Topological Interpretations of the Gap Free Betweenness Axiom

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1. The Gap Free Axiom.

For us “betweenness” is a pre-theoretical term, which may be given a precise meaning in a variety of ways.

The first-order language of betweenness has a single ternary predicate symbol $[\cdot,\cdot,\cdot]$, and we read $[a,c,b]$ as saying: “c lies between a and b” (with $c \in \{a,b\}$ permitted).

Gap freeness says that any two points have a third point between them; this is expressed formally as

- Gap Freeness:
  $\forall\, ab\,(a \neq b \to \exists\, x\,([a,x,b] \wedge x \neq a \wedge x \neq b))$

For example, if we start with a totally ordered set $\langle X, \leq \rangle$ and define $[a, c, b]$ to mean $(a \leq c \leq b) \vee (b \leq c \leq a)$, then gap freeness in this interpretation means that the ordering is dense.

We'll be talking today about gap free betweenness relations naturally arising in the context of connected topological spaces.

A connected space that is also compact Hausdorff is called a **continuum**; a continuum that is contained in a space is a **subcontinuum** of the space.

2. Three Topological Interpretations.

We highlight three such interpretations for a connected space X and points $a, b, c \in X$. Assuming $c \notin \{a, b\}$, we define:

- $[a, c, b]_Q$ if there's a disconnection $\langle A, B\rangle$ of $X \setminus \{c\}$ such that $a \in A$ and $b \in B$ (i.e., a and b lie in different quasicomponents of $X \setminus \{c\}$);

- $[a, c, b]_C$ if no connected subset of $X \setminus \{c\}$ contains $\{a, b\}$ (i.e., a and b lie in different components of $X \setminus \{c\}$); and

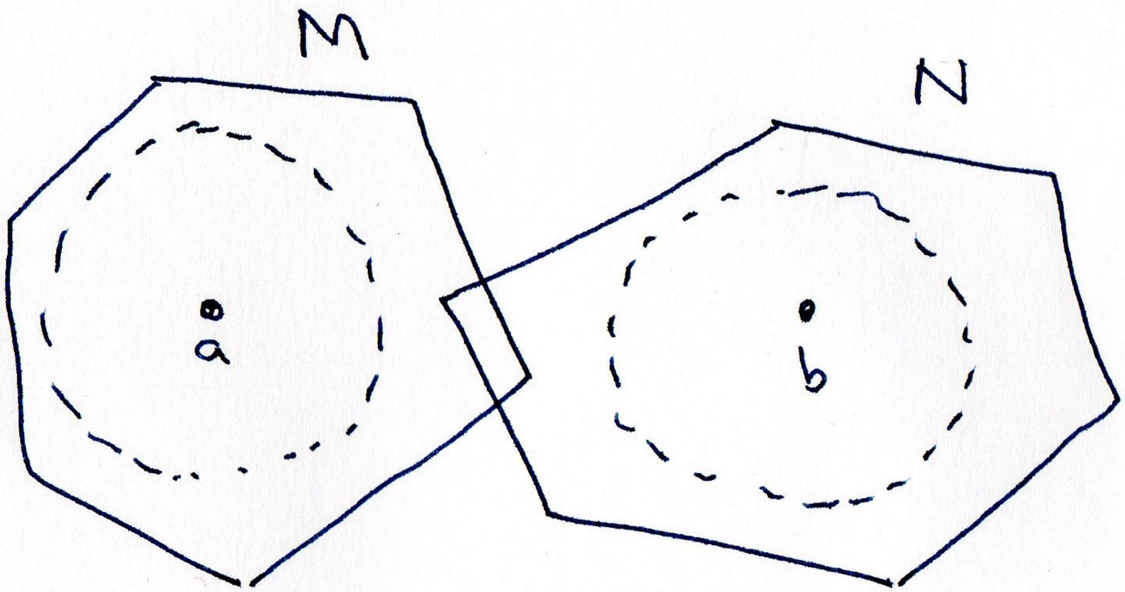
- $[a, c, b]_K$ if no subcontinuum of $X \setminus \{c\}$ contains $\{a, b\}$ (i.e., a and b lie in different continuum components of $X \setminus \{c\}$).

Clearly $[\cdot,\cdot,\cdot]_Q \subseteq [\cdot,\cdot,\cdot]_C \subseteq [\cdot,\cdot,\cdot]_K$; hence

Q-gap free $\Rightarrow$ C-gap free $\Rightarrow$ K-gap free.

So what about instances where betweenness interpretations agree?

