Continuity Properties of Betweenness

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1. Betweenness Relations Induced by Other Structure

1.1. Total Orderings (Huntington and Kline, 1917)

Given a totally ordered set $\langle X, \leq \rangle$, say c **lies between** a and b (in symbols: $[a, c, b]_{\mathsf{O}}$) if either $a \leq c \leq b$ or $b \leq c \leq a$.

1.2. Metric Spaces (Menger, 1928)

Given a metric space $\langle X, \varrho \rangle$, $[a, c, b]_{\mathsf{M}}$ holds if

$$\varrho(a, b) = \varrho(a, c) + \varrho(c, b).$$

1.3. Topological Spaces (Ward, 1954; PB, 2013)

Given a topological space $\langle X, \mathcal{T} \rangle$, $[a, c, b]_{\mathsf{T}}$ holds if every connected subset of X containing $\{a, b\}$ contains c as well.

(Other versions include: (1) a and b lie in different sets forming a separation of $X \setminus \{c\}$; and (2) c lies in every connected closed subset of X containing $\{a, b\}$.)

Consider the closed unit interval $[0, 1]$ in the real line, regarded as a totally ordered set, a metric space, or a topological space.

All three notions of betweenness coincide:

$$[a, c, b]_O \Leftrightarrow [a, c, b]_M \Leftrightarrow [a, c, b]_T.$$

(This includes the topological variants of betweenness mentioned in the last slide.)

2. Betweenness Relations On Their Own

These are ternary structures $\langle X, [\cdot, \cdot, \cdot] \rangle$ satisfying the **basic axioms**:

(Inclusivity) $[a, a, b] \wedge [a, b, b]$

(Symmetry) $[a, c, b] \Rightarrow [b, c, a]$

(Uniqueness) $[a, c, a] \Rightarrow a = c$

Given a basic ternary structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ and pair $\{a, b\} \subseteq X$ (a can equal b, and we still call it a *pair*) the **interval bracketed by a and b** is the set

$$[a, b] := \{c \in X : [a, c, b] \text{ holds}\}.$$

In interval notation, the basic axioms become:

(Inclusivity) $\{a, b\} \subseteq [a, b]$

(Symmetry) $[a, b] = [b, a]$

(Uniqueness) $[a, a] = \{a\}$

3. Betweenness Functions

Given a basic ternary structure $\langle X, [\cdot,\cdot,\cdot]\rangle$, the associated **betweenness function** is given by the assignment

$$\{x, y\} \mapsto [x, y].$$

The domain of this function is the **symmetric square**

$$\mathcal{F}_2(X) := \{\{a, b\} : a, b \in X\},$$

and the codomain is the power set

$$\wp(X) := \{Y : Y \subseteq X\}.$$

Our interest lies in conditions making the betweenness function "continuous" at a given pair, and for this it makes sense to assume X is topologized so that intervals are always closed. In this case we call $\langle X, [\cdot,\cdot,\cdot]\rangle$ a **closed basic ternary structure**.

For any topological space $X = \langle X, \mathcal{T} \rangle$, we denote by 2^X the set of closed nonempty subsets of X. The default topology on this set is that provided by Vietoris: for each $n \in \mathbb{N} := \{1, 2, \ldots\}$ and n-tuple $\langle U_1, \ldots, U_n \rangle$, the set $\|U_1, \ldots, U_n\|$ is defined to be

$$\{A \in 2^X : A \subseteq U_1 \cup \cdots \cup U_n \text{ and } A \cap U_i \neq \emptyset, (1 \leq i \leq n)\}.$$

Sets of this form give an open-set base for the **Vietoris topology**.

When X is a Hausdorff space, $\mathcal{F}_2(X)$–with the obvious topology induced by the cartesian product X^2–is a closed subset of 2^X.

Why the Vietoris topology is so important is partially explained by the following, where we assume X is Hausdorff:

1. 2^X is Hausdorff.

