A Menagerie of Non-Cut Points in Continua

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We take our initial motivation from convexity theory:

Let X be a real (topological) vector space. If $a, b \in X$, $[a, b]_{\mathsf{L}}$ denotes the line segment $\{(1 - s)a + sb : 0 \leq s \leq 1\}$ determined by a and b. For K a convex subset of X, we say $e \in K$ is an **extreme point** of K if "e is never properly between two points of K;" i.e., whenever $a, b \in K$ and $e \in [a, b]_{\mathsf{L}}$, it follows that $e = a$ or $e = b$.

In this setting, we have the famous Krein-Milman theorem: *If K is a compact convex subset of a locally convex tvs, then K is the closed convex hull of its set of extreme points.*

So how do we carry this notion over to the context of continua (= connected compact Hausdorff spaces)?

First we need something to correspond to *closed convex hull*, and for this we take the **subcontinuum hull** $[S]$ of a subset S of continuum X to be the intersection of all subcontinua of X containing S. ($[S]$ is always a compact subset of X, but can easily fail to be connected.)

When $S = \{a, b\}$, we write $[a, b]$ for $[S]$, the **subcontinuum interval** determined by a and b. We now say that $e \in X$ is an **extreme point** of X if whenever $a, b \in X$ are such that $e \in [a, b]$, it follows that either $a \in [e, b]$ or $b \in [a, e]$.

Why the more complicated–but obviously weaker–conclusion? (Bear with me.)

The two guiding questions are these:

Question A. *How does* extreme, *as a point type, relate to other–better known–point types for continua?*

Question B. *Discover interesting classes* $\mathfrak{C}$ *of continua for which a Krein-Milman theorem analogue applies; e.g., for each* $X \in \mathfrak{C}$, X *is the subcontinuum hull of its set of extreme points.*

Intuitively, extreme points are "at the edge" of a continuum.

Recall that a point c of continuum X is a **cut point** if $X \setminus \{c\}$ is disconnected; a **non-cut point** otherwise.

We will see that extreme points are non-cut; the point type *non-cut* satisfies the Krein-Milman property above for all continua; namely we have the well-known non-cut point existence theorem, due to R. L. Moore and G. T. Whyburn. *For any continuum X, X is the subcontinuum hull of its set of non-cut points.*

By way of terminology, we say X is **irreducible about** $S \subseteq X$ (or, S **spans** X) if $X = [S]$. A continuum is **irreducible** if it is irreducible about some two-point subset.

A space is **continuumwise connected** if each pair of points is contained in a subcontinuum. Each Hausdorff space is partitioned into its **continuum components**; i.e., maximal continuumwise connected subsets.

If $X \setminus \{c\}$ is not only connected, but continuumwise connected, then we call c a **strong non-cut point** of X. So c not being a strong non-cut point is called being a **weak cut point**. To paraphrase–or to say the same thing in a different way–c is a weak cut point of X iff $c \in [a, b] \setminus \{a, b\}$ for some $a, b \in X$.

Thus we have: *A point $c \in X$ is a strong non-cut point iff whenever $a, b \in X$ and $c \in [a, b]$ it follows that $c = a$ or $c = b$.*

This is more like the convexity theory definition of *extreme point*. If we'd used the weaker conclusion originally in the convexity theory definition, we would have the same notion because $[\cdot, \cdot]_{\mathsf{L}}$ satisfies the *antisymmetry axiom* of betweenness:

$$(c \in [a, b]_{\mathsf{L}} \;\&\; b \in [a, c]_{\mathsf{L}}) \Rightarrow b = c.$$

A continuum X is **antisymmetric** if, given any triple $\langle a, b, c \rangle$ of points, with $b \neq c$, we have a subcontinuum containing a and exactly one of b, c. A continuum is antisymmetric iff its subcontinuum betweenness interpretation satisfies the antisymmetry condition. (And, yes, this notion is related to antisymmetry in binary relations.)

1. Proposition. *If X is an antisymmetric continuum, every extreme point is a strong non-cut point (and* vice versa*).*

We will later see that extreme points can easily be weak cut.