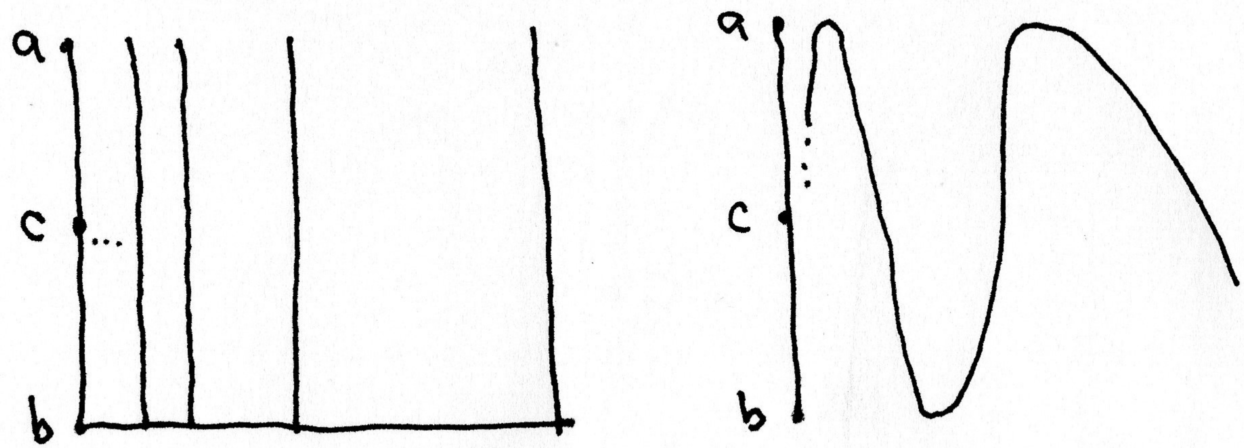
A continuum is **aposyndetic** (after F. B. Jones, 1941) if for each two of its points, one lies in the interior of a subcontinuum that excludes the other.



Aposyndesis

2.1 Theorem (PB, unpublished). *If X is an aposyndetic continuum, then $[\cdot,\cdot,\cdot]_K = [\cdot,\cdot,\cdot]_C$. If X is also locally connected, then $[\cdot,\cdot,\cdot]_K = [\cdot,\cdot,\cdot]_Q$.* $\square$

As for disagreement, any comb space or $\sin(\frac{1}{x})$-continuum serves to show that $[\cdot,\cdot,\cdot]_C$ needn't coincide with $[\cdot,\cdot,\cdot]_K$.



$[a, c, b]_K$, but $\neg [a, c, b]_C$

However, we have no example of a continuum for which $[\cdot,\cdot,\cdot]_C \neq [\cdot,\cdot,\cdot]_Q$. A connected metrizable–but not compact–example of this inequality may be described as follows:

