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THE TOTAL NEGATION OF A TOPOLOGICAL PROPERTY

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0. Introduction

The central theme in this paper is the uniform generation of new topological properties from old. Two of the best known properties obtained in this way are total disconnectedness (deriving from connectedness) and scatteredness (deriving from perfectness, i.e. having no isolated points). A third property, lesser known but interesting in its own right, is *pseudofiniteness* (the *cf-spaces* studied in [8], [9], [10], [12]) or the class of spaces whose compact subsets are finite. This last-mentioned property derives from compactness in the manner we will explore here.

In general, given a class K of topological spaces (K is closed under homeomorphism) we define the class Anti (K) in such a way that "totally disconnected" is co-extensive with "Anti (connected)" and so on. The "anti-property" of most interest to us here is pseudofiniteness which we henceforth relabel "anticompactness". We will also be interested in related anti-properties (Anti (sequentially compact), Anti (Lindelöf), etc.) but they will recieve secondary emphasis. The general behavior of the operation Anti $(\cdot)$ itself will occupy some of our attention. However at this stage there are many more questions than answers, so our general treatment will be sketchy, serving mainly to tie together ideas which otherwise may appear to be unrelated.

Our set-theoretic conventions are as follows: (i) ω_α denotes the αth infinite initial ordinal, where α is any ordinal. Since we assume the Axiom of Choice throughout, we identify ω_α with the cardinal $\aleph_\alpha$. $\omega = \omega_0$. (ii) An ordinal α is the set of its predecessors. Greek letters near the beginning of the alphabet will usually denote ordinals, while the letters κ, λ, μ will be reserved for cardinals. (iii) The ordinal successor of α is $\alpha+1=\alpha\cup\{\alpha\}$, the cardinal successor of κ is κ^+. (iv) If A is any set $P(A)$ denotes the power set of A. (v) B^A is the set of all maps f: $A \to B$. The cardinality of A is written $|A|$. (vi) If κ is a cardinal then $\exp(\kappa) = |2^\kappa| = |P(\kappa)|$. $\exp(\omega)$ is usually denoted by c. (vii) The cartesian product of a family $\langle A_i\colon i \in I\rangle$ of sets is denoted $\prod_I A_i$. If $A_i = A$ for all $i \in I$ then the set A^I will also at times be denoted $\prod_I (A)$. Further notations will be introduced as they arise in the discussion. The referee's kind suggestions regarding exposition are gratefully acknowledged.

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1. A Description of the process

Let K be a topological class. The *spectrum* Spec (K) of K is the class of cardinal numbers κ such that any topology on a set of power κ lies in K. For example, any topology on a finite set must be compact; and any infinite set supports noncompact topologies. Thus Spec ({compact spaces}) $= \omega$. Other spectra can be computed quite readily, such as Spec ({connected spaces}) $= 2(=\{0, 1\})$, and Spec ({perfect spaces}) $= 1$.

Now let K be a class of spaces and define Anti (K) to be the class of spaces X such that whenever $Y \subset X$, $Y \in K$ iff $|Y| \in$ Spec (K). Thus $X \in$ Anti (K) iff the only subspaces of X which are in K are those which "have to be" on account of their cardinalities. Clearly Anti (K) is always hereditary.

1.1 Proposition. Anti (K) *is never empty.*

Proof. If Anti (K) were empty for some K then for every X there would be a subspace Y with $Y \in K$ and $|Y| \notin$ Spec (K). Pick $X = \emptyset$. Then $\emptyset \in K$ but $0 \notin$ Spec (K). This is nonsense since there is only one topology on the empty set. ■

Remark. (1.1) shows that not every hereditary class need be of the form Anti (K). In a private communication, B. Scott has proved the following (the proof to appear elsewhere) Theorem: Let L be a hereditary class. Then L is not of the form Anti (K) iff there is an $n < \omega$ with $n \in$ Spec (L) and $X \notin L$ for all spaces X of power at least $n+1$. Moreover if L is of the form Anti (K) then K can be chosen to have empty spectrum; and if L contains spaces of all cardinalities then K can be taken to be the complement of L. ■

The reader can easily check that Anti ({connected spaces}) = {totally disconnected spaces}, Anti ({perfect spaces}) = {scattered spaces}, and Anti ({compact spaces}) = {pseudofinite spaces}. Other "anti-classes" are easy to compute as well.