The point types *non-cut* and *strong non-cut* are at the extremes of a menagerie of point types that say "at the edge."

Define a continuum X to be **aposyndetic** (after F. B. Jones) if, given any two of its points, each is in the interior of a subcontinuum that excludes the other. Aposyndetic continua can be shown to be antisymmetric; so there is no distinction between extreme and strong non-cut. But more is true: for aposyndetic continua, "at the edge" has just one meaning, thanks to the following.

2. Proposition (G. T. Whyburn). *Every non-cut point of an aposyndetic continuum is a strong non-cut point.*

So, addressing the Krein-Milman issue (Question B), we have a trivial corollary of the results of Moore and Whyburn.

3. Corollary. *Every aposyndetic continuum is irreducible about its set of extreme points.*

Two important point types interpolating between *strong non-cut* and *non-cut* are the following.

A point c in continuum X is a:

- **non-block point** if $X \setminus \{c\}$ has a continuum component which is dense in X.

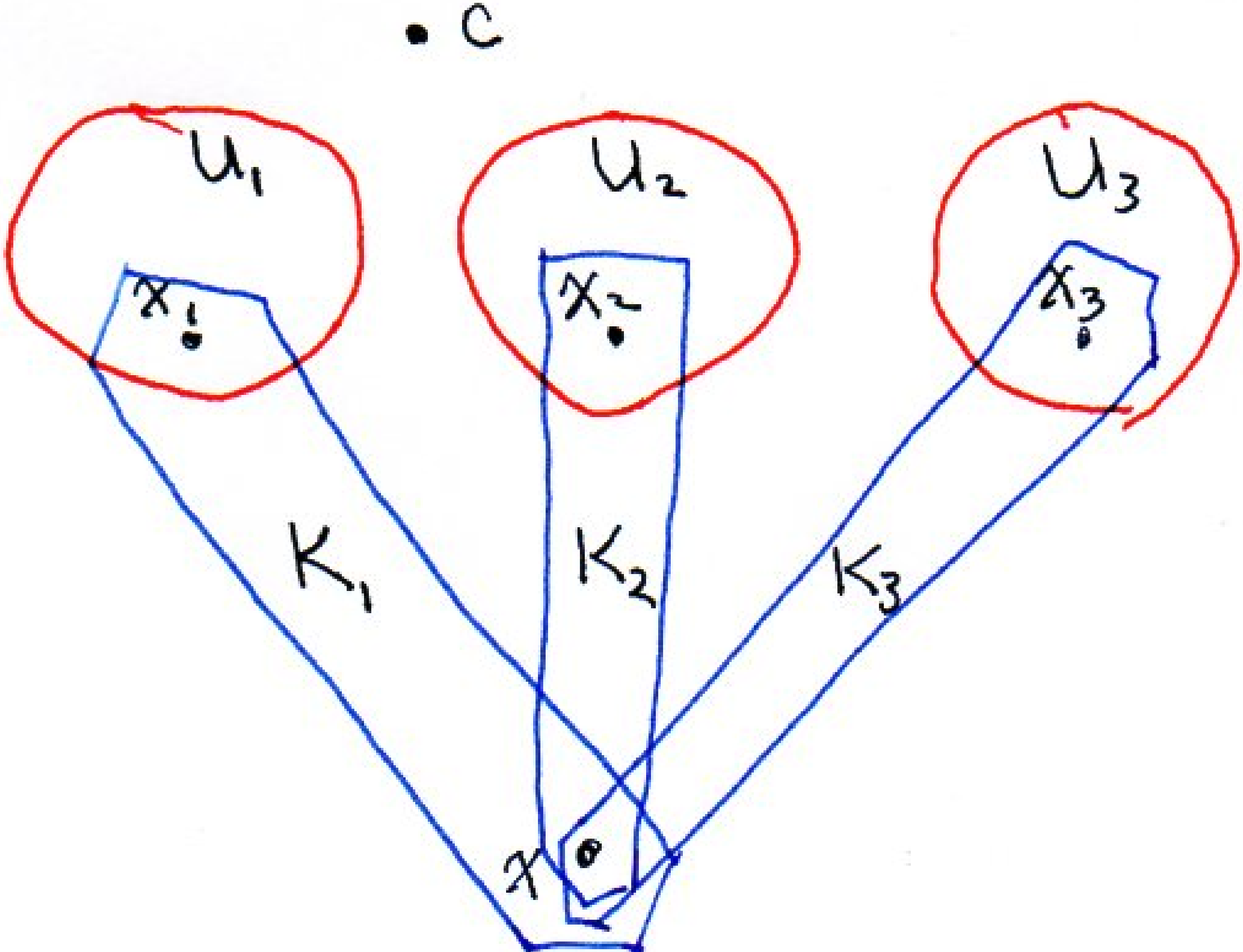
- **shore point** if for any finite family $\mathcal{U}$ of nonempty open sets of X, there is a subcontinuum of $X \setminus \{c\}$ which intersects each $U \in \mathcal{U}$.