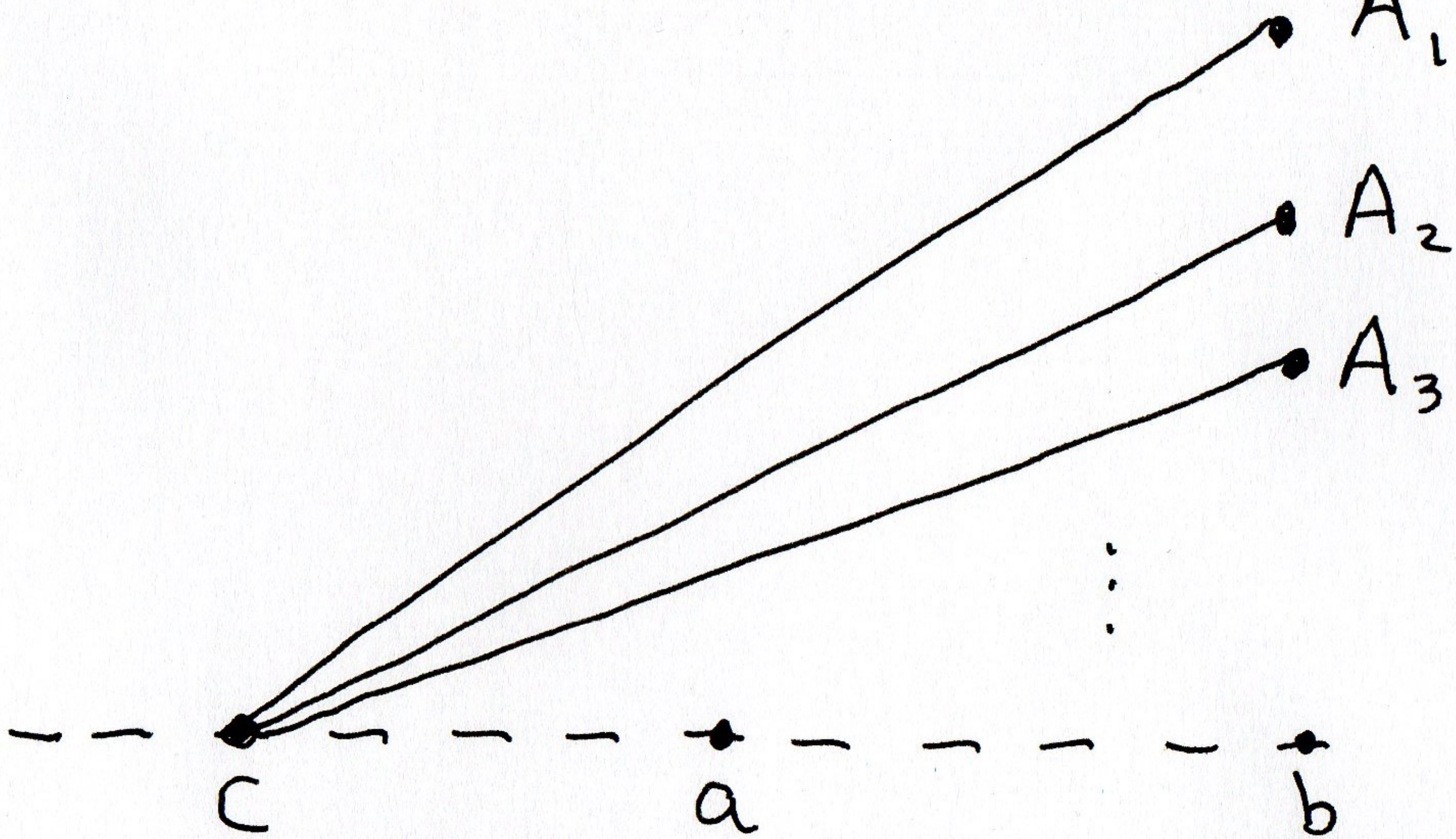
In the plane, let $a = \langle \frac{1}{2}, 0\rangle$, $b = \langle 1, 0\rangle$, and $c = \langle 0, 0\rangle$. For $n = 1, 2, \ldots,$ let

$$A_n = \{\langle t, \frac{t}{n}\rangle : 0 \leq t \leq 1\},$$

and set

$$X = (\bigcup_{n=1}^{\infty} A_n) \cup \{a, b\}.$$

Then $\{a\}$ and $\{b\}$ are components of $X \setminus \{c\}$; so we have $[a, c, b]_C$ holding. However, if U is any clopen subset of $X \setminus \{c\}$ with $a \in U$, then U also contains almost all sets A_n. Hence $b \in U$ as well; so $[a, c, b]_Q$ does not hold.



$[a,c,b]_C$, but $\neg[a,c,b]_Q$

Added post-talk: We can now produce a continuum example for which $[\cdot,\cdot,\cdot]_C \neq [\cdot,\cdot,\cdot]_Q$: Back in the plane, let $A_n = [0,1) \times \{\frac{1}{n}\}$, $n = 1,2,\ldots,$ with $A = [0,\frac{1}{2}) \times \{0\}$ and $B = (\frac{1}{2},1) \times \{0\}$. Pick $a \in A$ and $b \in B$. Then A and B are the components of a and b, respectively, but a and b lie in the same quasicomponent of X, namely $A \cup B$. X is locally compact, and none of its components is compact. Hence its one-point compactification $Y = \alpha(X)$ is a continuum. Let $c \in Y$ be the "point at infinity." Then, in Y, we have $[a,c,b]_C$, but not $[a,c,b]_Q$. Y is not C-gap free, however, so we still do not know whether C-gap free imples Q-gap free.

3. Q-gap Freeness.

Q-gap freeness is the defining condition for a continuum being a **dendron**. Dendrons are locally connected (L. E. Ward, 1954); hence Q=C=K for them (Theorem 2.1).

(Dendrites, the locally connected metrizable continua containing no simple closed curves, are just the metrizable dendrons.)

A topological space satisfies the **connected intersection property** (cip) if the intersection of any two of its connected subsets is connected. The following generalizes a well-known characterization of dendrites.

3.2 Theorem (Ward, 1991). *A continuum satisfies the cip if and only if it is a dendron.* □

4. C-gap Freeness.

Currently we do not know of any literature on the C-interpretation of betweenness, so here is an opportunity to ask some questions, especially in relation to continua:

- Do the Q- and the C-interpretations of betweenness agree for continua? [No, see note added post-talk above.]

