The Non-Cut Point Existence Theorem Almost A Century Later

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Birmingham Topology Seminar,
3 March, 2015.

1. Moore's Theorem.

• A point c of a connected topological space X is a **non-cut point** if $X \backslash \{c\}$ is connected; otherwise c is a **cut point** of X.

• Every point of the real line is a cut point; no point of the real plane is. (This is one way of showing new topology students that $\mathbb{R}$ and $\mathbb{R}^2$ are non-homeomorphic, despite being of the same cardinality.)

• Define a **continuum** to be a topological space that is both connected and compact. A continuum is **nondegenerate** if it has at least two points.

• Theorem 1.1 (R. L. Moore, 1920). *Every nondegenerate metrizable continuum has at least two non-cut points.*

• Moore actually proves a stronger result: *If a nondegenerate metrizable continuum has no more than two non-cut points, then it must be an arc.*

The purpose of this talk is to survey some of the work that has grown out of Moore's theorem in the 95 years since its publication.

2. Whyburn's Theorem.

• Theorem 2.1 (G. T. Whyburn, 1968). *Every nondegenerate* T_1 *continuum has at least two non-cut points.*

Note that any infinite set with the cofinite topology is a T_1 continuum. In this case it's easy to see that every point is non-cut.

A continuum X is **irreducible** about a subset S if no proper subcontinuum of X contains S.

Whyburn, by judicious use of Zorn's lemma, proved that if c is a cut point of X and $\langle U, V \rangle$ is a disconnection of $X \setminus \{c\}$ into disjoint nonempty open sets (open in X because of T_1), then each of U and V contains a non-cut point of X. As a consequence, we have

• Corollary 2.2. *A nondegenerate* T_1 *continuum is irreducible about its set of non-cut points.*

Proof. Let N be the set of non-cut points of X, with K a proper subcontinuum of X containing N. Let $c \in X \setminus K$. Then c is a cut point of X; hence there is a disconnection $\langle U, V \rangle$ of $X \setminus \{c\}$. But $K \subseteq U \cup V$ can't intersect both U and V because of being connected; say $K \cap V = \emptyset$. Whyburn's theorem tells us there is a non-cut point in V, a contradiction. Hence no proper subcontinuum of X can contain all the non-cut points of X $\Box$

3. Further Developments I: Shore Points.

In their study of dendroids and dendrites, I. Puga-Espinosa et al (1990s) introduced the notion of *shore point*; and in her 2014 paper, *Shore points of a continuum*, R. Leonel took the study of shore points into the realm of general metrizable continua.

• Let X be a metric continuum. $p \in X$ is a **shore point** if for any $\epsilon > 0$ there is a subcontinuum $K \subseteq X \setminus \{p\}$ which is ϵ-close to X, relative to the Hausdorff metric on the hyperspace of subcontinua of X.

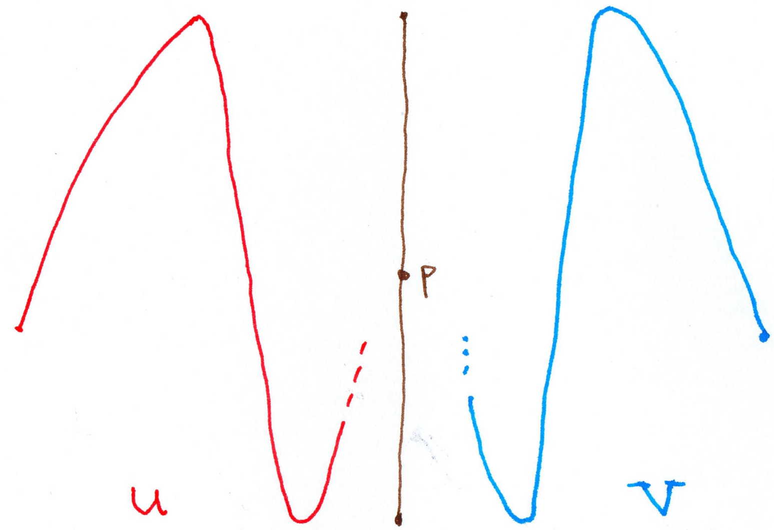
Because, for metric continua, the Hausdorff metric gives rise to the Vietoris topology, we quickly have the following, which allows the definition of *shore point* to make sense for any topological space.