We mention some of the general properties of the operation Anti $(\cdot)$. The proofs are straightforward.

1.2 Proposition. (i) *If* $K \subset L$ *then* Spec $(K) \subset$ Spec (L).

(ii) *If* $K \subset L$ *and* Spec (K) = Spec (L) *then* Anti $(K) \supset$ Anti (L). Anti (K) *and* Anti (L) *can be unrelated, however.*

(iii) Anti $(\cdot)$ *is not idempotent.*

(iv) *If* K *is hereditary then* $K \subset$ Anti (Anti (K)).

(v) Anti $(K) \subset$ Anti (Anti (Anti (K))) *for all* K.

Proof. (i) This is obvious.

(ii) Let K, L be as in the hypothesis, with $X \notin$ Anti (K). Let $Y \subset X$ be such that $Y \in K$ but $|Y| \notin$ Spec (K). Then $Y \in L$ but $|Y| \notin$ Spec (L); whence $X \notin$ Anti (L).

Let K = {compact spaces}, L = {Lindelöf spaces}. Then $K \subset L$ but Anti $(K) \not\subset$ Anti (L) and Anti $(L) \not\subset$ Anti (K) by (1.3(iii)).

(iii) Let $K = \{\text{compact spaces}\}$ again. Then $X \in \text{Anti}(\text{Anti}(K))$ iff every infinite subset of X contains an infinite compact set. The ordinal space $[0, \omega]$ $(= \omega + 1$ with the order topology) is in K but not in $\text{Anti}(\text{Anti}(K))$.

(iv) Let K be hereditary, with $X \notin \text{Anti}(\text{Anti}(K))$. Then for some $Y \subset X$, $|Y| \notin \text{Spec}(\text{Anti}(K))$ but $Y \in \text{Anti}(K)$. Since K is hereditary, we have $\text{Spec}(K) \subset \text{Spec}(\text{Anti}(K))$. Thus $|Y| \notin \text{Spec}(K)$. But $Y \in \text{Anti}(K)$; whence $Y \notin K$, so $X \notin K$.

(v) $\text{Anti}(K)$ is always hereditary, so use (iv). ■

In the remainder of this paper we will concentrate on $\text{Anti}(K)$ where K is one of the properties "compact", "sequentially compact", "Lindelöf".

1.3 PROPOSITION. (i) *If X is anti-Lindelöf, anti-(sequentially compact), and T_2 then X is anticompact.*

(ii) *If X is anticompact and T_1 then X is anti-(path connected).*

(iii) *Anticompactness and anti-Lindelöfness are implicationally unrelated.*

(iv) [12] *Anticompact spaces are anti-(sequentially compact). The converse is false.*

(v) *A T_2 space is discrete iff it is an anticompact k-space.*

Proof. (i) A T_2 space is anti-(sequentially compact) iff it contains no embedded copies of the compact ordinal space $[0, \omega]$. To see this, suppose first that $[0, \omega]$ is embedded in X. Then X would contain an infinite sequentially compact subspace. On the other hand suppose X fails to be anti-(sequentially compact). Let $Y \subset X$ be infinite and sequentially compact. Since Y is T_2, it contains a discrete sequence $\langle y_0, y_1, \ldots \rangle$. By sequential compactness there is a convergent subsequence, hence an embedded copy of $[0, \omega]$.

Now suppose X satisfies the hypotheses of (i). If $Y \subset X$ is compact then Y is also Lindelöf, hence countable. Thus, if Y isn't finite, Y is infinite compact metric and must contain a copy of $[0, \omega]$, an impossibility.

(ii) Let $f: [0, 1] \to X$ be a path in X. Since X is anticompact, range $(f) \subset X$ is a connected finite set. Since X is also T_1, range (f) must be a singleton.