2. X is compact iff 2^X is compact.

3. X is connected iff 2^X is connected.

4. X is metrizable iff 2^X is metrizable.

Of particular interest is the following famous theorem of Curtis and Shori:

3.1 Theorem. 2^X *is homeomorphic to the Hilbert cube* $[0,1]^{\mathbb{N}}$ *iff* X *is a nondegenerate Peano continuum (i.e., connected, compact, and metrizable, with more than one point).*

We take the betweenness function

$$[\cdot,\cdot] : \mathcal{F}_2(X) \to 2^X$$

to be **continuous** if it is continuous relative to the Vietoris topology.

This notion naturally splits in two as follows. Suppose $F : Y \to 2^X$ is a (multivalued) function. F is **upper** (resp., **lower**) **semicontinuous** at $a \in Y$ if for every open set $U \subseteq X$ such that $F(a) \subseteq U$ (resp., $F(a) \cap U \neq \emptyset$), there is an open neighborhood of a in Y such that for each $b \in V$, $F(b) \subseteq U$ (resp., $F(b) \cap U \neq \emptyset$). Upper semicontinuity is abbreviated usc; lower semicontinuity is abbreviated lsc.

It is easy to check that F is continuous at $a \in Y$ iff it is both usc and lsc at a.

The intuition behind usc (resp., lsc) at a is that:

As $y \to a$ in Y, and the values of $F(y)$ are “large” (resp., “small”), then the value of $F(a)$ is “large” (resp., “small”).

(In the usc (resp., lsc) case, “large” means not being contained within (resp., intersecting) a given open set.)

4. Betweenness Functions for Metric Spaces

We first concentrate on semicontinuity for the **Menger betweenness function**; i.e., the one for metric spaces. (So $[a,b] := [a,b]_{\mathsf{M}}$.)

Note that if $\langle X, \varrho \rangle$ is a metric space, then $\langle X, [\cdot,\cdot,\cdot]_{\mathsf{M}} \rangle$ is a closed basic structure. The three axioms constituting *basic* are obvious; as for *closed*, note that for each $a, b \in X$, $[a,b]$ is the zero set of the continuous map

$$x \mapsto \varrho(a,x) + \varrho(x,b) - \varrho(a,b).$$

4.1 Example. Let $X = \mathbb{S}^1$, the closed unit circle in the euclidean plane. If ϱ is the metric on X induced by the euclidean metric, then for any a, b, c distinct,

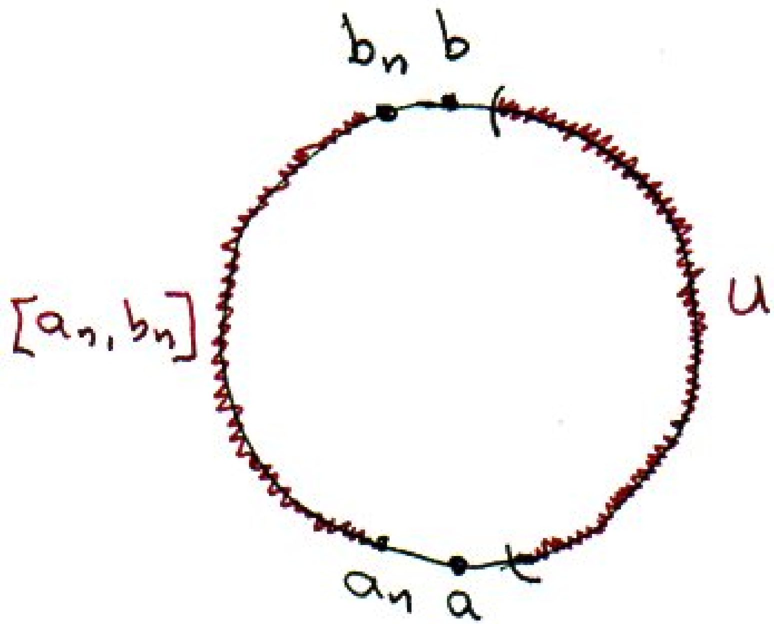
$$\varrho(a,b) < \varrho(a,c) + \varrho(b,c).$$

Thus for each $a, b \in X$, we have $[a,b] = \{a,b\}$. This tells us that the betweenness function is just the inclusion map from $\mathcal{F}_2(X)$ to 2^X, obviously continuous.