The point c being shore means, intuitively, that "there are arbitrarily large subcontinua missing c."

4. Proposition. *Strong non-cut* $\Rightarrow$ *non-block* $\Rightarrow$ *shore* $\Rightarrow$ *non-cut.*

Proof. The first implication is trivial; the third is almost trivial: If c is a cut point, let U, V partition $X \setminus \{c\}$ into two disjoint nonempty open sets. Then no subcontinuum of X intersecting both U and V can miss c. As for the middle implication, suppose c is non-block, say A is a continuum component of $X \setminus \{c\}$, with $x \in A \subseteq A^- = X$. Let $U_1, \ldots, U_n$ be nonempty open sets, and fix $x_i \in A \cap U_i$, $1 \leq i \leq n$. Then for each i we have a subcontinuum $K_i \subseteq A$ containing $\{x, x_i\}$. Hence $\bigcup_{i=1}^n K_i$ is a subcontinuum of $X \setminus \{c\}$ which intersects each U_i. $\square$

There are known metric examples to show that none of these implications can be reversed.


c
U_1
U_2
U_3
x_1
x_2
x_3
K_1
K_2
K_3
x

The following is an important result for us.

5. Lemma (R. H. Bing, 1948). *If X is a metrizable continuum and S is a nonempty proper subset, there is a point $c \in X$ such that the union of all subcontinua that intersect S and exclude c is dense in X.*

The proof relies on the Baire category theorem, as well as the second countability of X. And while D. Anderson has shown Bing's argument can be modified so that only the separability of X need be assumed, the result is not true for all continua.

From here it's a short hop to the following analogue of the Krein-Milman theorem, which is due to R. Leonel for shore points in the metrizable case, J. Bobok et al for non-block points in the metrizable case, and to D. Anderson for non-block points in the separable case.

6. Proposition. *Every separable continuum is irreducible about its set of non-block points.*

Proof. Suppose N is any set of non-block points of X, with $K \supseteq N$ a proper subcontinuum of X. Then, by (the separable version of) Bing's Lemma 5, there is a non-block point of X in $X \setminus K$. Hence the full set of non-block points cannot be contained within a proper subcontinuum. $\square$

A continuum is **decomposable** if it is the union of two proper subcontinua, and **indecomposable** otherwise.

Given a point a of a continuum X, the **composant** $\kappa(a)$ of a in X is the union of all proper subcontinua of X containing a. Composants are continuumwise connected dense subsets of X; and when X is indecomposable, the composants are pairwise disjoint.

The number of composants of a nondegenerate metrizable continuum is $\mathfrak{c}$, but D. Bellamy showed in the 1970s that there are indecomposable continua, of weight $\aleph_1$, which have just one composant. Clearly an indecomposabe continuum is irreducible iff it has at least two composants. So we refer to an indecomposable continuum which is not irreducible as a **Bellamy continuum**.

Bellamy continua play an important role in the problem of whether extreme points are always non-block.

7. Proposition. *If an indecomposable continuum is irreducible, then every one of its points is a weak cut point, as well as a non-block point.*

Proof. Suppose X is an indecomposable continuum with at least two separate composants. Given $c \in X$, first find $a \in \kappa(c) \setminus \{c\}$, then let $b \in X \setminus \kappa(c)$. Then any subcontinuum of X containing both a and b is all of X; hence c is a weak cut point.

