Some Mapping Properties of Arcs and Pseudo-Arcs

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1. Amalgamation.

A mapping diagram is called a **wedge** if it is of the form

$$Y \xrightarrow{f} X \xleftarrow{g} Z$$

or

$$Y \xleftarrow{f} X \xrightarrow{g} Z.$$

In the first instance the diagram is **inward**; in the second, it is **outward**. In either case, the object X is the **vertex** of the wedge. An **amalgamation** of the inward wedge above is an outward wedge

$$Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$$

such that the resulting mapping square commutes (i.e., $f \circ \varphi = g \circ \gamma$.)

$$
\begin{tikzcd}
Y \arrow[d, "f"'] & W \arrow[l, dashed, "\varphi"'] \arrow[d, dashed, "\gamma"] \\
X & Z \arrow[l, "g"]
\end{tikzcd}
$$

In algebra and model theory we usually study amalgamating outward wedges of embeddings with inward ones, but in topology it is often the reverse. In this talk we will look at the amalgamation of inward wedges of compacta, where all mappings are quotients.

Let $\mathfrak{P}$ be a topological property. A compactum X with property $\mathfrak{P}$ is a $\mathfrak{P}$-**base** if for every inward wedge with vertex X and compacta with property $\mathfrak{P}$, there is an amalgamation whose vertex also has property $\mathfrak{P}$.

Our main goal is to describe the $\mathfrak{P}$-bases, where $\mathfrak{P}$ is either no restriction at all or is the property of connectedness. The first part is easy:

2. Base Compacta.

Every compactum is a base compactum; the reason is the pullback construction.

Given quotient maps $f : Y \to X$ and $g : Z \to X$ between compacta, the associated **pullback** is a triple $\langle P, p, q \rangle$, where:

- P is the compactum $\{\langle y, z \rangle \in Y \times Z : f(y) = g(z)\}$; and
- $p : P \to Y$ and $q : P \to Z$ are the coordinate projections. p is surjective because g is; q is surjective because f is.