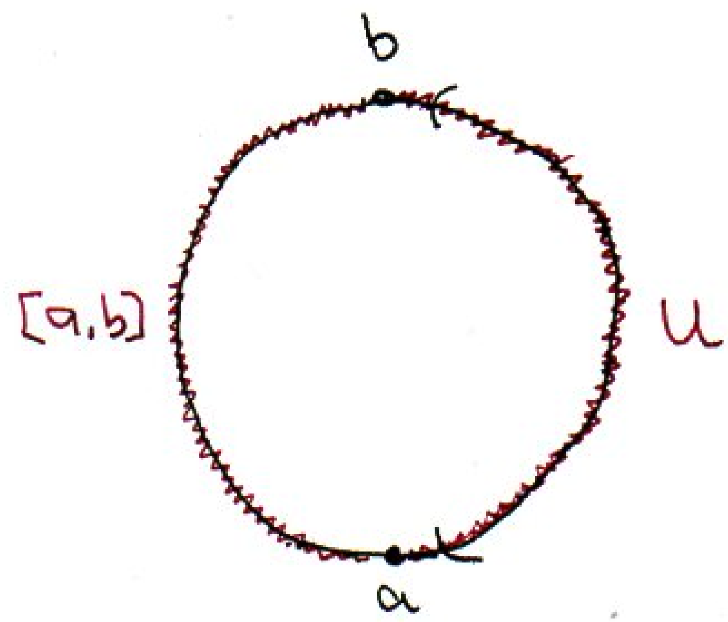
On the other hand, suppose ϱ is the metric on X that measures the length of the shortest circular arc containing a given pair. Then the associated betweenness function is lower semicontinuous (lsc) only at pairs which are not antipodal.

To see this, note that if a and b are not antipodal, then there is only one shortest circular arc, so $[a, b]$ consists of the points on that arc. It is easy to check lsc at such pairs. However, if a and b are antipodal, then $[a, b] = X$.

Wlog, suppose $a = \langle 0, -1 \rangle$ and $b = \langle 0, 1 \rangle$, with U the intersection of X with the open right half-plane. If $a_n \to a$ and $b_n \to b$ are two sequences with entries in the left half-plane, then $[a_n, b_n] \cap U = \emptyset$ for $n \in \mathbb{N}$. However, we have $[a, b]$ not only intersecting U, but containing U. This violates lsc at $\{a, b\}$: the sequence of intervals $[a_n, b_n]$ consists of "small" sets (in the sense of being disjoint from U), while $[a, b]$ is "large" (i.e., intersects U).



$[a_n, b_n] \cap U = \emptyset$



$[a, b] = X \supseteq U$

The next result shows that the Menger betweenness function is *upper* semicontinuous in this example, even at antipodal pairs.

4.2 Theorem. *Suppose* $\langle X, \varrho \rangle$ *is a proper metric space; i.e., closed bounded subsets are compact. Then the Menger betweenness function on* X *is usc at each pair.*

Proof: Suppose, on the contrary, that Menger betweenness is not usc at $\{a, b\}$. This means that we have $U \subseteq X$, open in X and containing $[a, b]$, with sequences $a_n \to a$ and $b_n \to b$ such that $[a_n, b_n] \setminus U \neq \emptyset$ for arbitrarily large $n \in \mathbb{N}$. Let c_n witness this fact for each n; wlog we may assume that $\varrho(a_n, a), \varrho(b_n, b) \leq 1/n$. Then $\varrho(c_n, a) \leq \varrho(c_n, a_n) + \varrho(a_n, a) \leq \varrho(a_n, b_n) + \varrho(a_n, a) \leq \varrho(a, b) + 2/n + 1/n$. From this we infer that the sequence c_n is bounded; hence–because the metric is proper–it has a convergent subsequence. Wlog, assume $c_n \to c \in X$. Since $\varrho(a_n, c_n) + \varrho(c_n, b_n) = \varrho(a_n, b_n)$ for each $n \in \mathbb{N}$, we know $c \in [a, b]$. But this means $c_n \in U$ for arbitrarily large n, a contradiction. $\square$

It is easy to show that Menger intervals are not only closed, but bounded as well (indeed, the diameter of $[a, b]$ is $\varrho(a, b)$). So if the metric is proper, intervals are compact.

