We immediately have  $w(\sum_{\mathcal{D}} X) = c$  by (2.3.3). Thus  $|\sum_{\mathcal{D}} X| \leq \exp(c)$ . Finally,  $|\sum_{\mathcal{D}} X| \geq \exp(c)$  by (2.3.14). ■

**2.3.16. REMARK.** We can apply (2.3.15) to disprove a conjecture which naturally arises in connection with (1.12). Given a space  $X$ , define the  $P$ -subspace  $P(X)$  of  $X$  to be the set of  $P$ -points of  $X$ . Thus if  $\mathcal{D}$  is a countably incomplete ultrafilter and  $X \in KH$  then  $P(\sum_{\mathcal{D}} X) \supseteq \prod_{\mathcal{D}} X$ . Strict inclusion can occur, as witnessed by the following. Let  $\mathcal{D}$  and  $\mathcal{E}$  be free ultrafilters on  $\omega$  and let  $X = 2^\omega$ . The iteration theorem (2.1.1) tells us that  $\sum_{\mathcal{D}, \mathcal{E}} X \simeq \sum_{\mathcal{D}} \sum_{\mathcal{E}} X$ ; hence  $|P(\sum_{\mathcal{D}, \mathcal{E}} X)| \geq |\prod_{\mathcal{D}} \sum_{\mathcal{E}} X| = \exp(c)$ , by (2.3.15). However  $|\prod_{\mathcal{D}, \mathcal{E}} X| = c$ .

Goodness of ultrafilters is also related to “Baire category” properties in ultraproducts and ultracoproducs. A space is  $\kappa$ -Baire if intersections of  $< \kappa$  dense open sets are dense. In [8] it was proved that  $\prod_{\mathcal{D}} X$  is always  $\kappa$ -Baire when  $\mathcal{D}$  is  $\kappa$ -good countably incomplete. Since  $\prod_{\mathcal{D}} X$  is dense in  $\sum_{\mathcal{D}} X$ , the same is true for ultracopowers. However the presence of compactness allows for a stronger conclusion.

**2.3.17. THEOREM.** *Let  $\mathcal{D}$  be a  $\kappa$ -good countably incomplete ultrafilter, and let  $X \in KH$ . Then  $\sum_{\mathcal{D}} X$  is  $\kappa^+$ -Baire.*

PROOF. (This is similar to the proof of Theorem 2.4(ii) in [10].) Let  $X^\pi = \prod_{\mathcal{D}} X$ ,  $X^\sigma = \sum_{\mathcal{D}} X$ , and let  $\mathcal{B}$  be an open basis for  $X^\pi$  satisfying the  $\kappa$ -intersection condition. (E.g., we could take  $\mathcal{B}$  to be the collection of open ultraboxes and use the  $\kappa$ -saturatedness of the lattice  $\prod_{\mathcal{D}} F(X)$ .) Let  $\langle U_\xi: \xi < \kappa \rangle$  be a family of dense open subsets of  $X^\sigma$ , with  $S = \bigcap_{\xi < \kappa} U_\xi$ . We need to show  $S$  is dense in  $X^\sigma$ , so let  $V \subseteq X^\sigma$  be nonempty open. To show  $V \cap S \neq \emptyset$ , use induction on  $\kappa$ . We construct a decreasing chain  $\langle B_\xi: \xi < \kappa \rangle$  ( $(\cdot)^-$  denotes closure in  $X^\sigma$ ) where  $B_\gamma \supseteq B_\xi$  for  $\gamma < \xi < \kappa$ ,  $\emptyset \neq B_\xi \in \mathcal{B}$  for  $\xi < \kappa$ , and  $B_\xi^- \subseteq V \cap (\bigcap_{\gamma < \xi} U_\gamma)$ . This is possible because  $X^\pi$  is dense in  $X^\sigma$ ,  $X^\sigma$  is a regular  $T_1$ -space, and  $\mathcal{B}$  satisfies the  $\kappa$ -intersection condition. Using compactness, we get  $\emptyset \neq \bigcap_{\xi < \kappa} B_\xi^- \subseteq V \cap S$ . ■

**2.4. Ultracopowers over countable index sets.** In this section, all ultrafilters  $\mathcal{D}, \mathcal{E}$ , etc. are assumed to be free on a countable set. Following [18], define a *Parovičenko space* to be a self-dense Boolean space of weight  $c$  which is also an  $F$ -space and an almost- $P$ -space.

Let  $\omega^*$  denote  $\beta(\omega) \setminus \omega$ , the Stone-Čech remainder of the integers. The following well-known results concerning this space depend on C.H.:

(i)  $\omega^*$  is a *Parovičenko space*. (A celebrated result of S. Shelah says that  $\omega^*$  need not have any  $P$ -points. Of course  $\omega^*$  satisfies all the other clauses in the definition of “Parovičenko space”.)