The continuum components of $X \setminus \{c\}$ consist of the continuum components of $\kappa(c) \setminus \{c\}$, as well as the composants of X other than $\kappa(c)$. There is at least one of these, and it is dense in X. Thus c is a non-block point. $\square$

We now turn to Question A. We already know *strong non-cut* $\Rightarrow$ *extreme*, and it is relatively easy to show that *extreme* $\Rightarrow$ *non-cut*. The question we want to consider in the rest of this talk is whether *extreme* $\Rightarrow$ *non-block*. Here is our first partial answer.

8. Proposition. *Every extreme point is a shore point.*

Proof. Suppose $e \in X$ is an extreme point, with $\{U_1, \ldots, U_n\}$ a finite family of nonempty open subsets of X. Let $\mathcal{A}$ be the family of continuum components of $X \setminus \{e\}$. Then A^-, for $A \in \mathcal{A}$, is a subcontinuum containing e. (This is an easy application of boundary bumping.) Now suppose there are $A, B \in \mathcal{A}$ with incomparable closures. Let $a \in A \setminus B^-$ and $b \in B \setminus A^-$. Then $e \in [a, b]$, but B^- (resp., A^-) witnesses that $a \notin [e, b]$ (resp., $b \notin [a, e]$), so e is not an extreme point of X, and we have a contradiction.

Thus, if $e \in X$ is an extreme point, the family $\mathcal{A}^- := \{A^- : A \in \mathcal{A}\}$ is nested. For $1 \leq i \leq n$, let $x_i \in U_i \backslash \{e\}$, with $A_i \in \mathcal{A}$ such that $x_i \in A_i$. WLOG, assume A_1^- contains each of the other A_i^-; in particular, we know $\{x_1, \ldots, x_n\} \subseteq A_1^-$. Thus there is some $y_i \in A_1 \cap U_i$ for each $1 \leq i \leq n$. Fix $x \in A_1$ and subcontinua $K_i \subseteq A_1$ such that $\{x, y_i\} \subseteq K_i$. Then $\bigcup_{i=1}^n K_i$ is a subcontinum that misses e and intersects each U_i. This makes e a shore point of X. $\square$

In the proof above we identified a new point type. Call $c \in X$ **nested** if the family of closures of the continuum components of $X \setminus \{c\}$ is nested. So we know that *extreme* $\Rightarrow$ *nested* $\Rightarrow$ *shore*.

9. Question. *Is every extreme (or nested) point non-block?* [After the talk: It is consistent with ZFC that a nested point can also be a block point.]

We will see below that a universal *yes* answer would solve a long-standing open problem. On the other hand, it is relatively easy to see that shore (even non-block) points needn't be nested and that nested points needn't be extreme.