• Proposition 3.1. *In a metric continuum* X, p *is a shore point of* X *iff whenever* $\mathcal{U}$ *is a finite family of nonempty open subsets of* X, *there is a subcontinuum* $K \subseteq X \setminus \{p\}$ *intersecting each set in* $\mathcal{U}$.

• Proposition 3.2. *A shore point of a* T_1 *continuum is a non-cut point; the converse fails in general for metrizable continua.*

Proof of Proposition 3.2. If $c \in X$ is a cut point, we have a disconnection $\langle U, V \rangle$ of $X \setminus \{c\}$. U and V are both open in X because $\{c\}$ is closed. No connected subset of $X \setminus \{c\}$ can intersect both U and V; hence c cannot be a shore point of X.

An example of a metrizable continuum with a a non-cut point that is not a shore point is depicted on the next slide. $\square$



p is non-cut; any subcontinuum intersecting both open sets U and V must contain p. ∴ p isn't shore.

Leonel proved that every metrizable continuum has at least two shore points, as a consequence of a 1948 result of Bing. First some notation:

• If A and P are subsets of X, denote by $\kappa(A;P)$ the **relative composant**, consisting of the union of all proper subcontinua of X that contain A and are disjoint from P.

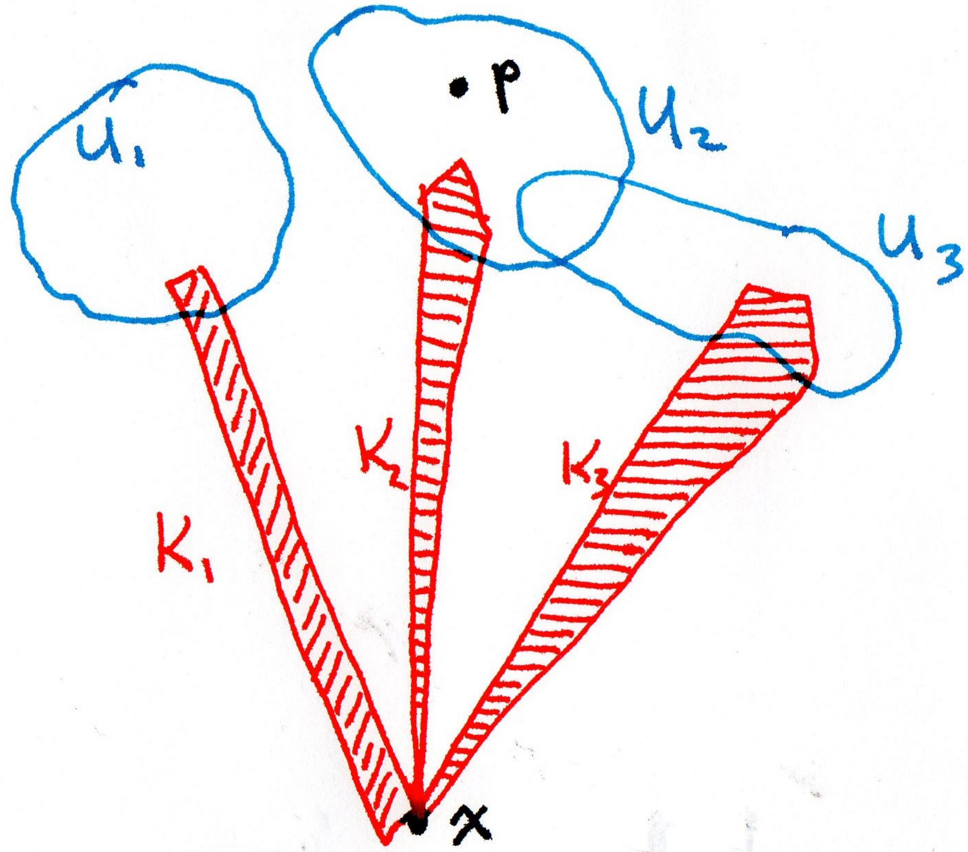
Note that if $A=\{a\}$, then $\kappa(\{a\};\emptyset)$ is the usual composant $\kappa(a)$ of X containing the point a. If X is a Hausdorff continuum and $a\in X$, $\kappa(a)$ is well known to be dense in X; the same argument shows $\kappa(A):=\kappa(A;\emptyset)$ to be dense whenever A is a proper subcontinuum of X.