- Or, failing that, does C-gap freeness imply Q-gap freeness?

- Assuming Q- and C-gap freeness are distinct notions for continua, are there any well-known consequences of Q-gap freeness that are also consequences of C-gap freeness? (E.g.: local connectedness, aposyndesis, hereditary unicoherence, hereditary decomposability).

- Or, is there some weakened form of the cip that characterizes C-gap freeness?

We will return to this later on.

5. K-gap Freeness.

Given a continuum X and $a, b \in X$, let $\mathcal{K}(a,b)$ constitute the subcontinua of X that contain both a and b. Then the **K-interval** $[a,b]_K$ **bracketed by** a **and** b is defined to be $\bigcap \mathcal{K}(a,b)$. Hence $[a,c,b]_K$ holds iff $c \in [a,b]_K$.

The following is straightforward.

5.1 Proposition. *A continuum is hereditarily unicoherent iff each of its K-intervals is a subcontinuum.* $\square$

Hereditary unicoherence clearly implies K-gap freeness, and it is natural to ask whether this weakening of the cip is actually a characterization.

The answer turns out to be NO.

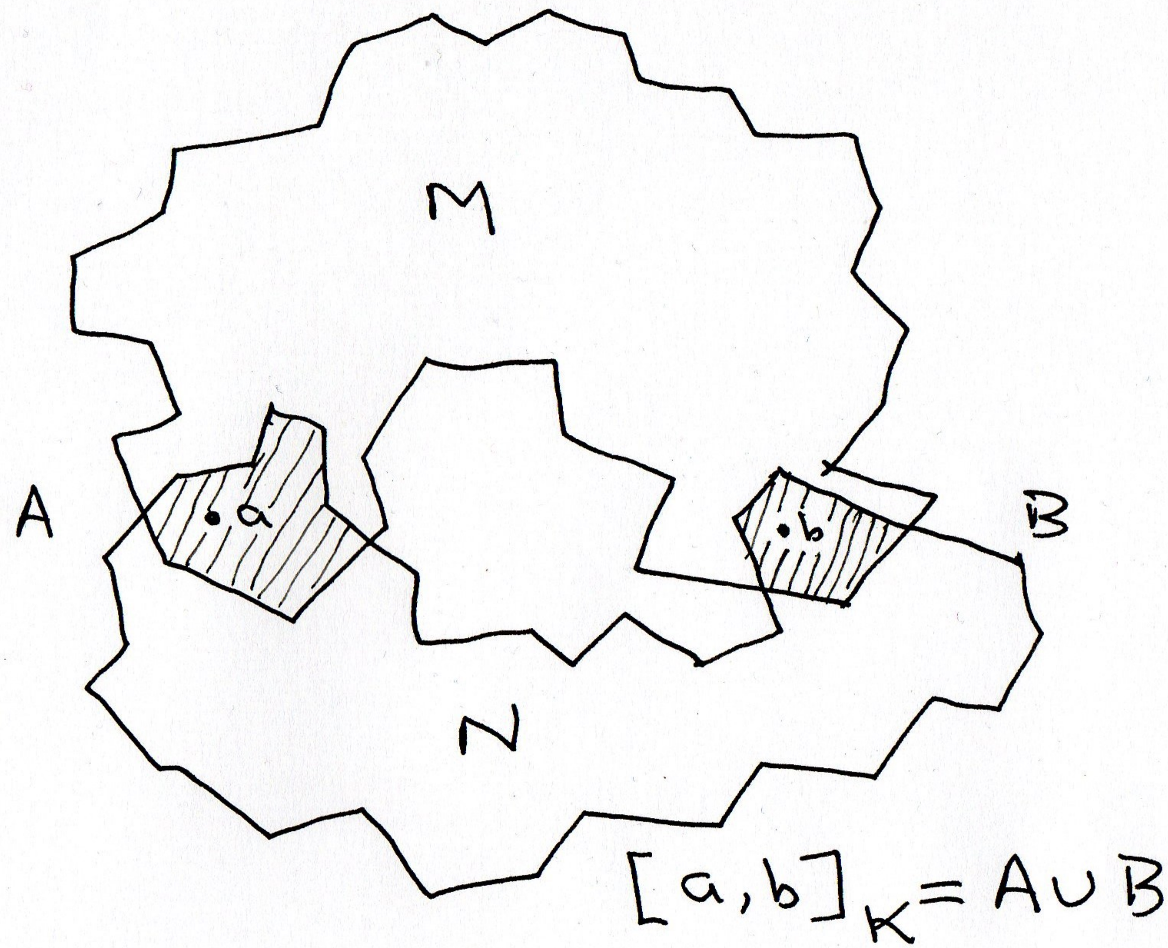
A continuum X is a **crooked annulus** if it has a decomposition $X = M \cup N$ into subcontinua such that:

- Both M and N are hereditarily indecomposable; and

- $M \cap N = A \cup B$, where A and B are disjoint nondegenerate subcontinua.