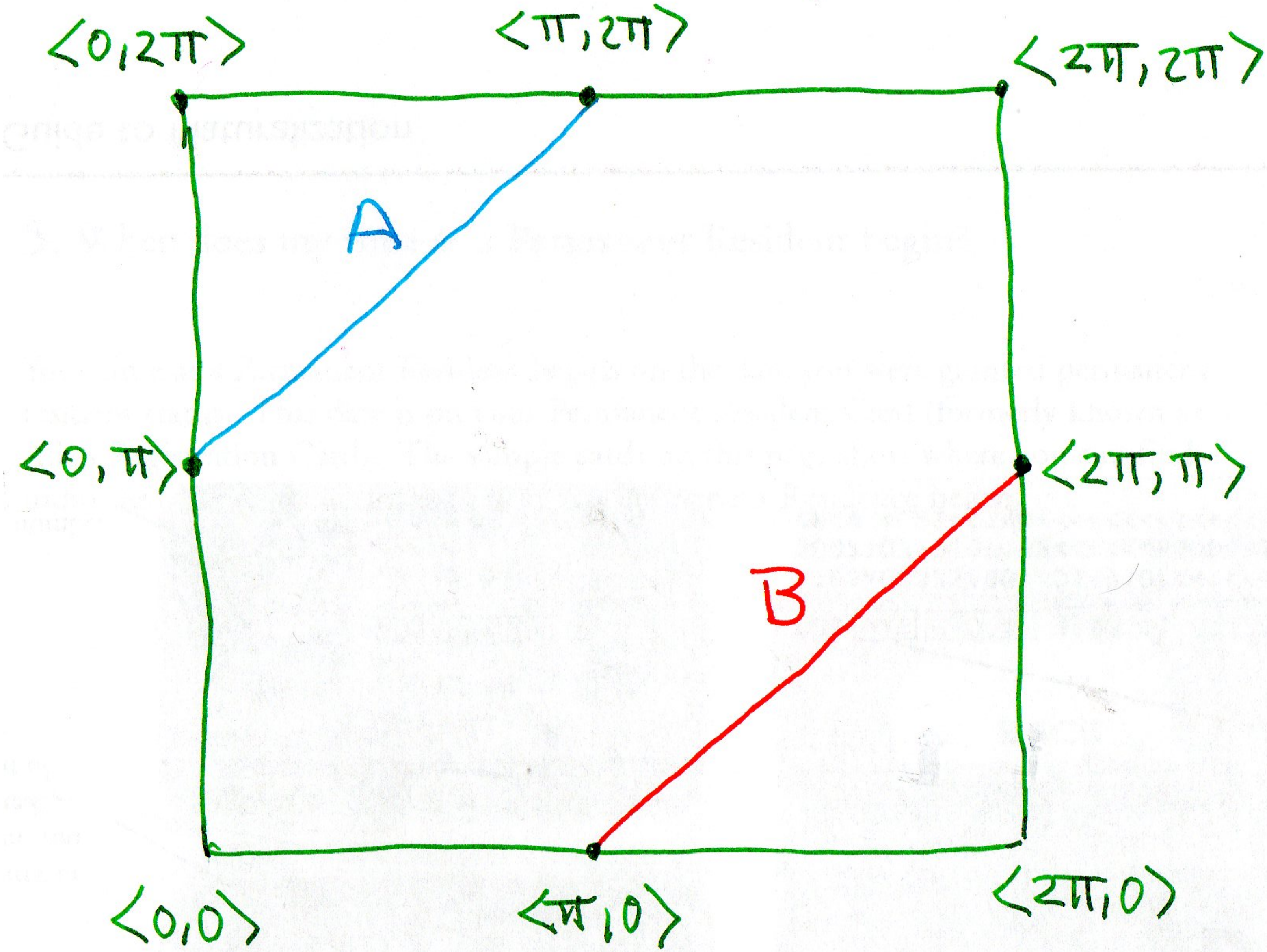
$$
\begin{tikzcd}
Y \arrow[dd, "f"'] & & W \arrow[ll, dashed, "\varphi"'] \arrow[dl, dashed, "\lambda"'] \arrow[dd, "\gamma"] \\
& P \arrow[ul, "p"'] \arrow[dr, "q"] & \\
X & & Z \arrow[ll, "g"]
\end{tikzcd}
$$

The pullback not only witnesses this kind of amalgamation, it is "minimal," in the following sense: Given $\langle X, f, Y, g, Z\rangle$ as above, as well as the amalgamation $\langle W, \varphi, \gamma\rangle$, there is a unique continuous mapping $\lambda : W \to P$–given by $\lambda(w) = \langle\varphi(w), \gamma(w)\rangle$–such that $p \circ \lambda = \varphi$ and $q \circ \lambda = \gamma$.