• Theorem 3.3 (R. H. Bing, 1948). *If X is a metrizable continuum and A is a proper subcontinuum of X, then there exists a point $p\in X$ with $\kappa(A;p):=\kappa(A;\{p\})$ dense in X.*

• Corollary 3.4 (R. Leonel, 2014). *Each nondegenerate metrizable continuum contains at least two shore points.*

Proof. Pick $x \in X$ arbitrary; by Bing's theorem 3.3, pick $p \in X$ with $\kappa(x; p)$ dense in X. If $U_1, \ldots, U_n$ are nonempty open sets, use density to find subcontinua $K_1, \ldots, K_n$ such that: for each $1 \leq i \leq n$, K_i is a subcontinuum that contains x, doesn't contain p, and intersects U_i. Then $K = K_1 \cup \cdots \cup K_n$ is a subcontinuum that doesn't contain p, and which intersects each U_i, $1 \leq i \leq n$.

Hence we have one shore point $p \in X$. Now use Bing's theorem again to find $q \in X$ such that $\kappa(p; q)$ is dense in X. $\square$



$K_1 \cup K_2 \cup K_3$ intersects each of U_1, U_2, U_3.

Bing's theorem 3.3 actually shows more.

• Corollary 3.5. *A nondegenerate metrizable continuum is irreducible about its set of shore points.*

Proof. Let A be a proper subcontinuum of X. By Bing's theorem, there is a point $p \in X$ with $\kappa(A; p)$ dense in X. By a simple argument similar to the above, we see that p is a shore point; hence no proper subcontinuum can contain all shore points of a metrizable continuum. $\square$

What is significant in its absence is an analogue of Whyburn's Theorem 2.1. We restrict ourselves to the Hausdorff case and state the following.

• Open Problem 3.6. *Does Bing's Theorem 3.3 work for Hausdorff continua?*

If so, then one can show that every nondegenarate Hausdorff continuum is irreducible about its set of shore points.

4. Further Developments II: A Reduction Process.

• Define a Hausdorff continuum X to be **coastal** about a proper subcontinuum $A \subseteq X$ if $\kappa(A;p)$ is dense in X for some $p \in X$. X is **coastal** if it is coastal about each of its proper subcontinua.

Bing showed each metrizable continuum to be coastal; we do not know whether this is still true for an arbitrary Hausdorff continuum.

• Theorem 4.1 (D. Anderson, 2015). *If X is a Hausdorff continuum that fails to be coastal at proper subcontinuum A, then there is a continuous surjection $f : X \to Y$ where: (i) Y is an indecomposable Hausdorff continuum with exactly one composant; and (ii) Y fails to be coastal at the proper subcontinuum $f[A]$, which may be taken to be a single point.*

Note that two points lie in the same composant of a continuum if there is a proper subcontinuum containing them both. A continuum is indecomposable iff this binary relation is transitive; i.e., an equivalence relation. Hence indecomposable continua sporting more than one composant have to be coastal since composants are dense.

For any space X, let $d(X)$ be its **density**; i.e., the minimal cardinality of a dense subset. A space X is d-**Baire** if intersections of at most $d(X)$ dense open subsets are dense. The Baire category theorem says that separable compact Hausdorff spaces are d-Baire.