5.2 Theorem (PB, 2013). *A crooked annulus is K-gap free without being even unicoherent, let alone hereditarily so.* $\square$

In a crooked annulus one can show that each nondegenerate K-interval $[a, b]_K$ contains two nondegenerate subcontinua, one containing a and the other containing b. (E.g., if $a \in A$ and $b \in B$, then $[a, b]_K = A \cup B$.) This clearly gives us K-gap freeness.


M
A
a
b
B
N
$[a,b]_K = A \cup B$

6. Strong K-gap Freeness.

Recall the first-order statement of gap freeness from above.

- Gap Freeness:
  $\forall\, ab\, (a \neq b \rightarrow \exists\, x\, ([a, x, b] \wedge x \neq a \wedge x \neq b))$

If we replace negations of equality in the conclusion with negations of betweenness, we obtain a stronger property (when betweenness is interpreted properly).

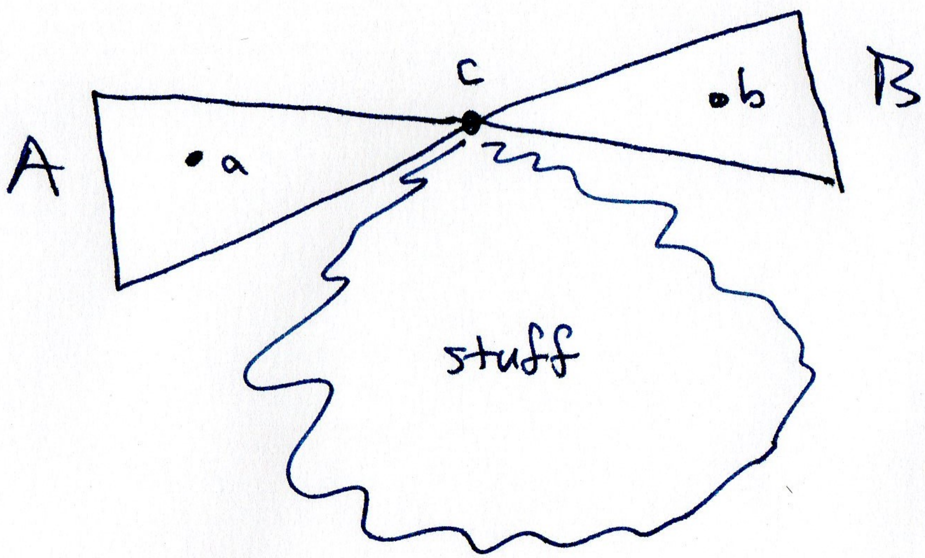
- Strong Gap Freeness:
  $\forall\, ab\, (a \neq b \rightarrow \exists\, x\, ([a, x, b] \wedge \neg[x, a, b] \wedge \neg[a, b, x]))$

With the Q- and the C-interpretations, strong gap freeness is not really stronger than gap freeness because these interpretations satisfy

- Antisymmetry:
  $\forall\, abc\, (([a,b,c] \wedge [a,c,b]) \rightarrow b = c)$

Antisymmetry in a "reasonable" betweenness interpretation amounts to saying that each binary relation $\leq_a$, given by $x \leq_a y$ iff $[a,x,y]$ holds, is antisymmetric in the usual sense. When this happens, the relation $\leq_a$ is a tree ordering, with root a.

To see why the C-interpretation is antisymmetric, suppose $[a, c, b]_C$ and $b \neq c$. We want to show that $[a, b, c]_C$ fails. If $c = a$ then clearly $\neg[a, b, c]_C$; so assume $c \notin \{a, b\}$. Then there are components A and B of $X \setminus \{c\}$ with $a \in A$ and $b \in B$. Thus, by an old theorem of K. Kuratowski, $X \setminus B$ is a connected subset of $X \setminus \{b\}$ containing a and c; so $\neg[a, b, c]_C$. The Q-interpretation is antisymmetric as well because it is finer than the C-interpretation.