2. Base Continua.

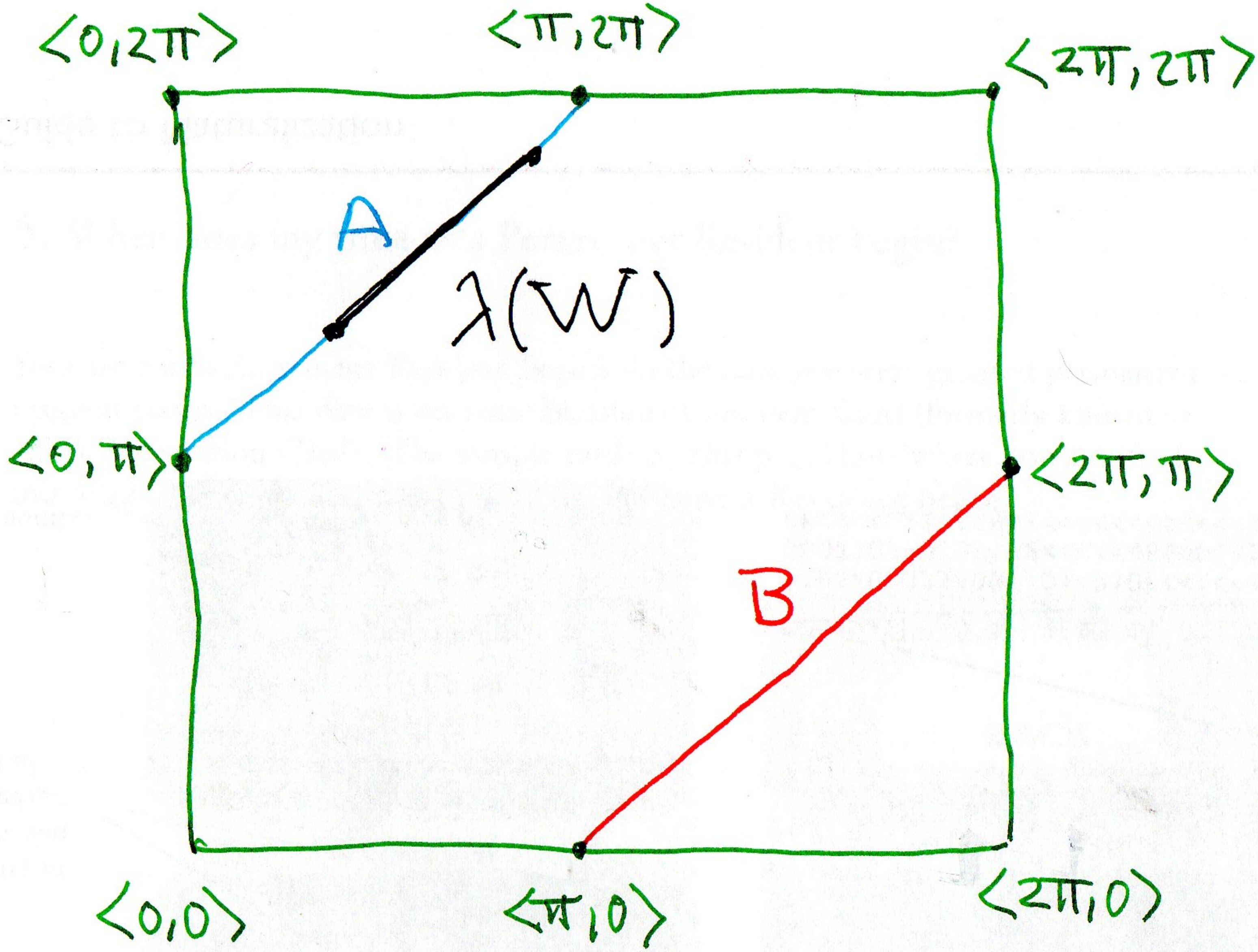
Not every continuum is a base continuum; the following example was suggested recently by Logan Hoehn.

2.1 Example. *Let X be a simple closed curve, modeled by the unit circle in the complex plane; and let Y and Z be arcs, both modeled by the closed unit interval* $[0, 2\pi]$ *in the real line. Let $f : Y \to X$ be the map $t \mapsto \cos t + i \sin t$, and let $g = -f$. The pullback P of this mapping diagram is the union $A \cup B$ of two disjoint line segments in the square* $[0, 2\pi] \times [0, 2\pi]$*, defined by the equations $y = x \pm \pi$. It is therefore disconnected, despite the fact that X, Y and Z are continua. Here's a picture.*



⟨0, 2π⟩
⟨π, 2π⟩
⟨2π, 2π⟩
A
⟨0, π⟩
⟨2π, π⟩
B
⟨0,0⟩
⟨π,0⟩
⟨2π,0⟩

But we can say more: If $\varphi : W \to Y$ and $\gamma : W \to Z$ are continuum mappings such that $f \circ \varphi = g \circ \gamma$, then $\lambda(W)$ is contained in one of A or B; say it's A. But p does not map A onto Y. Hence φ fails to map W onto Y. This shows that simple closed curves are not base continua.