(ii) All *Parovičenko spaces* are homeomorphic with  $\omega^*$ . (This is an important theorem of Parovičenko [33]. E. K. van Douwen and J. van Mill [18] proved that, under not-C.H., there are two nonhomeomorphic Parovičenko spaces.)

(iii)  $\omega^*$  has  $\exp(c)$   $P$ -points. (This is an old result of W. Rudin [33].)

(iv) If  $p \in \omega^*$  then  $\omega^* \setminus \{p\}$  is not normal. (The proof when  $p$  is not a  $P$ -point is due to L. Gillman; the  $P$ -point case is due independently to M. Rajagopalan and N. Warren [33].)

**2.4.1. PROPOSITION.** *Let  $X \in KH$ . Then  $\sum_{\mathcal{D}} X$  is a Parovičenko space iff  $X$  is a self-dense Boolean space of weight  $\leq c$ .*

PROOF. For any  $X$  and  $\mathcal{D}$ ,  $X$  is Boolean iff  $\sum_{\mathcal{D}} X$  is Boolean;  $X$  is self-dense iff  $\sum_{\mathcal{D}} X$  is self-dense. If  $\sum_{\mathcal{D}} X$  is Parovičenko then  $X$  is self-dense and Boolean. Since  $X$  is a continuous image of  $\sum_{\mathcal{D}} X$ , its weight cannot exceed  $c$ . Now if  $X$  is a self-dense Boolean space of weight  $\leq c$  then  $w(\sum_{\mathcal{D}} X) = c$  by (2.3.3).  $\sum_{\mathcal{D}} X$  is an  $F$ -space and an almost- $P$ -space by (2.3.7). ■

We call  $\sum_{\mathcal{D}} X$  a *Parovičenko ultracopower* when it is a Parovičenko space. By Parovičenko's theorem there is a connection, under C.H., between  $\omega^*$  and such spaces as  $2^\omega, 2^{\omega_1}$ , and in fact the space  $\mathbf{Q}$  of rationals (because  $\sum_{\mathcal{D}} \mathbf{Q} \simeq \sum_{\mathcal{D}} \beta(\mathbf{Q})$ ). Under C.H.,  $\prod_{\mathcal{D}} \mathbf{Q}$  has the ordered field structure of Hausdorff's canonical  $\eta_1$ -field [10]. Moreover, its subspace topology in  $\sum_{\mathcal{D}} \mathbf{Q}$  is the order topology. Thus we can conclude that  $\omega^*$  has a dense set of  $P$ -points whose subspace topology derives from the canonical  $\eta_1$ -field structure.

**2.4.2. PROPOSITION.** *Let  $\sum_{\mathcal{D}} X$  be a Parovičenko ultracopower.*

(i) *If  $X$  is metrizable then  $\sum_{\mathcal{D}} X$  has a dense set of  $P$ -points whose subspace topology derives from the order structure of an  $\eta_1$ -field of cardinality  $c$ .*

(ii) *If  $\mathcal{D}$  is of the form  $\mathcal{E} \cdot \mathcal{F}$  then  $\sum_{\mathcal{D}} X$  has  $\exp(c)$   $P$ -points.*

(iii)  *$\sum_{\mathcal{D}} X$  is  $\omega_2$ -Baire.*

PROOF. (i) Let  $X$  be metrizable. Since  $X$  is separable and self-dense,  $\mathbf{Q}$  embeds densely in  $X$ . Thus  $\prod_{\mathcal{D}} \mathbf{Q}$  embeds densely in  $\prod_{\mathcal{D}} X$ , so  $\sum_{\mathcal{D}} X$  contains a dense copy of  $\prod_{\mathcal{D}} \mathbf{Q}$ . This space is an  $\eta_1$ -field of cardinality  $c$ .

(ii) This is an inessential generalization of (2.3.16).

(iii) This is a special case of (2.3.17). ■

**2.4.3. QUESTIONS.** (i) Can one find two nonhomeomorphic Parovičenko ultracopowers under not-C.H.?

(ii) Can one remove a point from a Parovičenko ultracopower and still preserve normality?

Let us now look briefly at ultracopowers  $\sum_{\mathcal{D}} X$  where  $X$  is connected. An obvious analogue to  $\omega^*$  in this setting is  $[0, 1]^*$ , where  $[0, 1)$  denotes the half-open unit interval. Is it possible, using C.H. perhaps, to represent  $[0, 1]^*$  as an ultracopower  $\sum_{\mathcal{D}} X$ ?