Note that in the proof of Theorem 4.2 we needed the metric to be proper to get the sequence $c_1, c_2, \ldots$ to have a convergent subsequence. If the pair $\{a, b\}$ is a singleton, though, it is easy to see that $c_n \to a = b$ without any further assumptions at all.

Furthermore, it is a triviality that the Menger betweenness function is lsc at singletons. Hence we have

4.3 Corollary. *Suppose* $\langle X, \varrho \rangle$ *is any metric space. Then the Menger betweenness function on* X *is continuous at each singleton.*

5. Betweenness Functions for Geodesic Spaces

Recall from basic calculus the definition of *arc length* using the definite integral. This actually makes sense in the general metric space setting, as follows.

If $\langle X, \varrho \rangle$ is a metric space and $p : [0, 1] \to X$ is a path from point a to point b, the **length** $L(p)$ of p is the supremum of all finite sums

$$\sum_{i=1}^{n-1} \varrho(p(s_i), p(s_{i+1})),$$

where $n \in \mathbb{N}$ and $0 = s_1 < s_2 < \cdots < s_n = 1$.

If $L(p) = \varrho(a, b)$, we call p a **geodesic** from a to b. $\langle X, \varrho \rangle$ is a **geodesic space** if there is a geodesic from any point to any other.

The **support** of a path is the image of the path as a mapping. It is a not-too-surprising fact that a geodesic between two distinct points is a topological arc, with the two points as end points.

5.1 Theorem. *If* $\langle X, \varrho \rangle$ *is a geodesic space, then each Menger interval* $[a, b]$ *is the union of the supports of all geodesics from* a *to* b *(and is hence connected).*

A geodesic space is **unique-geodesic** at the pair $\{a, b\}$ if any two geodesics from a to b have the same support (equal to $[a, b]$). The space is **unique-geodesic** if it is unique-geodesic at each pair.

In the example above, where X is the unit circle and ϱ is the length of the shortest circular arc joining two points, we have a geodesic space which is unique-geodesic precisely at the nonantipodal pairs.

5.2 Theorem. *For a proper geodesic space, being unique-geodesic at a pair of points implies that the Menger betweenness function is continuous at that pair.*

By Theorem 4.2, we already have upper semicontinuity; so what is needed is a proof for lower semicontinuity. This requires a fairly involved argument.

6. Betweenness Functions for Normed Vector Spaces

Given a normed vector space $\langle X, \| \cdot \| \rangle$ over the real field, the **norm metric** is given by

$$\varrho(a, b) := \|a - b\|.$$

We denote by B_X (resp., S_X the **unit ball** (resp., **sphere**) of X; namely those points of norm ≤ 1 (resp., $= 1$).

With the norm metric, a normed space is a geodesic space, as straight line segments are always geodesics.

To see this, let $p : [0,1] \to X$ be given by $p(s) = sa+(1-s)b$. If $0 = s_1 < s_2 < \cdots < s_n = 1$, then

$$\sum_{i=1}^{n-1} \varrho(p(s_i), p(s_{i+1})) = \sum_{i=1}^{n-1} (s_{i+1} - s_i)\|a-b\| = \|a-b\|.$$

Thus $L(p) = \varrho(a,b)$, as desired.

Let $[a,b]_L = \lfloor p \rfloor$, where p is as above. Then, by Theorem 5.1, we have $[a,b]_L \subseteq [a,b]$.

A normed space is called **strictly convex** (or **rotund**) if its unit sphere contains no nondegenerate line segments.