(iii) The ordinal space $[0, \omega]$ is anti-Lindelöf but not anticompact. To get a space which is anticompact but not anti-Lindelöf, let $R^{\#}$ denote the real numbers with the topology basically generated by sets of the form (open interval)–(countable set). This is the classical example of a hereditarily Lindelöf nonseparable T_2 space. Since $|R^{\#}| = c$, this space cannot be anti-Lindelöf. However let $K \subset R^{\#}$ be compact. If K is infinite then K contains a countable subset. But such sets are closed discrete in $R^{\#}$. Thus K is finite and $R^{\#}$ is anticompact.

(iv) This is proved in [12] (c.f. the observation after their Theorem 5).

(v) Let X be T_2. If X is discrete it is clearly anti-compact as well as compactly generated. Conversely if X is an anticompact k-space and $Y \subset X$

then $Y \cap K$ is closed in K whenever K is compact (hence finite discrete) in X. Thus Y is closed in X, whence X is discrete. ■

Remark. The class of k-spaces is quite large, including the first countable spaces as well as the p-spaces of Archangel'skiĭ. Thus it doesn't take much to force an anticompact space to be discrete. We will see in §2, however, that nondiscrete anticompact spaces abound.

2. Anticompactness and connectedness

In view of the fact that anticompact T_1 spaces have trivial path components, one might suspect that connectedness and anticompactness are mutually inconsistent properties for reasonably nice (with regard to separation axioms) spaces. However our only nontrivial example, up to this point, of an anticompact space turns out also to be connected T_2 (but not regular). The space $R^{\#}$, originally suggested to me by F. Galvin, is connected for the following reason. Let $\mathfrak{U} = \langle U_\alpha - A_\alpha : \alpha < \kappa \rangle$ be a basic open cover of $R^{\#}$. That is, U_α is a nonempty open interval and A_α is countable for each $\alpha < \kappa$. It will suffice to show that $\mathfrak{U}$ is "connected" in the sense that whenever V, $W \in \mathfrak{U}$ there is a "simple chain" $M_1, \ldots, M_n \in \mathfrak{U}$ with $V \cap M_1 \neq \emptyset$, $M_n \cap W \neq \emptyset$, and $M_i \cap M_{i+1} \neq \emptyset$ for $1 \leq i \leq n-1$. But this is clearly true for $\mathfrak{U}$ since R is connected in its usual topology and the overlap of two open intervals is either empty or uncountable.

Let us now look at two important sources of anticompact examples, the P-spaces and the MI-spaces. X is a *P-space* if intersections of countably many open sets are open. A P-space which is also T_1 must be anticompact since countable subsets are always closed discrete. As far as existence is concerned, these spaces are quite common and there are many ways of systematically constructing them (see [2], [3], [4], [11], [13], [14], also §3). In [11] Misra constructs a connected T_2 P-space, so again connectedness and anticompactness co-occur in the presence of the Hausdorff axiom. The inevitable question then is whether there are any regular connected anticompact spaces. $R^{\#}$ is well-known to be nonregular. Moreover no regular P-space with more than one point is connected since, as can be seen in [2], [11], such spaces are always strongly zero-dimensional.

This brings us to our second source, namely the MI-spaces of Hewitt [7]. X is an *MI-space* if it is perfect, Hausdorff, and "sub-maximal" in the sense of [5], i.e. every dense subset is open. There are several ways of constructing these spaces (see [1], [5], [7], [9], [10]); and in [1] Anderson gives a uniform way of constructing connected examples. To complete the picture, Kirch [9] shows that MI-spaces are anticompact. To the best of our knowledge, however, it is an open question whether a connected MI-space can be regular.

As a side remark, the space $R^{\#}$ is neither a P-space nor an MI-space. For on the one hand each of its points is a G_δ; and on the other hand, $R^{\#}$ is "resolvable" into a disjoint union of two dense subsets (see [7]).