⟨0, 2π⟩
⟨π, 2π⟩
⟨2π, 2π⟩
A
λ(W)
⟨0, π⟩
⟨2π, π⟩
B
⟨0, 0⟩
⟨π, 0⟩
⟨2π, 0⟩

Simple closed curves are not base continua; here's a positive result.

2.2 Theorem (J. Krasinkiewicz, 2000). *Arcs are base continua.*

The rest of this talk is devoted to the unearthing of more base continua, but first we need to talk about what it means for a continuum to be "co-existentially closed."

3. Ultracopowers and Co-Existential Maps.

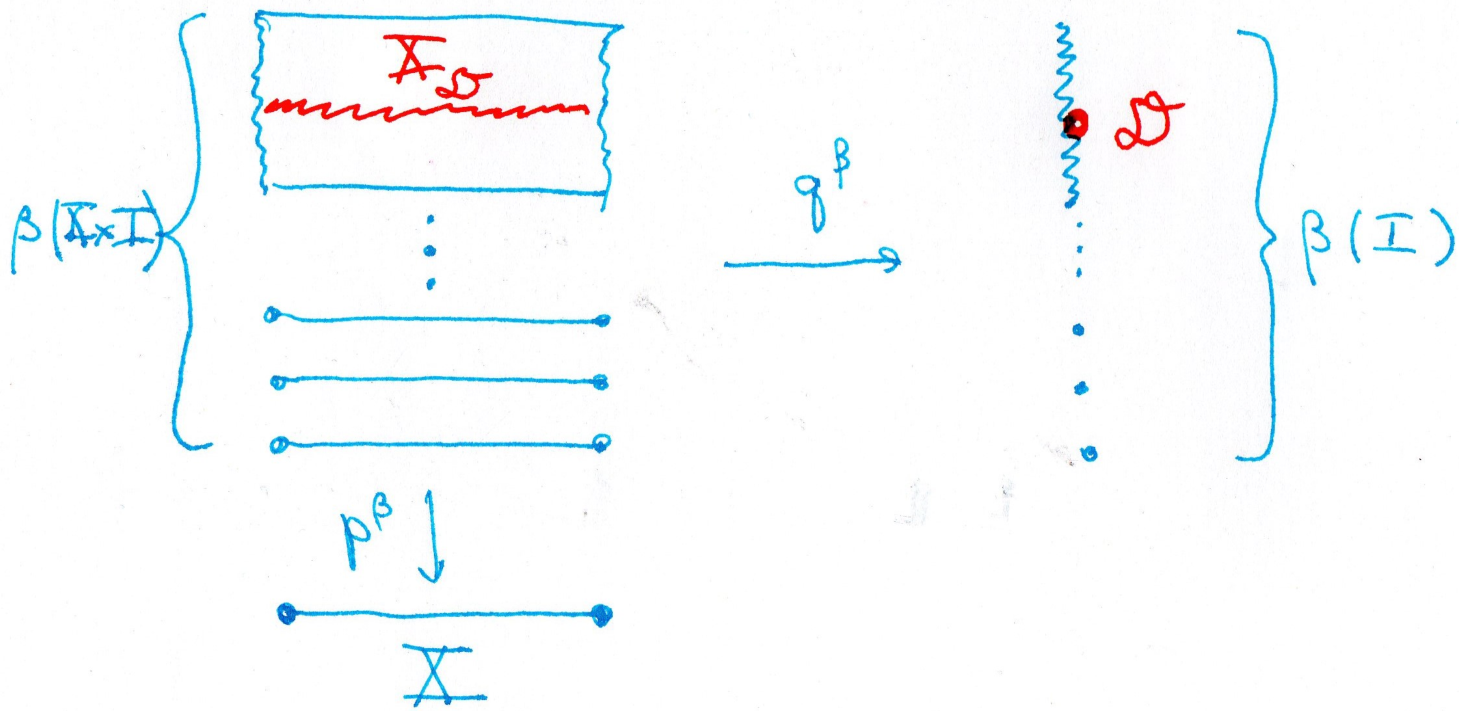
Given a compactum X and (discrete) set I, first form the cartesian product $X \times I$, with coordinate maps $p : X \times I \to X$ and $q : X \times I \to I$. Next apply the Stone-Čech functor, obtaining the following diagram.