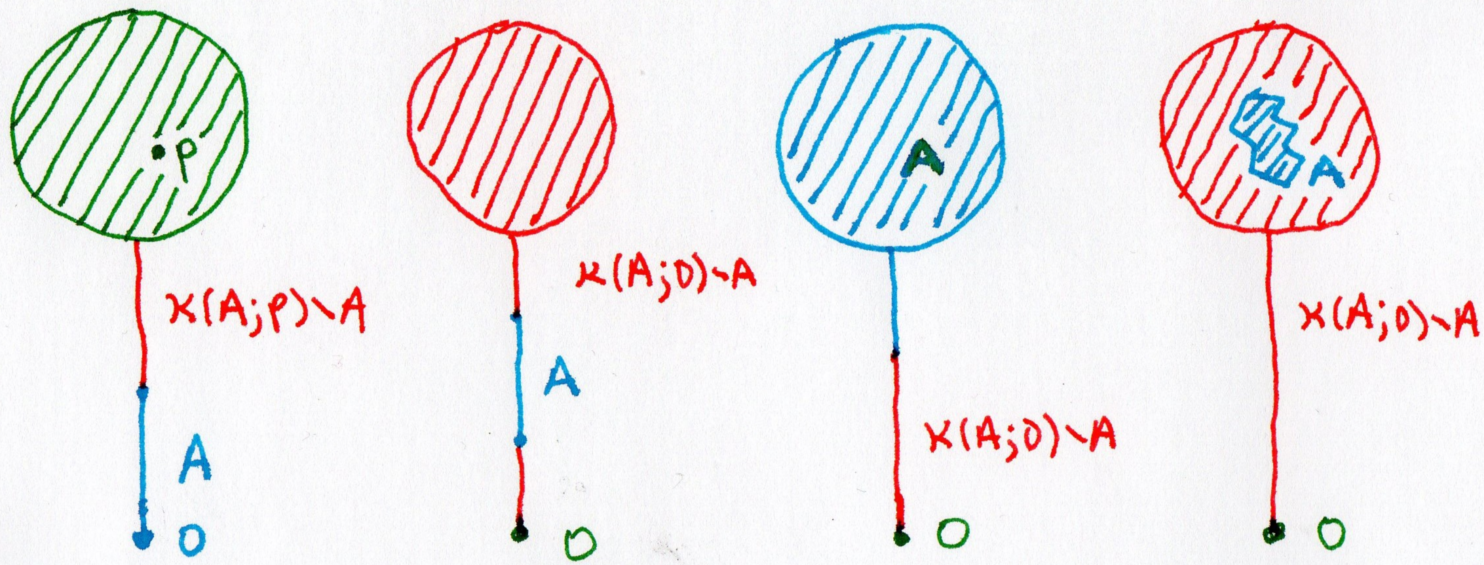
Using Theorem 4.1, Anderson has been able to show that separable Hausdorff continua are coastal, and hence irreducible about their sets of shore points. Indeed, he proves the more general result:

• Theorem 4.2 (D. Anderson, 2015). *Each d-Baire Hausdorff continuum is coastal.*

An interesting example of a separable continuum that isn't metrizable is the Stone-Čech compactification $\beta[0,1)$ of the half-open unit interval. Theorem 4.2 implies $\beta[0,1)$ is coastal; however we can see this directly:

If $A \subseteq \beta[0,1)$ is a proper subcontinuum that contains 0, then $A \subseteq [0,1)$ and thus $\kappa(A;p) = [0,1)$ is dense for any $p \in [0,1)^* := \beta[0,1) \setminus [0,1)$.

If A doesn't contain 0, then $\kappa(A;0) = \beta[0,1) \setminus \{0\}$ is dense.



The four representative scenarios for a proper subcontinuum A of $\beta[0,1)$.

What is even more interesting is the question of whether $[0,1)^*$ is coastal: It was shown in 1971 by D. Bellamy that $[0,1)^*$ is an indecomposable continuum. Hence its composants are pairwise disjoint. If there are at least two composants, then $[0,1)^*$ is clearly coastal, as mentioned earlier.

On the other hand, it is consistent with ZFC (see, e.g., A. Blass' paper on the Near Coherence of Filters axiom) that $[0,1)^*$ has exactly one composant. Is it still coastal?

5. Further Developments III: Distal Points.

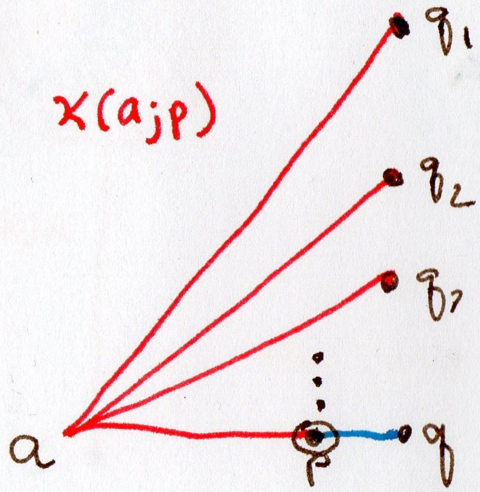
• Given continuum X and proper subcontinuum A, let $\mathcal{K}(A)$ be the family $\{\kappa(A;q) : q \in X\}$, partially ordered by set inclusion. Define $p \in X$ to be A-**distal** if $\kappa(A;p)$ is maximal in $\mathcal{K}(A)$. A point is **distal** if it is A-distal for some proper subcontinuum A.