**2.4.4. PROPOSITION.** *Let  $X$  be a "decomposable continuum" (i.e.  $X$  is connected compact Hausdorff and  $X = K \cup L$  for some proper subcontinua  $K, L$  of  $X$ ). Then so is  $\sum_{\mathcal{D}} X$ .*

PROOF. Let  $X = K \cup L$ , where  $K$  and  $L$  are proper subcontinua. Then  $\sum_{\mathcal{D}} X = (\prod_{\mathcal{D}} K)^\# \cup (\prod_{\mathcal{D}} L)^\#$ . By compactness of  $K$  and  $L$ ,  $(\prod_{\mathcal{D}} K)^\# \simeq \sum_{\mathcal{D}} K$  (ditto for  $L$ ). Since these subsets of  $\sum_{\mathcal{D}} X$  are clearly proper, this shows  $\sum_{\mathcal{D}} X$  is decomposable. ■

**2.4.5. QUESTION.** Is the converse of (2.4.4) true?

Now by a result of D. Bellamy and R. G. Woods,  $[0, 1]^*$  is indecomposable. Thus if  $[0, 1]^*$  is to be represented as an ultracopower  $\sum_{\mathcal{D}} X$ ,  $X$  will have to be indecomposable too, by (2.4.4). In particular we cannot hope that  $[0, 1]^*$  will be homeomorphic with, say,  $\sum_{\mathcal{D}} [0, 1]$ . (The possibility that  $[0, 1]^*$  will be homeomorphic with

some  $\sum_{\mathcal{D}} X$  for metrizable  $X$  remains:  $[0, 1]^*$  is of covering (and large inductive) dimension one; hence  $X$  would have to be a one-dimensional indecomposable metric continuum. (A pseudoarc perhaps?)

**2.5. First-order representations.** Let  $R$  be a “first-order representation” on  $KH$  in the sense of [12] and [13]. (That is,  $R$  assigns to each  $X \in KH$  a relational structure of a given finitary type in such a way that homeomorphic spaces get sent to isomorphic structures.) We wish to compare  $R(\sum_{\mathcal{D}} X)$  and  $\prod_{\mathcal{D}} R(X)$  for various instances of  $R$ .

**2.5.1. REMARK.**  $B(\sum_{\mathcal{D}} X) \cong \prod_{\mathcal{D}} B(X)$  by (1.5). This is definitely the tidiest relation we know of.

**2.5.2. REMARK.** By (1.2) and the initial remarks of §2.1,  $F(\sum_{\mathcal{D}} X)$  can be viewed as the meet-completion of the lattice  $\prod_{\mathcal{D}} F(X)$ . If  $\mathcal{D}$  is countably incomplete and  $X$  is infinite then  $\prod_{\mathcal{D}} F(X)$ , being infinite and  $\omega_1$ -saturated, is not even countably meet-complete. Thus it is almost never the case that  $F(\sum_{\mathcal{D}} X) \cong \prod_{\mathcal{D}} F(X)$ . Even the cardinal inequality  $|F(\sum_{\mathcal{D}} X)| \geq |\prod_{\mathcal{D}} F(X)|$  is hard to improve on. For if  $X = [0, 1]$ , say, and  $D$  is countably incomplete on  $\omega$  then  $|F(\sum_{\mathcal{D}} X)| \geq \exp(c)$  by (2.3.15). However  $|\prod_{\mathcal{D}} F(X)| = c$  by (2.3.2).

**2.5.3. REMARK.** In contrast with our usage in the proof of (2.3.3), we let  $C(X)$  now denote the ring of continuous real-valued functions on  $X$ . In [12] we show how  $C(\sum_{\mathcal{D}} X)$  may be obtained from  $\prod_{\mathcal{D}} C(X)$  by taking a quotient of a subring (by “throwing away the infinite elements and factoring out the infinitesimals”). In particular, the inequality  $|C(\sum_{\mathcal{D}} X)| \leq |\prod_{\mathcal{D}} C(X)|$  is always true. By taking  $X$  to be finite, strict inequality is easy to come by. (Also  $C(\sum_{\mathcal{D}} X)$  is hardly ever isomorphic with  $\prod_{\mathcal{D}} C(X)$  since rings of continuous functions are never  $\omega_1$ -saturated [14].)

**2.5.4. REMARK.** The most problematic first-order representation which we consider is  $Z$ . As in (2.5.2),  $Z(\sum_{\mathcal{D}} X)$ , being a countably meet-complete lattice [19], is rarely isomorphic with  $\prod_{\mathcal{D}} Z(X)$ ; and in fact we know of no general method of obtaining one from the other as we did in (2.5.2) and (2.5.3). However, if  $\mathcal{D}$  is a regular ultrafilter then  $|Z(\sum_{\mathcal{D}} X)| = |\prod_{\mathcal{D}} Z(X)|$ . To see this, first note that there is nothing to prove when  $X$  is finite. When  $X$  is infinite we use (2.3.3) plus the fact that for any regular Lindelöf space  $Y$ ,  $|Z(Y)| \leq w(Y)^\omega$  (since every cozero-set is an  $F_\sigma$ -set). Thus  $|Z(\sum_{\mathcal{D}} X)|$ ,  $w(Z_{\mathcal{D}} X)$ ,  $w(X)^{|\mathcal{I}|}$ , and  $|\prod_{\mathcal{D}} Z(X)|$  are all equal.