With the above ample introduction aside, we now answer our "inevitable" question in the affirmative with the following offering. This example owes its beginnings to an elightening conversation with E. K. van Douwen and M. E. Rudin.

2.1 *Example.* A connected anticompact Tichonov space which is resolvable, hence neither an MI-space nor a P-Space.

Construction. Let $A=[0,1)$ be the half-open unit interval with $A^*=\beta A-A$ its Stone-Čech remainder. We construct a subspace Σ of A^* and show that Σ, automatically a Tichonov space, satisfies the remaining conditions. The basic facts we use are the following:

(i) A^* is an indecomposable continuum (see Walker [15]).

(ii) There are exp (c) infinite closed subsets of A^*, and each has exp (c) points (again, see [15]).

(iii) If X is any connected T_1 space and $p\in X$ is a cutpoint of X then there are disjoint nonempty open sets U, V in X with $U\cup V=X-\{p\}$ and $U\cup\{p\}$, $V\cup\{p\}$ connected (see Ward [16]).

We first prove the claim that if X is a nondegenerate indecomposable continuum and $F\subset X$ is finite then $X-F$ is connected. Induct on $|F|$. If $|F|=0$ there's nothing to prove, so suppose $X-F$ is connected with $p\in X-F$. We show $X-(F\cup\{p\})$ is connected by proving that p isn't a cutpoint of $X-F$. For if it were then (since finite sets are closed) there would be disjoint nonempty open sets U, V of X with $U\cup V=X-(F\cup\{p\})$ and $U\cup\{p\}$, $V\cup\{p\}$ connected. Now $X-(F\cup\{p\})$ is dense in X. Thus

$$\mathrm{Cl}\,(U\cup\{p\})\cup\mathrm{Cl}\,(V\cup\{p\})=X.$$

But neither subcontinuum is all of X. This contradicts indecomposability.

Now, using (ii) above, let $\langle F_\alpha : \alpha<\exp(c)\rangle$ be a well ordering in type exp (c) of the infinite closed subsets of A^*. By induction we can pick distinct points

$$x_\alpha^{(i)}\in F_\alpha-\bigcup_{\beta<\alpha}\{x_\beta^{(1)},x_\beta^{(2)},x_\beta^{(3)}\},\quad i=1,2,3,\ \alpha<\exp(c).$$

Let $X^{(i)}=\{x_\alpha^{(i)}:\alpha<\exp(c)\}$, $i=1,2,3$, and set $\Sigma=X^{(1)}\cup X^{(2)}$. We check that Σ has the properties we want.

Let U be nonempty open in A^*, $x\in U$. Let V be open in A^* with $x\in V\subset\mathrm{Cl}\,(V)\subset U$. Then $\mathrm{Cl}\,(V)$ is infinite closed in A^* so hits each $X^{(i)}$, $i=1,2,3$; whence each $X^{(i)}$ is dense in A^*. Thus Σ is resolvable into the disjoint union of the dense subsets $X^{(1)}$, $X^{(2)}$, so isn't an MI-space. Σ is anticompact since if K is a compact subset then K is closed in A^*. Since infinite closed sets share points with $X^{(3)}\supset A^*-\Sigma$, K must be finite. To see that Σ is connected let $\mathfrak{U}$ be a collection of open subsets of A^* which covers Σ. We show the relativized cover $\mathfrak{U} \restriction \Sigma$ to be connected. Since Σ is dense in A^*, it will suffice to show that $\mathfrak{U}$ itself is connected. But $A^* - \bigcup \mathfrak{U}$ is finite; and hence by (i), (iii) above, $\bigcup \mathfrak{U}$ is a connected set. The argument is completed by noting that Σ is not a P-space since it is regular and not zero-dimensional. ■

We end this section with some open questions.

2.2 *Problems.* (i) Are there nontrivial examples of connected anticompact spaces which are normal? paracompact? We don't know whether Σ above is normal.

(ii) Are there regular connected anticompact spaces of power c? Our space Σ has power $\exp(c)$.

(iii) Are regular anticompact spaces always Tichonov? We know anticompactness fails to collapse any of the other pairs of well-known separation axioms.