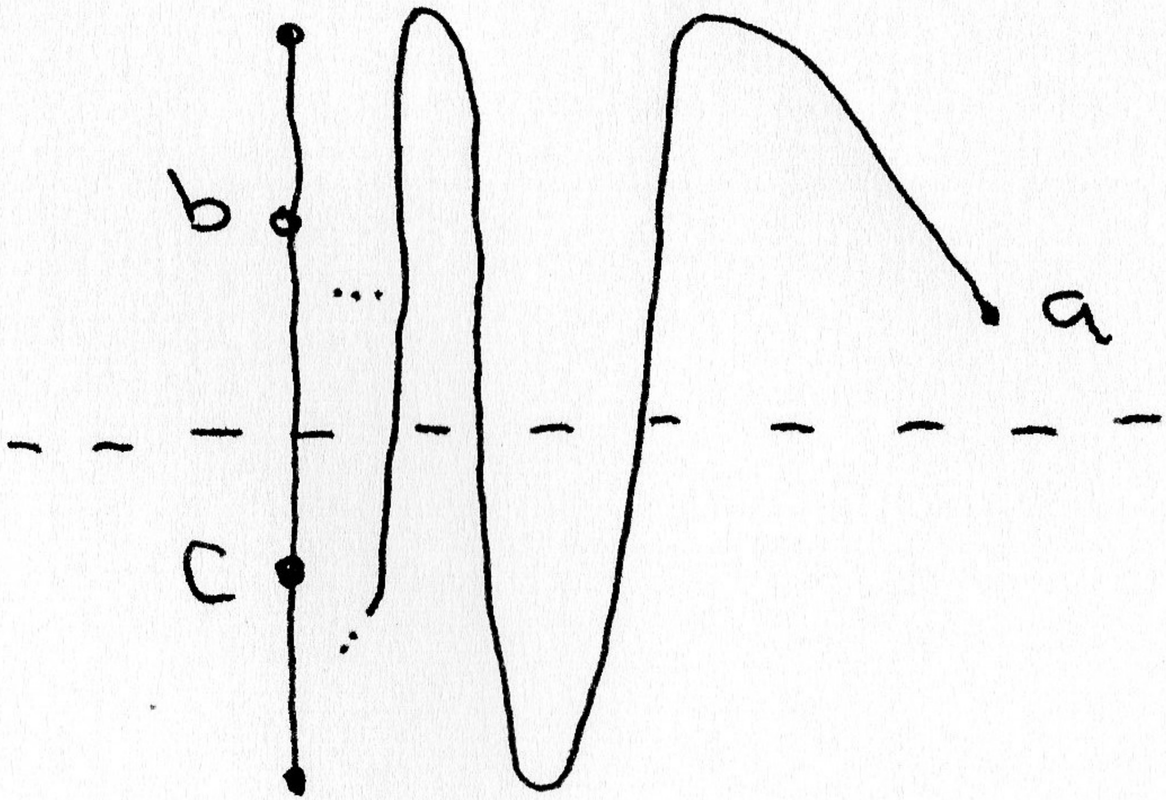


$a, c \in X \setminus B$, $b \notin X \setminus B$

By Theorem 2.1 (aposyndetic $\Rightarrow$ C=K), aposyndetic continua are K-antisymmetric. The converse is not true, as the comb space is K-antisymmetric without being aposyndetic.

The $\sin(\frac{1}{x})$-continuum is not K-antisymmetric: if a is any point on the graph of $y = \sin(\frac{1}{x})$, $0 < x \leq 1$, and b and c are any two points on the line segment $\{0\} \times [-1, 1]$, then both $[a, c, b]_K$ and $[a, b, c]_K$ hold.

[Indeed, for a continuum X to be K-antisymmetric it is necessary for $|X \setminus C| \leq 1$ for each composant C of X.]



$[a, c, b]_K$ and $[a, b, c]_K$

Recall Ward's result (Theorem 3.1) that Q-gap freeness in continua is equivalent to the cip, but (Theorem 5.2) that K-gap freeness is strictly weaker than hereditary unicoherence. We coin the term λ-**arboroid**–inspired by a 1974 paper of Ward–to refer to a continuum that is both hereditarily unicoherent and hereditarily decomposable. (So that what is commonly known as a λ-dendroid is just a metrizable λ-arboroid.)