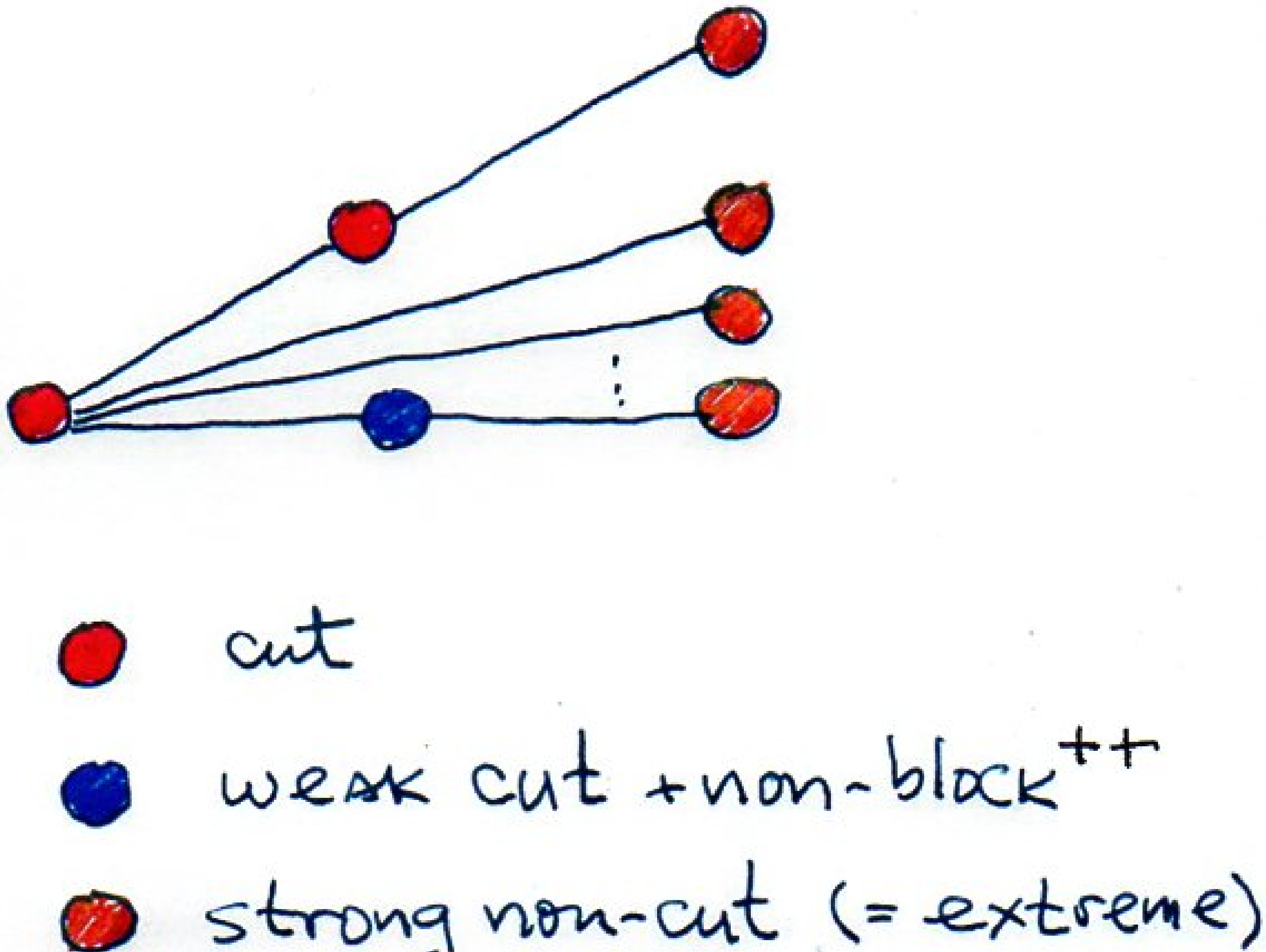
A continuum is **unicoherent** if it is not the union of two subcontinua whose overlap is disconnected; it is **hereditarily unicoherent** if every subcontinuum is unicoherent.

Fact: *A continuum X is hereditarily unicoherent iff $[S]$ is connected for any $S \subseteq X$.*