6.1 Lemma (folklore). *A normed space is strictly convex iff* $[a, b] = [a, b]_L$ *for all pairs* $\{a, b\}$*. Thus strictly convex is equivalent to being unique-geodesic for a normed space.*

Inner product spaces are strictly convex; hence all Menger intervals are compact. However, for any normed space, the norm metric is proper iff the vector space dimension is finite. We will be interested in the relationship between: (1) the Menger betweenness function is usc at $\{a, b\}$; and (2) $[a, b]$ is compact.

6.2 Theorem. *If* $\langle X, \|\cdot\|\rangle$ *is a unique-geodesic normed space, then the Menger betweenness function is continuous at each pair.*

6.3 Example. For $1 \leq p \leq \infty$, let $\mathbb{R}_p^2$ be the euclidean plane with the p-**norm**; i.e., for $p < \infty$, we have, for each $\vec{x} = \langle x_1, x_2 \rangle$,

$$\|\vec{x}\|_p := (|x_1|^p + |x_2|^p)^{1/p};$$

and

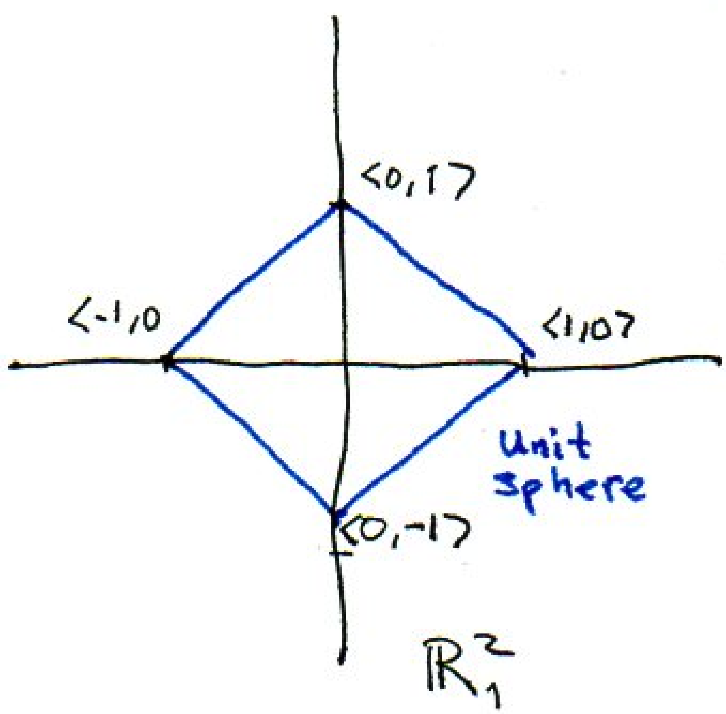
$$\|\vec{x}\|_\infty := \max\{|x_1|, |x_2|\}.$$

The notation is justified by the observation that

$$\|\vec{x}\|_\infty = \lim_{p \to \infty} \|\vec{x}\|_p$$

for each $\vec{x} \in \mathbb{R}^2$.

When $p = 1$ or $p = \infty$, the unit sphere is a square, and hence $\mathbb{R}_p^2$ is not strictly convex.