**§3. Coelementary equivalence and coelementary mappings.** We were originally motivated to study ultracoproducts in  $KH$  because we thought that there was a good chance a valuable tool could be developed to analyze the structure of this important category. We now know we were right to an extent, and we understand a little better what kinds of duality theorems  $KH$  can be involved in.

**3.1. Background: the duality question.** The guiding problem is this: The full subcategory  $BS$  of  $KH$  is linked, via Stone duality, to the category  $BA$  of Boolean algebras and homomorphisms. As a class of finitary relational structures,  $BA$  enjoys the property of being an equational class of algebras (= a variety); so in particular it is a  $P$ -class, an  $S$ -class (= closed under substructures), and an elementary class (= the class of models of a set of first-order sentences). ( $BA$  is also closed under homomorphic images, but we have not developed the tools to analyze this phenomenon.) Can any of these properties be carried over to a duality involving all of  $KH$ ? (We are ignoring dualities, such as the one studied in [4], in which the dual

category cannot be easily interpreted as a class of finitary relational structures, plus all attendant homomorphisms.) This problem was (we believe) first posed in [9], and the following summarizes what we now know.

**3.1.1. THEOREM.** *Suppose  $\mathcal{K}$  is a class of finitary relational structures, and suppose  $KH$  is dual to  $\mathcal{K}$ . Then:*

(i) (B. BANASCHEWSKI [2])  *$\mathcal{K}$  can be a  $P$ -class (e.g. the class of “[0, 1]-lattices” of continuous interval-valued functions).*

(ii) [9]  *$\mathcal{K}$  cannot be an elementary  $P$ -class with a representable underlying set functor. In particular,  $\mathcal{K}$  cannot be a universal Horn class (= a quasivariety).*

(iii) [9]  *$\mathcal{K}$  cannot be either an elementary class or an  $S$ -class which has fewer than  $c$  distinguished symbols and which has a representable underlying set functor.*

(iv) (B. BANASCHEWSKI [1])  *$\mathcal{K}$  cannot be an  $SP$ -class.*

(v) (B. BANASCHEWSKI [3], and independently, J. ROSICKÝ [31])  *$\mathcal{K}$  cannot be an elementary  $P$ -class.*

The deepest result in this connection is definitely (3.1.1(v)), which improves on (3.1.1(ii)), and we will return to the duality question from time to time in the sequel. The question of whether  $KH$  can be dual to an elementary  $P$ -class has generated many further questions concerning how ultracoproducts behave in  $KH$ , and has given rise to what we call, for want of a better term, “dualized model theory in  $KH$ ”.

**3.2. Coelementary equivalence.** Two compact Hausdorff spaces  $X$  and  $Y$  are coelementarily equivalent (in symbols  $X \equiv Y$ ) if there are ultrafilters  $\mathcal{D}$  and  $\mathcal{E}$  such that  $\sum_{\mathcal{D}} X \simeq \sum_{\mathcal{E}} Y$ . Note that, thanks to Stone duality and the ultrapower theorem [16], Boolean spaces  $X$  and  $Y$  are coelementarily equivalent iff their clopen algebras  $B(X)$  and  $B(Y)$  are elementarily equivalent in the usual sense of model theory. (We write  $B(X) \equiv B(Y)$  as per tradition. The notation “ $X \equiv Y$ ” is only a slight abuse; observe that in [23], “ $X \equiv Y$ ” means  $F(X) \equiv F(Y)$ .) Also note that, if  $R: KH \rightarrow \mathcal{K}$  were a category duality onto an elementary  $P$ -class then  $X \equiv Y$  iff  $R(X) \equiv R(Y)$ . Of course we know such a duality does *not* exist by (3.1.1(v)), so we are at pains to determine topologically what might otherwise be trivial consequences of model-theoretic lore.

**3.2.1. THEOREM.** *Coelementary equivalence is an equivalence relation.*

**PROOF.** We need check transitivity only. Suppose  $X \equiv Y$  (say  $\sum_{\mathcal{D}} X \simeq \sum_{\mathcal{E}_1} Y$ ), and  $Y \equiv Z$  (say  $\sum_{\mathcal{E}_2} Y \simeq \sum_{\mathcal{F}} Z$ ). By (2.1.1) there is an ultrafilter  $\mathcal{G}$  such that  $\sum_{\mathcal{G}} \sum_{\mathcal{E}_1} Y \simeq \sum_{\mathcal{G} \cdot \mathcal{E}_1} Y \simeq \sum_{\mathcal{G} \cdot \mathcal{E}_2} Y \simeq \sum_{\mathcal{G}} \sum_{\mathcal{E}_2} Y$ . Adding to this the fact that homeomorphisms between spaces lift to homeomorphisms between corresponding ultracoproducts, we have  $\sum_{\mathcal{G} \cdot \mathcal{D}} X \simeq \sum_{\mathcal{G} \cdot \mathcal{F}} Z$ ; so  $X \equiv Z$ . ■