(iv) Find an interesting class of spaces (not contained in the class of k-spaces) whose intersection with the anticompact spaces is contained within the totally disconnected spaces.

3. Preservation of anticompactness

In this section we consider questions involving the preservation of anticompactness under topological operations. For instance anticompactness is trivially preserved by open bijections (e.g. expansion of topologies). Also anticompactness is preserved by "compact covering maps" (i.e. continuous maps such that compact sets in the range are images of compact sets in the domain).

We next turn our attention to the preservation of anticompactness under various topological product formations. The following is stated in [12].

3.1 Proposition. *The Tichonov product of topological spaces is anticompact iff all of the factors are anticompact and all but finitely many of them are singletons.* ■

A generalization of the Tichonov product is the "λ-box product" where λ is an infinite cardinal. Specifically let $\langle X_i : i \in I\rangle$ be a collection of spaces with $\prod_I X_i$ denoting the cartesian product of the underlying sets of the X_i's. An *open λ-box* is a product $\prod_I U_i$ where U_i is open in X_i and $|\{i: U_i \neq X_i\}| < \lambda$. The *$\lambda$-box product*, denoted by $\prod_I^\lambda X_i$, is the space with underlying set $\prod_I X_i$ and topology generated by the open λ-boxes. The Tichonov product is then $\prod_I^\omega X_i$; and the full box product is $\prod_I^{|I|^+} X_i$, denoted $\prod_I^\infty X_i$.

3.2 Theorem. *Let $\langle X_i : i \in I\rangle$ be a collection of T_1 spaces with $\omega_1 \leq \lambda \leq \infty$. Then $\prod_I^\lambda X_i$ is anticompact iff each X_i is anticompact.*

Proof. Suppose $\prod_I^\lambda X_i$ is anticompact, $j \in I$. Then X_j embeds in $\prod_I^\lambda X_i$ and is thus anticompact.

Conversely suppose each X_i is anticompact. Since $\lambda \geq \omega_1$ and anticompactness is preserved under expansion of topologies, we need only prove the assertion for $\lambda = \omega_1$. So let K be compact in $\prod_I^{\omega_1} X_i$ with K_j the jth projection of K. Then each K_i is compact, hence finite; and is therefore discrete since X_i is T_1. So $K \subset \prod_I^{\omega_1} K_i$, an ω_1-box product of discrete spaces. But such spaces are clearly P-spaces. Since they are also T_1, they are therefore anticompact. Thus K is finite. ■

Another generalized product (generalizing not the Tichonov product but the box product) is the topological reduced product. This construction, borrowed from model theory, is studied in [2], [3], [4] and is defined as follows. Let $\langle X_i : i \in I \rangle$ be a collection of spaces with D a filter of subsets of I. In $\prod_I^\infty X_i$ define the equivalence relation $x \sim_D y$ if $\{i: x_i = y_i\} \in D$. Let $\prod_D X_i$ be the resulting quotient space. This space is the *D-reduced product* of the X_i's. Clearly $\prod_{\{I\}} X_i = \prod_I^\infty X_i$; and if D is an ultrafilter on I then $\prod_D X_i$ is called the *D-ultraproduct* of the X_i's. We consider the preservation of anticompactness under reduced products. The reader is assumed to have a nodding acquaintance with some of the lore of measurable cardinals and of the set-theoretic properties of filters. In particular a filter D is *λ-complete* if D is closed under $<\lambda$ intersections. D is *λ-regular* if there is a set $E \subset D$ of power λ such that each $i \in I$ is contained in only finitely many members of E. An ultrafilter D is ω-regular iff it is ω_1-incomplete (i.e. countably incomplete). This is an important property of ultrafilters.