6.1 Theorem (PB, 2013). *A continuum is strongly K-gap free if and only if it is a λ-arboroid.* $\square$

7. Extra Strong K-gap Freeness.

By **extra strong gap freeness** in an interpretation of betweenness we mean that both gap freeness and antisymmetry hold. A continuum is **arcwise connected** if each two of its points constitute the noncut points of a subcontinuum; an **arboroid** is a hereditarily unicoherent continuum that is arcwise connected. (The dendroids are the metrizable arboroids; the dendrites are the locally connected dendroids, the locally connected λ-dendroids, as well as the metrizable dendrons. A comb space is a dendroid that is not a dendrite; a $\sin(\frac{1}{x})$-continuum is a λ-dendroid that is not a dendroid.)

We can now state an analogue of Theorem 6.1 for extra strong K-gap freeness.

7.1 Theorem (PB, unpublished). *A continuum is extra strongly K-gap free if and only if it is an arboroid.* $\square$

So, if we were to *define* being a dendron as satisfying the cik, our main gap free characterization results for continua could be summarized as:

Q-gap free $\Leftrightarrow$ dendron;

Extra strongly K-gap free $\Leftrightarrow$ arboroid; and

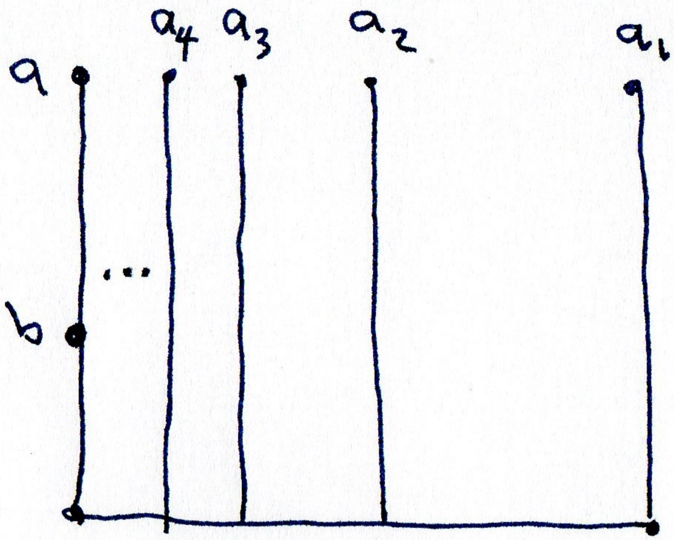
Strongly K-gap free $\Leftrightarrow$ λ-arboroid.

We currently have no characterizations of C-gap free or of K-gap free.

8. K-Closedness and C-Gap Freeness.

Define a continuum X to be **K-closed** if the ternary relation $[\cdot,\cdot,\cdot]_K$ is a closed subset of the cube X^3.

A comb space is not K-closed: indeed, if $a_1, a_2, \ldots$ are the end points of the "free teeth" of X and a is the end point of the "limit tooth," then we have $a = \lim_{n\to\infty} a_n$. If b any point on the limit tooth other than a, then $[a,b,\cdot]_K$ contains all the points a_n, but not a itself. Hence $[a,b,\cdot]_K$ is not closed in X.