**3.2.2. REMARKS.** (i) (1.10) tells us that, for any  $X, Y \in KH$ ,  $X \equiv Y$  if either  $F(X) \equiv F(Y)$  or  $Z(X) \equiv Z(Y)$ . (1.5) tells us further that  $B(X) \equiv B(Y)$  if  $X \equiv Y$ . None of the converses are true; for, as remarked in [13], we could let  $X$  and  $Y$  be self-dense Boolean spaces such that  $X$  is “extremally disconnected” (i.e. closures of open sets are open) and  $Y$  fails to be basically disconnected. Then  $X \equiv Y$  since  $B(X)$  and  $B(Y)$  are atomless Boolean algebras, hence elementarily equivalent. But clearly  $F(X) \not\equiv F(Y)$  and  $Z(X) \not\equiv Z(Y)$ . To see that  $B(X) \equiv B(Y)$  does not imply  $X \equiv Y$ , simply let  $X$  be a self-dense Boolean space and let  $Y$  be the disjoint union of a self-dense Boolean space and a nontrivial continuum.

(ii) A less trivial consequence of coelementary equivalence can be obtained by a combination of the main techniques of [12] and [24]: Let  $X$  and  $Y$  be

coelementarily equivalent compact Hausdorff spaces. Then their respective Banach spaces of continuous real-valued functions “approximately” satisfy the same positive-bounded sentences (where quantification is restricted to the closed unit ball).

**3.2.3. PROPOSITION.** *Suppose  $X \equiv Y$ . Then there is an ultrafilter  $\mathcal{F}$  such that  $\sum_{\mathcal{F}} X \simeq \sum_{\mathcal{F}} Y$ .*

**PROOF.** Suppose  $\sum_{\mathcal{G}} X \simeq \sum_{\mathcal{E}} Y$ . Using (2.1.1), find an ultrafilter  $\mathcal{G}$  such that  $\sum_{\mathcal{G} \cdot \mathcal{E}} Y \simeq \sum_{\mathcal{G} \cdot \mathcal{G}} Y$ . Set  $\mathcal{F} = \mathcal{G} \cdot \mathcal{E}$ . ■

**3.2.4. THEOREM.** *Let  $X \equiv Y$ . Then:*

- (i)  *$X$  is connected iff  $Y$  is connected.*
- (ii)  *$X$  is self-dense iff  $Y$  is self-dense.*
- (iii)  *$\dim(X) = \dim(Y)$ .*
- (iv)  *$X$  is Boolean iff  $Y$  is Boolean.*

**PROOF.** (i) This follows from (1.5).

(ii)  $X$  is self-dense iff  $B(X)$  is atomless.

(iii) This follows from (2.2.2).

(iv) Immediate from (iii). ■

**3.2.5. THEOREM.** *There are exactly  $c$  coelementary equivalence classes in  $KH$ ; only countably many of them in  $BS$ .*

**PROOF.** That there are at most  $c$  classes in  $KH$  is immediate from the above remark (3.2.2(i)). That there are countably many classes in  $BS$  is an immediate consequence of the well-known fact [16] that the theory of Boolean algebras has countably many complete extensions. To finish the proof, we will exhibit  $c$  compact metrizable spaces, no two of which are coelementarily equivalent.

Let  $S$  be the set of all sequences  $s: \{1, 2, \dots\} \rightarrow \{0, 1\}$ . (Of course  $|S| = c$ .) For each  $s \in S$ , let  $X_s$  be the one-point compactification of the disjoint union  $X_{1,s(1)} \dot{\cup} X_{2,s(2)} \dot{\cup} \dots$ , where  $X_{n,s(n)}$  is either a singleton or the cube  $[0, 1]^n$ , depending upon whether  $s(n)$  is 0 or 1. Suppose  $s$  and  $t$  are distinct in  $S$ , say  $s(k) = 1$  and  $t(k) = 0$ . For convenience let  $Y = X_s$ ,  $Z = X_t$ ,  $Y_n = X_{n,s(n)}$  and  $Z_n = X_{n,t(n)}$ . Assuming  $Y \equiv Z$ , we can find ultrafilters  $\mathcal{D}$ ,  $\mathcal{E}$ , and a homeomorphism  $\delta: \sum_{\mathcal{D}} Y \rightarrow \sum_{\mathcal{E}} Z$ . Let  $\delta'$  be the induced isomorphism of clopen algebras. It is easy to see that, for any  $u \in S$ ,  $B(X_u)$  is isomorphic to the finite-cofinite algebra on  $\omega$ , and its atoms are the clopen sets  $X_{n,u(n)}$ . Thus, by (1.5),  $B(\sum_{\mathcal{D}} X_u)$  has atoms corresponding to ultracoproducts of the spaces  $X_{n,u(n)}$ . Since  $\delta'$  takes atoms to atoms, we infer that  $\delta$  takes  $\sum_{\mathcal{D}} Y_k$  to an ultracoproduct of the  $Z_n$ 's. But, by (2.2.2),  $\dim(\sum_{\mathcal{D}} Y_k) = k$ . Since no  $Z_n$  has dimension  $k$ , it is impossible for any ultracoproduct to have that dimension. (Either the dimensions of the factors are bounded, in which case the dimension of the ultracoproduct is finite and different from  $k$ ; or there is no bound on the dimension. In that case one can embed ultracopowers of arbitrarily high dimensional cubes in the ultracoproduct, forcing the dimension to be infinite.) This brings us to a contradiction, and to the conclusion that  $X_s \not\equiv X_t$  for distinct  $s, t \in S$ . ■