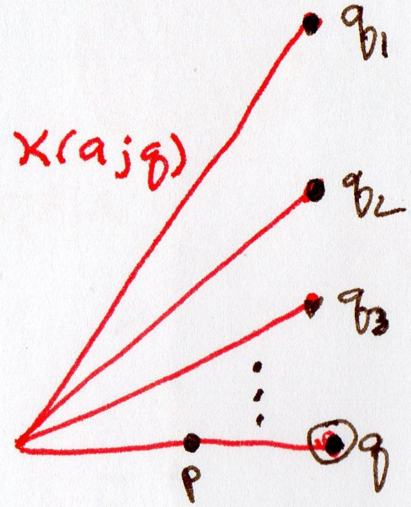
• Proposition 5.1. *If X is a Hausdorff continuum, A is a proper subcontinuum and $p \in X$ is A-distal, then $\kappa(A;p)$ is dense in X. Thus distal points are shore points; the converse fails in general for metrizable continua.*

Proof of Proposition 5.1. For any $p \in X \setminus A$, $\kappa(A;p)$ is a connected set that contains A but does not contain p. $\overline{\kappa(A;p)}$ is therefore a subcontinuum of X that contains A. If it did not contain p, then boundary bumping would allow a subcontinuum $M \subseteq X \setminus \{p\}$ that properly contains $\overline{\kappa(A;p)}$, a contradiction. Hence $p \in \overline{\kappa(A;p)}$.

Suppose $\kappa(A;p)$ is not dense in X and let $q \in X \setminus \overline{\kappa(A;p)}$. Then $\kappa(A;p) \subseteq \overline{\kappa(A;p)} \subseteq \kappa(A;q)$. The two relative composants can't be equal; otherwise we would have $q \notin \overline{\kappa(A;q)}$, a contradiction. Hence $\kappa(A;p)$ is not maximal in $\mathcal{K}(A)$.

That distal points are shore points immediately follows. A depiction of a shore point that is not distal is left to the next slide. $\square$





p is a shore point which is not distal.

The points $q_1, q_2, q_3, \dots, q$ are the distal points of $\mathbb{X}$.

• A point c of a connected topological space X is a **strong non-cut point** if $X \setminus \{c\}$ is continuumwise connected; otherwise c is a **weak cut point.**

• Proposition 5.2. *Every strong non-cut point in a connected topological space is distal; the converse fails in general for metrizable continua.*

Recall that a continuum is **aposyndetic** if for any two of its points, each is contained in the interior of a subcontinuum that does not contain the other.

• Theorem 5.3 (F. B. Jones, 1952). *In an aposyndetic continuum, every non-cut point is strong non-cut.*

In particular, in aposyndetic continua, the strong non-cut points, the distal points, the shore points, and the non-cut points are the same.

Proof of Proposition 5.2. Assume p is not a distal point of X and fix $a \in X \setminus \{p\}$. Then p is not $\{a\}$-distal; thus there is a point b with $\kappa(a;p) \subseteq \kappa(a;b)$, but $\kappa(a;b) \nsubseteq \kappa(a;p)$. Immediately we have from the second condition that $b \in X \setminus \{a,p\}$, so a,b,p are three distinct points. The first condition says that, since $b \notin \kappa(a;b)$, we know $b \notin \kappa(a;p)$. Hence any subcontinuum containing both a and b must also contain p. Thus $X \setminus \{p\}$ is not continuumwise connected, and p is therefore a weak cut point.

The pseudo-arc is a metrizable continuum in which each point is a weak cut point. However, because it contains at least two disjoint composants, every point is distal. $\Box$

The pseudo-arc stands in the way of a “strong non-cut point existence theorem.” So what remains is the following

• Open Problem 5.4. *If A is a proper subcontinuum of a Hausdorff (or even metrizable) continuum X, does there always exist an A-distal point?*

• The answer to Problem 5.4 is *yes* if the composant $\kappa(A)$ is proper: any $p \in X \setminus \kappa(A)$ is A-distal.

• Parting Comment. It is tempting to look for maximal elements in $\mathcal{K}(A)$ via Zorn's lemma. Assuming $\mathcal{C}$ to be a chain in $\mathcal{K}(A)$, it is easy to show there is an upper bound if $\bigcup\mathcal{C}$ fails to be dense in X: just pick $q \in X \setminus \overline{\bigcup\mathcal{C}}$, and an argument like that for the proof of Proposition 5.1 will show $\kappa(A; q)$ to be an upper bound of $\mathcal{C}$. The snag happens when $\bigcup C$ is dense in X.

THANK YOU!