3.3 Lemma. *Let $\langle X_i : i \in I \rangle$ be a collection of spaces with D a λ-regular filter on I. Then $\prod_D X_i$ is a P_{λ^+}-space (i.e. intersections of $\leq \lambda$ open sets are open).*

Proof. This is proved in [2] for D a λ-regular ultrafilter. The proof for arbitrary λ-regular D is identical. ■

3.4 Theorem. *Let $\langle X_i : i \in I \rangle$ be a collection of T_1 spaces, with D an ω-regular filter on I. Then $\prod_D X_i$ is anticompact.*

Proof. Clearly $\prod_D X_i$ is T_1. By (3.3) it is also a P_{ω_1}-space ($=P$-space); and is hence anticompact. ■

Thus the ω-regular reduced product formation not only preserves anticompactness of T_1 spaces, it confers the property for free. We next consider the behavior of countably complete filters.

3.5 Theorem. *Let $\langle X_i : i \in I \rangle$ be a collection of anticompact T_2 spaces with D a countably complete ultrafilter on I. Then $\prod_D X_i$ is anticompact.*

Proof. D is either fixed, in which case we're done, or free and μ-complete where μ is a measurable cardinal (see [6]). Let C be compact in

$\prod_D X_i$ and assume C is infinite. If $K \subset C$ is countable then of course $\mathrm{Cl}(K) \subset C$ is compact. We show first that $|\mathrm{Cl}(K)| < \mu$. Indeed reduced products preserve the Hausdorff axiom so $\mathrm{Cl}(K)$ is a separable T_2 space; and its cardinal therefore is $\leq \exp(c)$. This follows from a well-known general fact about T_2 spaces; for let X be T_2, let $S \subset X$ be dense, and let

$$\mathfrak{U}_x = \{U \cap S \colon U \text{ is an open neighborhood of } x \in X\}$$

Then $\mathfrak{U}_x \neq \mathfrak{U}_y$ whenever $x \neq y$, so we obtain a one-one map of X into

$$P(\{U \cap S \colon U \text{ open in } X\});$$

whence $|X| \leq \exp(\exp(d(X)))$, where $d(X)$ is the density character of X. Since measurable cardinals are inaccessible, we have $|\mathrm{Cl}(K)| < \mu$. Now let

$$K_i = \{x_i \colon \text{there is a } [f]_D \in \mathrm{Cl}(K) \text{ with } f_i = x_i\}.$$

We show $\mathrm{Cl}(K) = \prod_D K_i$. First it is clear that $\mathrm{CL}(K) \subset \prod_D K_i$ so suppose that $[f]_D \in \prod_D K_i - \mathrm{Cl}(K)$. Then $\{i \colon f_i \in K_i\} \in D$; and for each $[g]_D \in \mathrm{Cl}(K)$, $\{i \colon f_i \neq g_i\} \in D$. Since $|\mathrm{Cl}(K)| < \mu$ and D is μ-complete we know

$$\{i \colon f_i \in K_i \text{ and for all } [g]_D \in \mathrm{Cl}(K),\ f_i \neq g_i\} \in D.$$

In particular there is an $i \in I$ with $f_i \in K_i$ and $f_i \neq g_i$ for any $[g]_D \in \mathrm{Cl}(K)$, contradicting the definition of K_i.

Now since $\mathrm{Cl}(K) = \prod_D K_i$ is compact, it is a basic fact about topological ultraproducts (see [2]) that $\{i \colon K_i \text{ is compact}\} \in D$. Since each X_i is anticompact, $\{i \colon K_i \text{ is finite}\} \in D$. Finally, since D is countably complete, it follows that $\prod_D K_i$ is finite, a contradiction. Thus C must have been finite to begin with. ■

Now ultrafilters are either countably complete or ω-regular. Thus combining (3.4) and (3.5) gives:

3.6 Corollary. *Topological ultraproducts of anticompact T_2 spaces are anticompact.* ■

Remark. The argument in (3.5) clearly requires that the countably complete filter D be an ultrafilter and that the spaces X_i be Hausdorff (as opposed to T_1 for λ-box products and ω-regular reduced products). The inequality $|X| \leq \exp(\exp(d(X)))$ fails for T_1 spaces since the cofinite topology on any set is separable T_1.