**3.2.6. REMARK.** The author admits to being somewhat surprised that the number of coequivalence classes in  $KH$  is  $c$ . At one time he recklessly conjectured that there was no cardinality at all; especially since it was his cherished belief (verified in (3.1.1(v))) that  $KH$  is not dual to any elementary  $P$ -class.

It is well known [17] that, for infinite  $X \in BS$ ,  $|B(X)| = w(X)$ . (The analogous statement goes through also for Pontryagin duality [26].) Moreover it is proved in

[13] that if  $R: BS \rightarrow \mathcal{K}$  is any duality onto an elementary  $P$ -class in which equalizers are embeddings and coequalizers are surjections, then  $|R(X)| = w(X)$  for any infinite  $X$ . Thus a case can be made that the weight for compact Hausdorff spaces is the correct “dual” to cardinality from the standpoint of model theory. This, of course, suggests the following “Löwenheim-Skolem” problem.

**3.2.7. QUESTION.** Let  $X \in KH$ . Can one always find a second countable (= metrizable)  $Y \in KH$  with  $X \equiv Y$ ?

**3.2.8. REMARKS.** (i) Of course the answer to (3.2.7) is yes when  $X \in BS$ .

(ii) Let  $X$  be infinite and extremally disconnected. If  $Y \in KH$  and  $F(Y) \equiv F(X)$  then  $Y$  is also extremally disconnected; hence  $B(Y)$ , an infinite complete Boolean algebra, is of cardinality  $\geq c$  [29]. Thus  $w(Y) \geq c$ . If  $Z(Y) \equiv Z(X)$  then  $Y$  is basically disconnected. In this case  $B(Y)$  is an infinite countably complete Boolean algebra, and must therefore be uncountable. Thus  $w(Y) \geq \omega_1$ .

(iii) We could settle (3.2.7) in the negative if we could answer (2.2.4(i)) positively. In fact, all we need to know is that if  $\text{Ind}(\sum_{\mathcal{O}} X) \leq 1$  then  $\text{Ind}(X) \leq 1$  (we already know  $\text{Ind}(X) = 1$  implies  $\text{Ind}(\sum_{\mathcal{O}} X) = 1$ ). For then we could let  $X \in KH$  be such that  $\text{dim}(X) = 2$  and  $\text{Ind}(X) = 1$  [28]. If  $Y \equiv X$  then  $Y$  cannot be second countable since  $\text{dim}(Y) \neq \text{Ind}(Y)$ .

(iv) Another way we might try to answer (3.2.7) negatively is to try to show that hereditary normality is preserved by coelementary equivalence. However, as we saw in (2.3.12(i)) and the proof of (2.3.14), every infinite ultracopower via a countably incomplete ultrafilter contains a copy of  $\beta(\omega)$ . This space is known [22] not to be hereditarily normal.

(v) A positive answer to (3.2.7) would give a good indication of how much weaker coelementary equivalence is than elementary equivalence of closed (or zero-) set lattices. A negative answer could give insight into the question of whether  $KH$  can be category dual to an elementary class.

Call a compact metrizable space  $X$  *categorical* if, whenever  $Y$  is compact metrizable and  $Y \equiv X$ , then  $Y \simeq X$ .