<0,1>
<-1,0
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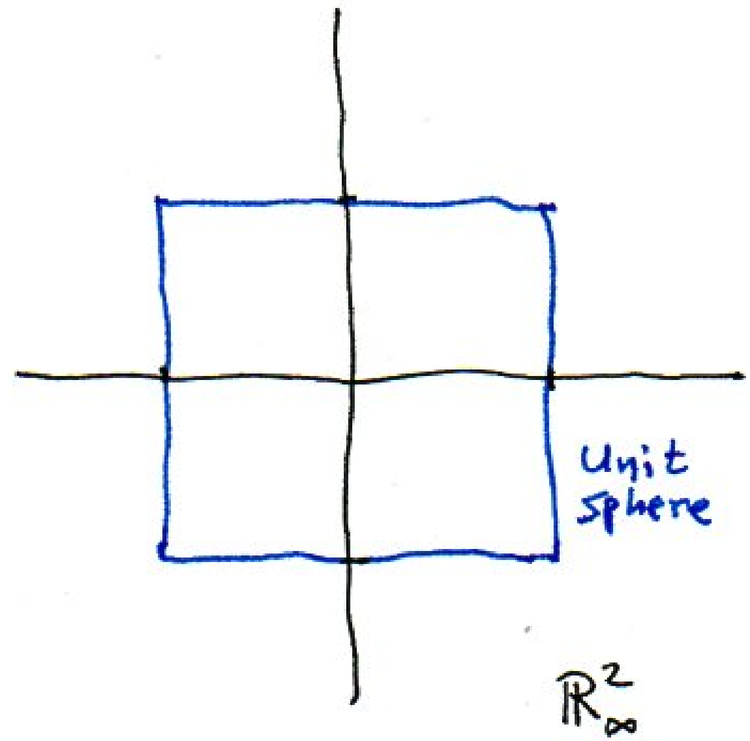
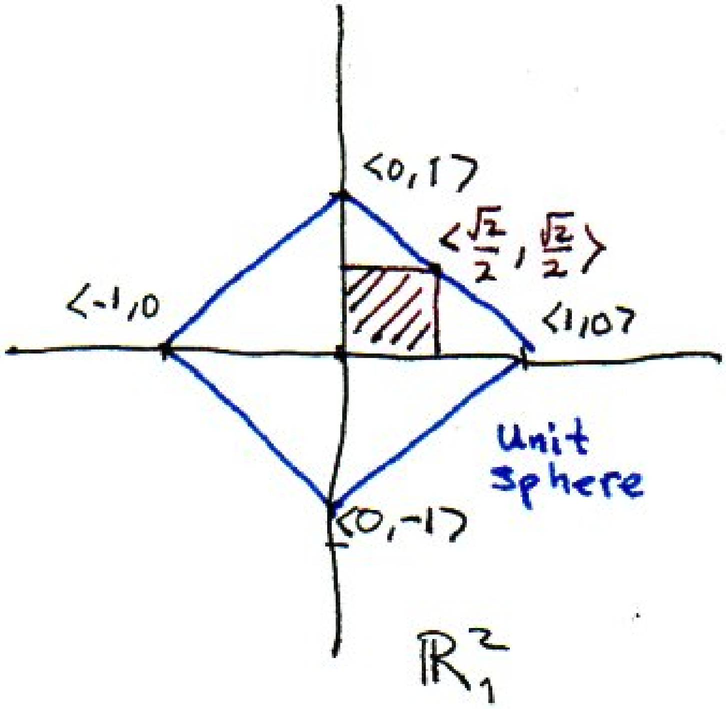
Unit sphere
$\mathbb{R}^2_1$