$$\begin{tikzcd}
\beta(\mathbb{X} \times I) \arrow[r, "q^\beta"] \arrow[d, "p^\beta"'] & \beta(I) \\
\mathbb{X} &
\end{tikzcd}$$

Now if $\mathcal{D}$ is an ultrafilter on I, then it may be viewed as a point in $\beta(I)$. Denote by $X_{\mathcal{D}}$ the pre-image of $\{\mathcal{D}\}$ under q^{β}. This is the $\mathcal{D}$-**ultracopower** of X.

When X is a continuum, these ultracopowers partition $\beta(X \times I)$ into its components.

The map $p_{\mathcal{D}} = p_{X,\mathcal{D}} = p^{\beta}|X_{\mathcal{D}} : X_{\mathcal{D}} \to X$ is a quotient map, called the **ultracopower co-diagonal map**.



$\beta(X \times I)$
$X_{\mathcal{D}}$
q^β
$\mathcal{D}$
$\beta(I)$
p^β
X

We now define a mapping $f : Y \to X$ between compacta to be **co-existential** if there is an ultracopower $X_{\mathcal{D}}$ and a quotient map $g : X_{\mathcal{D}} \to Y$ such that $f \circ g = p_{\mathcal{D}}$.

$$
\begin{tikzcd}
 & \mathbb{X}_\infty \arrow[dl, dashed, "g"'] \arrow[d, dashed, "p_\infty"] \\
\mathbb{Y} \arrow[r, "f"] & \mathbb{X}
\end{tikzcd}
$$

4. Co-Existentially Closed Continua.

A **co-existentially closed continuum** is a continuum X such that every continuous map from a continuum onto X is co-existential.

4.1 Theorem (PB, 2005).

- *Every co-existentially closed continuum is hereditarily indecomposable, as well as of covering dimension one. (In particular, it's nondegenerate.)*

- *Every nondegenerate continuum is a continuous image of a co-existentially closed continuum, of the same weight.*

- *There exists a metrizable co-existentially closed continuum that is not chainable. (In particular, it's not a pseudo-arc.)*

4.2 Remark. *Recall that a quotient map $f : Y \to X$ is* **confluent** *if whenever K is a subcontinuum of X and C is a component of $f^{-1}(K)$, then $f(C) = K$. X is* **confluently closed** *(aka Class(C)) if X is only a confluent image of other continua. A classic theorem of H. Cook, A. Lelek and D. Read characterizes the confluently closed continua as the hereditarily indecomposable ones. (In particular, the pseudo-arc is confluently closed.)*

Even though co-existentially closed continua are confluently closed, it is not *the case that co-existential maps are always confluent (P. B., K. P. Hart).*

We know the pseudo-arc is a confluently closed continuum, by virtue of its being hereditarily indecomposable. Is it also a co-existentially closed continuum?

Here's the reason for bringing up co-existentially closed continua in this talk.

4.3 Theorem (PB, 2005). *Every co-existentially closed continuum is a base continuum.*

Proof-by-picture:

$$
\begin{tikzcd}
Y \arrow[d, "f"'] & \\
X & Z \arrow[l, "g"]
\end{tikzcd}
$$

$$\begin{tikzcd}
Y \arrow[ddd, "f"'] & & & \\
& X_\xi \arrow[ul, "f'"'] \arrow[ddl, "\rho_{X,\xi}"] & & \\
& & X_\vartheta \arrow[dll, "\rho_{X,\vartheta}"] \arrow[dr, "g'"] & \\
X & & & Z \arrow[lll, "g"]
\end{tikzcd}$$