**3.2.9. PROPOSITION.**  $2^\omega$  is categorical.

**PROOF.** Suppose  $Y$  is compact metrizable,  $Y \equiv 2^\omega$ . Then, by (3.2.4),  $Y$  is also self-dense and Boolean. Thus  $Y \simeq 2^\omega$  by standard results [35]. ■

**3.2.10. QUESTION.** Is  $[0, 1]$  categorical?

**3.2.11. REMARKS.** (i) Of course if  $Y$  is compact metrizable and  $F(Y) \equiv F([0, 1])$  then  $Y \simeq [0, 1]$  because of the well-known characterization of the unit interval as the only compact metrizable space which is connected and has exactly two noncut points [35]. (In [23] it is shown that any metrizable  $Y$  must be homeomorphic with  $[0, 1]$  if  $F(Y) \equiv F([0, 1])$ ). This might tempt us to redefine “categorical” without the word “compact”. But then (3.2.9) would be false, since  $\mathbf{Q} \equiv \beta(\mathbf{Q}) \equiv 2^\omega$ ; and the answer to (3.2.10) would almost certainly be no.)

(ii) What makes (3.2.10) difficult is the fact that having exactly two noncut points is not preserved by coelementary equivalence. (If it were, then every ultracopower of  $[0, 1]$  would be linearly orderable. But then every ultracopower of  $[0, 1]$  would be hereditarily normal, contradicting Remark 3.2.8(iv).)

**3.3. Coelementary mappings.** To motivate the idea of “coelementary map”, let us look at the model-theoretic notion of “elementary embedding”. If  $A$  and  $B$  are two relational structures, a function  $\varepsilon: A \rightarrow B$  is an *elementary embedding* (in symbols

$\varepsilon: A < B$ ) if whenever  $\phi(v_1, \dots, v_n)$  is a formula with free variables among  $v_1, \dots, v_n$  and  $a_1, \dots, a_n \in A$  then the sentence  $\phi[a_1, \dots, a_n]$  (where  $a_i$  is “plugged in” for  $v_i$ ) is true in  $A$  iff the sentence  $\phi[\varepsilon(a_1), \dots, \varepsilon(a_n)]$  is true in  $B$ . This, of course, is equivalent to saying that the expanded structures, with constants naming each element of  $A$ , are elementarily equivalent (in symbols,  $(A, a)_{a \in A} \equiv (B, \varepsilon(a))_{a \in A}$ ). (Recall the proof of (2.1.1(iii)).) By the ultrapower theorem we therefore have

**3.3.1. PROPOSITION.**  $\varepsilon: A < B$  iff there are ultrafilters  $\mathcal{D}$  and  $\mathcal{E}$  and an isomorphism  $\delta: \prod_{\mathcal{D}} A \rightarrow \prod_{\mathcal{E}} B$  such that  $\delta \circ \Delta_{\mathcal{D}, A} = \Delta_{\mathcal{E}, B} \circ \varepsilon$ .

If  $X$  and  $Y$  are compact Hausdorff and  $\gamma: X \rightarrow Y$  is any function, call  $\gamma$  a *coelementary mapping* (in symbols  $\gamma: X > Y$ ) if there are ultrafilters  $\mathcal{D}$  and  $\mathcal{E}$  and a homeomorphism  $\delta: \sum_{\mathcal{D}} X \rightarrow \sum_{\mathcal{E}} Y$  such that  $\gamma \circ V_{\mathcal{D}, X} = V_{\mathcal{E}, Y} \circ \delta$ .

**3.3.2. THEOREM.** (i) Let  $\gamma: X > Y$ . Then  $\gamma$  is a continuous surjection which preserves covering dimension.

(ii) If  $X, Y \in BS$  and  $\gamma: X \rightarrow Y$  is continuous then  $\gamma$  is coelementary iff the Stone dual homomorphism  $B(\gamma): B(Y) \rightarrow B(X)$  is an elementary embedding.

(iii) Let  $\gamma: X \rightarrow Y$  and  $\delta: Y \rightarrow Z$  be functions. Then the coelementarity of  $\delta \circ \gamma$  (resp.  $\delta$ ) follows from the coelementarity of  $\gamma$  and  $\delta$  (resp.  $\gamma$  and  $\delta \circ \gamma$ ).

**PROOF.** (i) Let  $\mathcal{D}, \mathcal{E}$ , and  $\delta$  witness the coelementarity of  $\gamma$ . Then  $\gamma$  is obviously onto. If  $C \subseteq Y$  is a closed set then the equality  $\gamma^{-1}[C] = V_{\mathcal{D}}[\delta^{-1}[V_{\mathcal{E}}^{-1}[C]]]$  shows that  $\gamma^{-1}[C]$  must be closed in  $X$ . The fact that  $\gamma$  preserves dim follows from the coelementary equivalence of  $X$  and  $Y$ .

(ii) This follows directly from Stone duality and (3.3.1).

(iii) Suppose first that  $\gamma$  and  $\delta$  are coelementary. Draw the obvious mapping diagram and use the full power of (2.1.1) in a manner analogous to the way we proved (3.2.1).