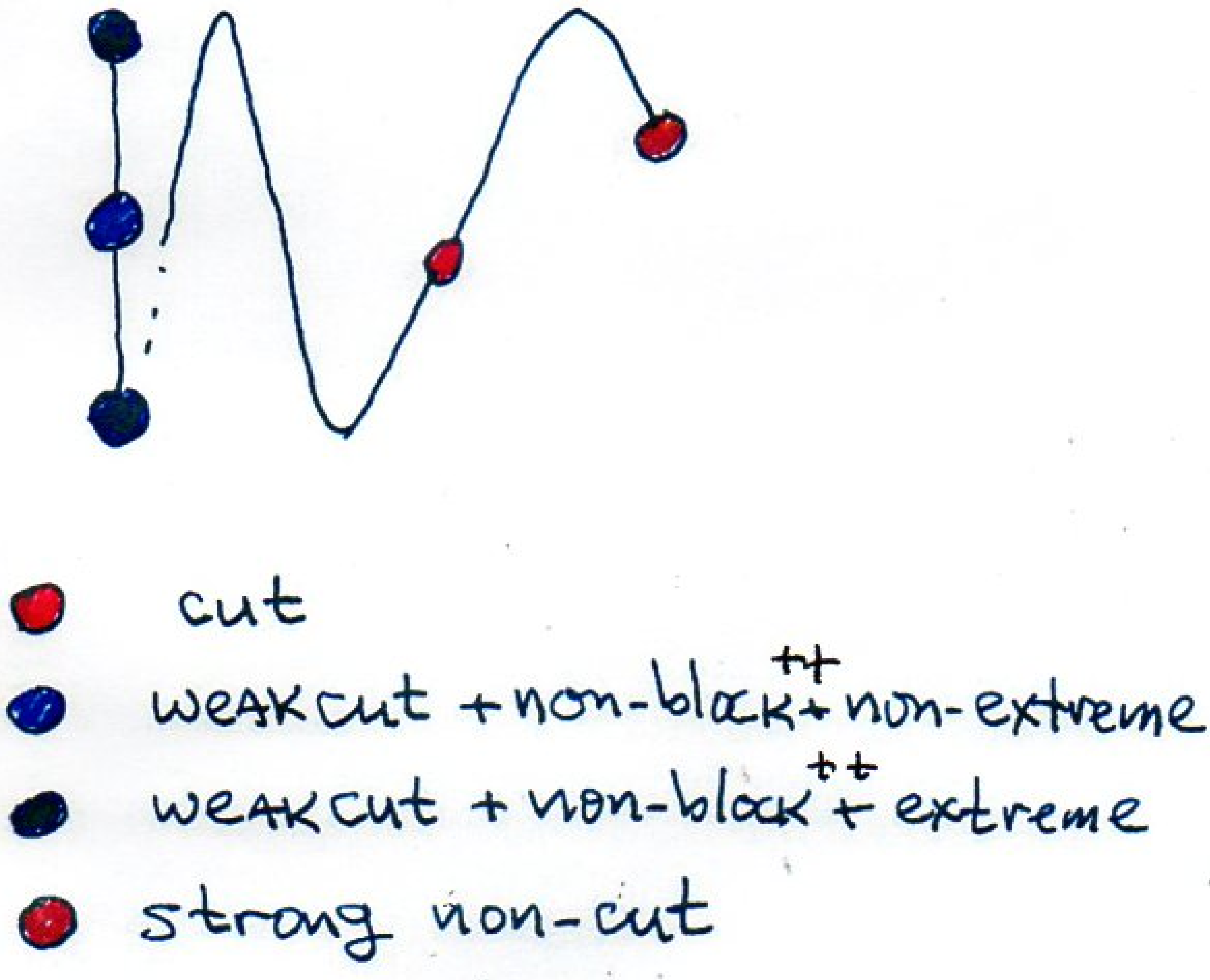
10. Proposition. *If X is hereditarily unicoherent, $e \in X$ is an extreme point, and K is a subcontinuum of X containing e, then e is an extreme (and hence a shore) point of K.*

This proposition may be used to show that certain continua–e.g., solenoids, $\mathbb{H}^*$–have no extreme points at all. Indeed, if c is any point of a solenoid X, then all the continuum components of $X \setminus \{c\}$ are dense in X; hence c is nested. So nested points needn't be extreme.

Here's a color-coded picture of the harmonic fan, which is antisymmetric without being aposyndetic. This is another instance where you can have a nested point which is not extreme. (Viz. the blue point.)


cut
weak cut + non-block++
strong non-cut (= extreme)

In the absence of antisymmetry, extreme points can be weak cut points. Here's a color-coded picture of the $\sin\frac{1}{x}$-continuum. Here you have a shore–indeed, non-block–point which is not nested. (Viz.–again–the blue point.)


cut
weak cut + non-block++ + non-extreme
weak cut + non-block++ + extreme
strong non-cut

An indecomposable continuum is **hereditarily indecomposable** if each of its nondegenerate subcontinua is indecomposable. This is equivalent to saying that if two subcontinua overlap, then one is contained in the other.

It is unknown whether a hereditarily indecomposable (nonmetrizable) continuum can have just one composant, but regardless of that we have the following easy result.