3.7 *Problem.* Do all reduced products preserve anticompactness? An interesting special case to consider might be whether the countably complete filter generated by the closed unbounded subsets of $[0, \omega_1)$ preserves anticompactness. Judging by the special properties of countably complete ultrafilters which we had to use in proving (3.5), it seems likely that a counterexample awaits discovery here.

4. The anti-Lindelöf property

Let κ be a cardinal, X a space. X is *κ-compact* if every open cover of X has a subcover of power $<\kappa$. Thus X is anti-κ-compact iff the only subsets of X which are κ-compact are those of power $<\kappa$. Of course compact $=\omega$-compact and Lindelöf $=\omega_1$-compact. We concern ourselves here with generalizing the results of Sections 2 and 3. Unfortunately many of these results have proved highly resistant to generalization, so there is no shortage of open questions in this connection. Since many of the difficulties which arise for general κ evidence themselves already in the case $\kappa=\omega_1$, this is the case which will receive most of our attention.

Let us first examine the existence question. The space consisting of $I\cup\{\infty\}$, where I is uncountable discrete, $\infty\notin I$, and the neighborhoods of ∞ are of the form $J\cup\{\infty\}$ where $I-J$ is countable, is Lindelöf and not anti-Lindelöf, as well as a paracompact P-space. So, not surprisingly, P-spaces do not provide us automatically with anti-Lindelöf examples. As regards the MI-spaces, less is known.

4.1 *Problem.* Can uncountable MI-spaces be Lindelöf?

4.2 PROPOSITION. *If X is a T_1 P_{λ^+}-space then X is anti-κ-compact for $\kappa\leq\lambda$.*

Proof. Every subset of X of power $\leq\lambda$ is closed discrete. ■

As we saw in (3.3), λ-regular reduced products of T_1 spaces provide an excellent source of anti-λ-compact examples.

4.3 *Example.* For each infinite cardinal λ, a space of power λ which is nondiscrete, paracompact, and anti-κ-compact for each $\kappa\leq\lambda$.

Construction. Let I be discrete of power λ and let $p\in\beta I-I$. Let $X=I\cup\{p\}\subset\beta I$. This space clearly has the desired properties. ■

The space $R^{\#}$ introduced in the proof of (1.3iii) motivates the next example.

4.4 *Example.* For each infinite cardinal λ, a connected $\exp(\lambda)$-compact T_2 space of power $\exp(\lambda)$ which is anti-κ-compact for $\kappa<\exp(\lambda)$.

Construction. Let X be the cube $[0,1]^{\lambda}$ where we allow sets of the form (open Tichonov box) $-$ (set of power $<\exp(\lambda)$) to form a topological basis. Thus sets of power $<\exp(\lambda)$ are automatically closed. Clearly X is T_2 and of power $\exp(\lambda)$. Also since sets of power $<\exp(\lambda)$ are discrete in X, this space is clearly anti-κ-compact for $\kappa<\exp(\lambda)$. X is connected for almost exactly the same reason that $R^{\#}$ is connected. X is $\exp(\lambda)$-compact by a straightforward argument using the fact that $[0,1]^{\lambda}$ with the Tichonov topology is compact. ■

4.5 *Problem.* Are there any regular (Tichonov) connected anti-Lindelöf spaces?

We next look for analogues of the theorems in §3 regarding preservation of anti-Lindelöfness. For example it is easy to see that this property is preserved under open bijections as well as "Lindelöf covering maps". Nonetheless it turns out that anti-Lindelöfness is generally a more difficult property to work with than is anticompactness.

The proof of the following is identical to that of (3.1) [12].