Next suppose  $\gamma$  and  $\delta \circ \gamma$  are coelementary. Here we apply the same tricks but in a different order. First find homeomorphisms  $\phi: \sum_{\mathcal{D}} X \rightarrow \sum_{\mathcal{E}} Y$  and  $\psi: \sum_{\mathcal{F}} X \rightarrow \sum_{\mathcal{G}} Z$  making the obvious diagram commute. Next apply (2.1.1) and obtain the inevitable homeomorphism  $\theta: \sum_{\mathcal{H} \cdot \mathcal{F}} X \rightarrow \sum_{\mathcal{H} \cdot \mathcal{D}} X$ . Then the required homeomorphism between  $\sum_{\mathcal{H} \cdot \mathcal{E}} Y$  and  $\sum_{\mathcal{H} \cdot \mathcal{G}} Z$  is  $(\sum_{\mathcal{H}} \psi) \circ \theta^{-1} \circ (\sum_{\mathcal{H}} \phi)^{-1}$ . That the appropriate square commutes is a straightforward computation. ■

**3.3.3. PROPOSITION.** Suppose  $\gamma: X > Y$ . Then there is an ultrafilter  $\mathcal{F}$  and a homeomorphism  $\delta: \sum_{\mathcal{F}} X \rightarrow \sum_{\mathcal{F}} Y$  such that  $V_{\mathcal{F}, Y} \circ \delta = \gamma \circ V_{\mathcal{F}, X}$ .

**PROOF.** Proceed as in the proof of (3.2.3). ■

**3.3.4. EXAMPLES.** (i) Codiagonal maps are the prototypical examples of coelementary mappings.

(ii) Since any embedding between atomless Boolean algebras is elementary, it follows that every continuous surjection between self-dense Boolean spaces is a coelementary mapping.

(iii) Since coelementary mappings preserve covering dimension, it is easy to find continuous surjections which are not coelementary. In particular, restrictions of coelementary mappings to closed subspaces are not necessarily coelementary onto their images.

(iv) For  $X \in KH$ , let  $X^g$  be the (Gleason) projective cover of  $X$ , with covering map  $\gamma: X^g \rightarrow X$  (i.e.  $X^g$  is the Stone space of the regular-open algebra of  $X$ , and  $\gamma(p)$  is the unique point of  $X$  to which the regular-open ultrafilter  $p$  converges). Suppose  $\gamma$  is

coelementary. Since  $X^\theta \in BS$ ,  $X \in BS$ . Thus  $B(\gamma): B(X) \rightarrow B(X^\theta)$  is an elementary embedding. Now  $X^\theta$  is extremally disconnected; hence  $B(X^\theta)$  is a complete Boolean algebra. Thus the set of atoms of  $B(X)$  has a supremum (since this is a first-order condition). On the topological side this says that the closure of the set  $X'$  of isolated points (= the derived set) of  $X$  is open. Suppose conversely that  $X \in BS$  and that  $\text{Cl}(X')$  is open in  $X$  (so  $X$  is a disjoint union  $Y \dot{\cup} Z$ , where  $Y$  is self-dense and  $Z'$  is dense in  $Z$ ). Then the atoms of  $B(X)$  have a supremum (i.e.  $B(X)$  is “separable” [27]). In [27] it is shown that the class of such Boolean algebras is elementary and admits elimination of quantifiers, by the addition of one predicate which says of an element that it is an atom, and other predicates which say that an element contains  $n$  atoms ( $n = 1, 2, \dots$ ). Embeddings of models of this expanded theory are thus elementary; in particular the canonical embedding of a separable Boolean algebra into its MacNeille completion (= injective hull) is elementary. To recap:  $\gamma: X^\theta \rightarrow X$  is coelementary iff  $X \in BS$  and  $\text{Cl}(X')$  is open.

**3.5.5. QUESTIONS.** (i) Let  $X \in KH$ . Is there always a metrizable  $Y$  and a coelementary mapping  $\gamma: X \succ Y$ ?

(ii) Find examples of coelementary mappings between metrizable continua.

**Parting remark.** One could approach the theory of ultracopowers through the methods of nonstandard analysis [30]. This has been done in [20] and [24]. The ultracopower is now viewed as a “nonstandard hull” of an enlarged Tichonov space (equipped with A. Robinson’s “ $Q$ -topology”). This hull looks very much like a “nonstandard Stone-Ćech compactification”. As far as we can tell there is little overlap between this approach and ours, other than the basic facts concerning the preservation of connectedness and 0-dimensionality.

**Added in revision.** The answers to (2.4.5), (3.2.7), and (3.2.10) are yes, yes, and no in that order. In a recent letter, R. Gurevič gave a complete answer to (2.4.5) and suggested how techniques in his (unpublished) manuscript, *Topological model theory and factorization theorems*, could be used to settle (3.2.7) and (3.2.10). We have verified Gurevič’s claim regarding (3.2.7), and in fact we can use techniques of this paper to settle (3.3.5(ii)) in the affirmative. (This, by Remark (3.2.8(iii)), gives a negative answer to (2.2.4(i)).)

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