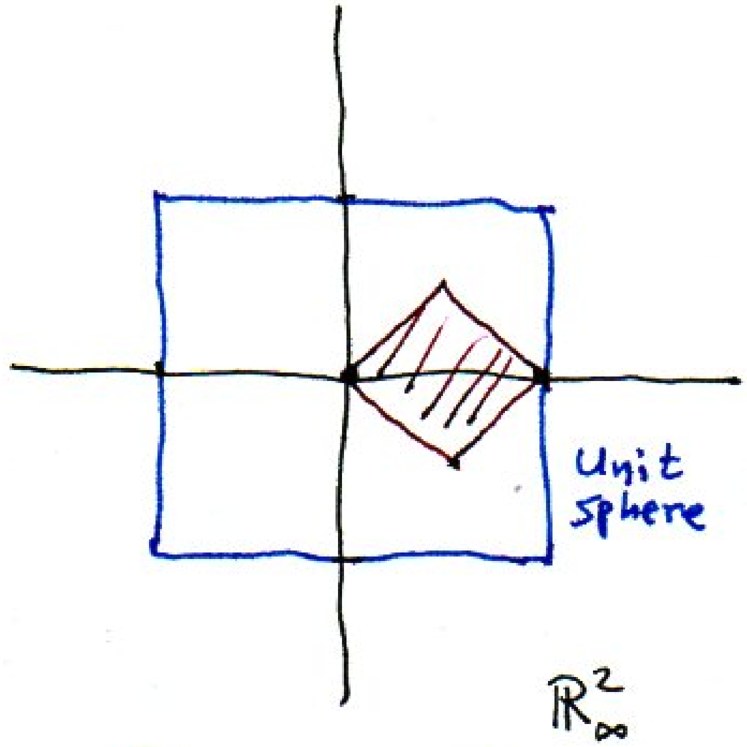

Unit sphere
$\mathbb{R}^2_\infty$

For $1 < p < \infty$, the unit sphere S is a (radially) symmetric simple closed curve which contains no line segments–e.g., S is the usual unit circle when $p = 2$–and hence $\mathbb{R}^2_p$ is strictly convex. This means that all Menger intervals are line segments.

In the extremal cases, Menger intervals are usually (solid) rectangles.



$[\langle 0,0\rangle, \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle]$



$[\langle 0,0\rangle, \langle 1,0\rangle]$

For any finite-dimensional normed space X, the norm metric is proper; hence–by Theorem 4.2–the Menger betweenness function is usc at at all pairs. But what about for infinite-dimensional spaces?

6.4 Theorem. *Let X be a normed space whose Menger betweenness function is usc at the pair $\{a, b\}$. Then the interval $[a, b]$ is compact.*

6.5 Example. Let X be the sequence space c_0–i.e., vectors are null sequences–with the supremum norm $\|\cdot\|_\infty$. Then no nondegenerate Menger interval is compact. Hence the Menger betweenness function for X is usc at $\{a, b\}$ iff $a = b$.

One can also equip c_0 with a norm topologically equivalent to $\| \cdot \|_\infty$, such that there exists a pair $\{a, b\}$ at which:

(1) the Menger interval $[a, b]$ is compact (indeed a line segment); but

(2) the betweenness function is not usc.

This tells us that the converse to Theorem 6.4 is false.

With regard to the issue of lower semicontinuity, we are 99% certain that the Menger betweenness function is lsc (and hence continuous) for every two-dimensional normed space. However, this does not hold in higher dimensions.

6.6 Example. There is a norm $\|\cdot\|$ on $\mathbb{R}^3$, with respect to which the Menger betweenness function is not always lsc. The unit ball B for this space agrees with that of the ∞-norm on the plane $z = 0$. However, all line segments in the unit sphere S lie on $z = 0$; hence all intervals $[\vec{0}, \vec{a}]$ are line segments whenever $\vec{a} \in S$ has nonzero third coordinate. Let $\vec{b} = \langle 1, 0, 0 \rangle$. Then $[\vec{0}, \vec{b}]$ is a square with corners $\langle 0, 0, 0 \rangle$, $\langle 1, 0, 0 \rangle$, and $\langle \frac{1}{2}, \pm\frac{1}{2}, 0 \rangle$. If $\vec{b}_n \to \vec{b}$ is a sequence of points on S, all with positive third coordinate, then the sequence $[\vec{0}, \vec{b}_n]$ witnesses that the betweenness function is not lsc at $\{\vec{0}, \vec{b}\rangle$.

7. Some Questions

7.1 *Is the Menger betweenness function usc at all pairs of a normed space if all intervals are compact?* (By Theorem 4.2, the answer is yes under the assumption that the metric is proper, but in the vector space context, this implies finite-dimensionality. Lots of infinite-dimensional normed spaces exist–e.g., Hilbert spaces–where all Menger intervals are not only compact, but line segments.)

7.2 *Is every Menger interval in a normed space linearly convex?* (We're pretty certain that in the 2-dimensional case, they're not only linearly convex, but M-convex: like usual, only with Menger intervals replacing line segments. M-convexity no longer holds for 3-dimensional normed spaces.)

THANK YOU!