11.Proposition. *Every point of a hereditarily indecomposable continuum is extreme, as well as weak cut.*

Proof. Start with $c \in X$ arbitrary. Then (boundary bumping) there is a proper nondegenerate subcontinuum K containing c. Let $a \in K \setminus \{c\}$, with $b \in X \setminus K$. If M is a subcontinuum containing both a and b, then M overlaps K, but is not contained in K. Thus $M \supseteq K$, and so $c \in M$. Thus $c \in [a, b] \setminus \{a, b\}$, making c a weak cut point of X. Also for any triple $\langle a, b, c\rangle$ from X, hereditary indecomposability implies that either $c \in [a, b]$ or $b \in [a, c]$. This trivially implies that any point of X is an extreme point. $\square$

There is also a partial converse to Proposition 11: *If X is hereditarily unicoherent and every point of X is extreme, then X is hereditarily indecomposable.* (You can't dispense with hereditary unicoherence, as any simple closed curve consists entirely of extreme points.)

The following is a contribution to answering Question A.

12. Proposition. *Suppose* $e \in X$ *is an extreme point which is also block. Then* X *is a Bellamy continuum.*

Proof Sketch. Let $\mathcal{A}$ be the family of continuum components of $X \setminus \{e\}$. Since e is extreme, we know–from the proof of Proposition 8 above–that $\mathcal{A}^-$ is a nested family of subcontinua containing e. Since $\bigcup \mathcal{A} = X \setminus \{e\}$ is dense, so too is $\bigcup \mathcal{A}^-$. And since e is a block point, each A^- is a proper subcontinuum; i.e., $\mathcal{A}^-$ has no $\subseteq$-maximal element.

We again use the fact that e is extreme to infer that if $A, B \in \mathcal{A}$ are such that $A^- \subsetneq B^-$, and if K is any subcontinuum of X which intersects both A and B, then $A^- \subseteq K$. From this we infer that if K is any subcontinuum with nonempty interior, then $A^- \subseteq K$ for all $A \in \mathcal{A}$. Since $\bigcup \mathcal{A}^-$ is dense, we conclude that $K = X$. Thus all proper subcontinua of X are nowhere dense; hence X is indecomposable. If X had more than one composant, all of its points would be non-block, by Proposition 7. Hence X is a Bellamy continuum. $\square$

13. Corollary. *Suppose X is a continuum which is either decomposable, irreducible, or metrizable. Then every extreme point of X is non-block*

So if we want a counterexample to the assertion *extreme* $\Rightarrow$ *non-block*, we need to look at Bellamy continua. But not any old Bellamy contnuum will do: $\mathbb{H}^*$ is consistently a Bellamy continuum, but has no extreme points at all. On the other hand, what if there were a Bellamy continuum that is hereditarily indecomposable. (Not known to exist; wide open problem studied by lots of people.)

14. Proposition. *Let X be a hereditarily indecomposable Bellamy continuum. Then every point of X is both extreme and block.*

Proof. We saw above (Proposition 11) that every point of X is extreme; so fix $c \in X$, with A a continuum component of $X \setminus \{c\}$. We may pick $a \in A$ and write $A = \bigcup \mathcal{K}$, where $\mathcal{K}$ is a family of subcontinua of A, all containing a. Since X is not irreducible, there is a proper subcontinuum $M \supseteq \{a, c\}$. If $K \in \mathcal{K}$ is arbitrary, we know–since $a \in K \cap M$, $c \in M \setminus K$, and X is hereditarily indecomposable–that $K \subseteq M$. Hence $A \subseteq M$. But then $X \setminus M$ is a nonempty open set disjoint from A; so A is not dense. Hence c is a block point. □

15. Parting Questions.

(i) *If* X *is nondegenerate and every point of* X *is both extreme and block, is* X *necessarily a hereditarily unicoherent Bellamy continuum?* (If so, X is also hereditarily indecomposable.)

(ii) *What are some interesting consequences of having a nested point which is also block? Are nested points in, say, metrizable continua necessarily non-block?* [After the talk: The continuum $\mathbb{H}^*$ has no extreme points, and every point is nested. Consistently, every point is block. So these facts do not seem to affect the question very much.]

THANK YOU!