$$
\begin{tikzcd}
Y \arrow[dddd, "f"'] & & & & \\
& & (X_{\mathcal{E}})_{\mathcal{F}} \arrow[dl, "\rho_{X_{\mathcal{E}},\mathcal{F}}"'] \arrow[dr, "h \text{ (homeomorphism)}"] & & \\
& X_{\mathcal{E}} \arrow[uul, "f'"] \arrow[ddl, "\rho_{X,\mathcal{E}}"'] & & (X_{\mathcal{D}})_{\mathcal{F}} \arrow[dl, "\rho_{X_{\mathcal{D}},\mathcal{F}}"'] & \\
& & X_{\mathcal{D}} \arrow[dll, "\rho_{X,\mathcal{D}}"] \arrow[drr, "g'"] & & \\
X & & & & Z \arrow[llll, "g"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y \arrow[ddd, "f"'] & Y_{\mathfrak{F}} \arrow[l, "\rho_{Y,\mathfrak{F}}"'] & & (X_{\mathfrak{E}})_{\mathfrak{F}} = W \arrow[ll, "f'_{\mathfrak{F}}"'] \arrow[dll, "\rho_{X_{\mathfrak{E}},\mathfrak{F}}"'] \arrow[dr, "h\ (\text{homeomorphism})"] \arrow[lll, bend right, "\varphi"'] \arrow[dddrr, bend left, "\gamma"] & & \\
& X_{\mathfrak{E}} \arrow[ul, "f'"] \arrow[ddl, "\rho_{X,\mathfrak{E}}"'] & & & (X_{\mathfrak{D}})_{\mathfrak{F}} \arrow[dll, "\rho_{X_{\mathfrak{D}},\mathfrak{F}}"'] \arrow[dr, "g'_{\mathfrak{F}}"] & \\
& & X_{\mathfrak{D}} \arrow[dll, "\rho_{X,\mathfrak{D}}"'] \arrow[drrrr, "g'"] & & & Z_{\mathfrak{F}} \arrow[d, "\rho_{Z,\mathfrak{F}}"] \\
X & & & & & Z \arrow[lllll, "g"]
\end{tikzcd}
$$

4.4 Remark. *A metrizable base continuum is a base metrizable continuum. That is, given the inward wedge*

$$Y \xrightarrow{f} X \xleftarrow{g} Z$$

of metrizable continua, the amalgamating outward wedge

$$Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$$

may be chosen such that W is metrizable too. This is because in the pullback

$$Y \xleftarrow{p} P \xrightarrow{q} Z,$$

P is clearly metrizable. Now, given any old amalgamation $\langle W, \varphi, \gamma \rangle$, take $\lambda(W)$ for our new W, and $p|\lambda(W)$, $q|\lambda(W)$, respectively, for our new φ and γ. The surjectivity of the old φ and γ ensures the surjectivity of the new.

Recently, using a continuous model theory approach to C^*-algebras, Christopher Eagle, Isaac Goldbring and Alessandro Vignati have proved the following.

4.5 Theorem. *The pseudo-arc is a co-existentially closed continuum.*

As an immediate corollary of this and Theorem 4.3, we know that the pseudo-arc is a base continuum; in fact a base metrizable continuum, by Remark 4.4. (Observe that being a metrizable base continuum is ostensibly stronger than being a base metrizable continuum.)

However, Logan Hoehn has recently informed us that it follows from a 1984 paper of L. Oversteegen and E. D. Tymchatyn that *every metrizable continuum of span zero is a base metrizable continuum.*

This includes arcs and pseudo-arcs, but this result does not show immediately that they are base continua.

Hoehn and Oversteegen have recently proved that any hereditarily indecomposable metrizable continuum of span (or surjective semispan) zero is chainable, and hence a pseudo-arc. Since we have the existence of non-chainable co-existentially closed metrizable continua, and all such are herediarily indecomposable, we have the existence of metrizable base continua that are not of span zero.

THANK YOU!