4.6 PROPOSITION. *The Tichonov product of topological spaces is anti-Lindelöf iff all of the factors are anti-Lindelöf and all but finitely many of them are singletons.* ■

In analogy with (3.2) we have:

4.7 THEOREM. *Let $\langle X_i : i \in I\rangle$ be a collection of T_1 P-spaces with $\omega_2 \le \lambda \le \infty$. Then $\prod_I^\lambda X_i$ is anti-Lindelöf iff each X_i is anti-Lindelöf.*

Proof. Mimic the proof of (3.2). At the stage where $K \subset \prod_I^{\omega_2} K_i$ (i.e. K is Lindelöf, K_i is the ith projection of K, and each K_i is discrete, owing to the fact that X_i is a T_1 P-space with K_i countable), we use the fact that ω_2-box products of discrete spaces are P_{ω_2}-spaces which in turn are anti-Lindelöf. Thus K is countable. ■

Concerning ω_1-box products of anti-Lindelöf spaces, the "real" analogue of (3.1) is:

4.8 THEOREM. *Suppose $\langle X_i : i \in I\rangle$ is a collection of T_1 spaces.*

(i) *If $|\{i : |X_i| \ge 2\}| \ge \omega_1$ then $\prod_I^{\omega_1} X_i$ is not anti-Lindelöf.*

(ii) *If $|\{i : |X_i| \ge 2\}| \le \omega$ and each X_i is an anti-Lindelöf P-space then $\prod_I^{\omega_1} X_i$ is anti-Lindelöf.*

Proof. (i) If each X_i is T_1 and uncountably many X_i's have more than one point then $\prod_I^{\omega_1} X_i$ contains a copy of $\prod_{\omega_1}^{\omega_1}(2)$ which, while not Lindelöf itself (it further contains a copy of $\prod_\omega^{\omega_1}(2)$, an uncountable closed discrete subset), fails to be anti-Lindelöf. To see this, let $(X)_\lambda$ denote the space obtained from X by closing up the topology of X under intersections of $<\lambda$ open sets. It is a straightforward matter to prove that if X is discrete and λ is a regular cardinal then for any index set J, $\prod_J^\lambda(X)$ is homeomorphic to $(\prod_J^\omega(X))_\lambda$. Thus $\prod_{\omega_1}^{\omega_1}(2) \simeq (\prod_{\omega_1}^{\omega}(2))_{\omega_1}$. Let $F : \omega_1 \to 2^{\omega_1}$ be defined by

$$F(\alpha)(\beta) = \begin{cases} 0 & \text{if } \beta < \alpha \\ 1 & \text{if } \beta \ge \alpha \end{cases}$$

Then range (F) is a discrete subset of $\prod_{\omega_1}^{\omega_1}(2)$ (i.e. F is an "ω_1-Cauchy sequence" in the sense of Sikorski (see [13], [14])), and it has precisely one limit point, namely the zero sequence. Thus range $(F) \cup$ {zero sequence} is a copy of the modified ordinal space $([0, \omega_1])_{\omega_1}$ which itself is uncountable Lindelöf.

(ii) A box product of countably many anti-Lindelöf P-spaces is anti-Lindelöf. For again look at the proof of (3.2). When we get to the stage $K \subset \prod_I^{\omega_1} K_i$ (where I is countable), each K_i is discrete and box products of discrete spaces are discrete. Thus K is countable. ■

4.9 *Problem.* Is the box product of countably many copies of $[0, \omega]$ anti-Lindelöf?

We now move to consider reduced products of anti-Lindelöf spaces. By (3.3) and (4.2), an ω_1-regular reduced product of T_1 spaces is anti-Lindelöf as well as anticompact. Moreover the analogy of (3.5) goes through intact. The proof is virtually the same (with the obvious minor adjustments).

4.10 THEOREM. *Let $\langle X_i \colon i \in I \rangle$ be a collection of anti-Lindelöf T_2 spaces with D a countably complete ultrafilter on I. Then $\prod_D X_i$ is anti-Lindelöf.* ■

4.11 COROLLARY. *Topological ultraproducts of anti-Lindelöf T_2 spaces are anti-Lindelöf, provided the ultrafilters are either countably complete or ω_1-regular.* ■

We leave the subject with the obvious question implied by (4.11), namely:

4.12 *Problem.* Decide whether free ultrafilters on the integers (as examples of ω-regular but not ω_1-regular ultrafilters) yield preservation of anti-Lindelöfness for T_2 spaces.

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