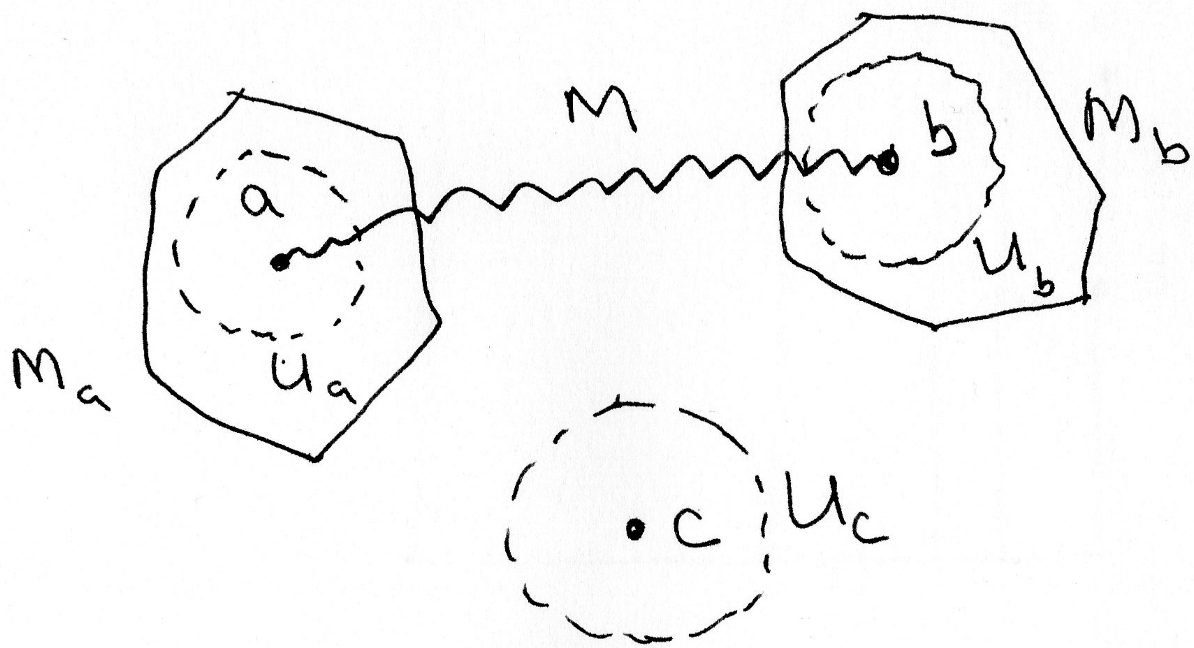


$$a \in \overline{[a, b, \cdot]}_K \setminus [a, b, \cdot]_K$$

We may relate K-closedness to properties previously discussed as follows:

8.1 Theorem (PB, unpublished). *Aposyndetic continua are K-closed; and K-closed hereditarily unicoherent continua are aposyndetic, as well as C-gap free.*

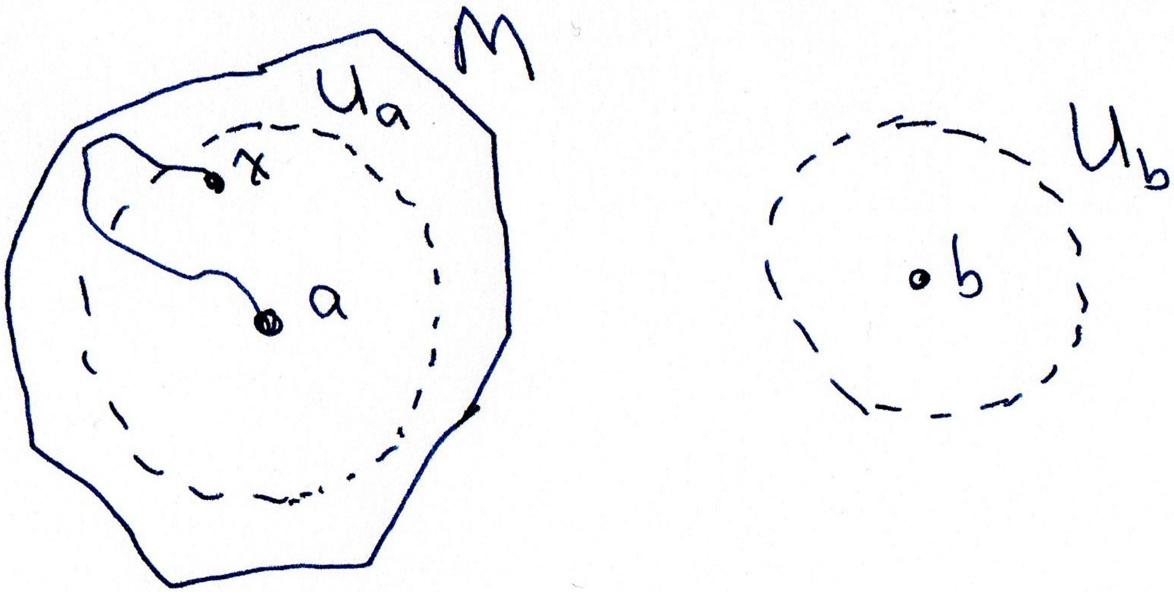
Proof (part 1). Suppose X is aposyndetic and that $\langle a, c, b\rangle \notin [\cdot,\cdot,\cdot]_K$; i.e., that $[a, c, b]_K$ does not hold. Then there is a subcontinuum $M \in \mathcal{K}(a, b)$ with $c \notin M$. Using aposyndesis, we have open sets U_a and U_b, and subcontinua M_a and M_b, with $a \in U_a \subseteq M_a \subseteq X \setminus \{c\}$ and $b \in U_b \subseteq M_b \subseteq X \setminus \{c\}$. Let U_c be an open neighborhood of c missing the subcontinuum $M_a \cup M \cup M_b$. Then $U_a \times U_c \times U_b$ is an open neighborhood of $\langle a, c, b\rangle \in X^3$ that does not intersect $[\cdot,\cdot,\cdot]_K$. Hence X is K-closed. $\square$


M
a
b
M_b
U_b
M_a
U_a
c
U_c

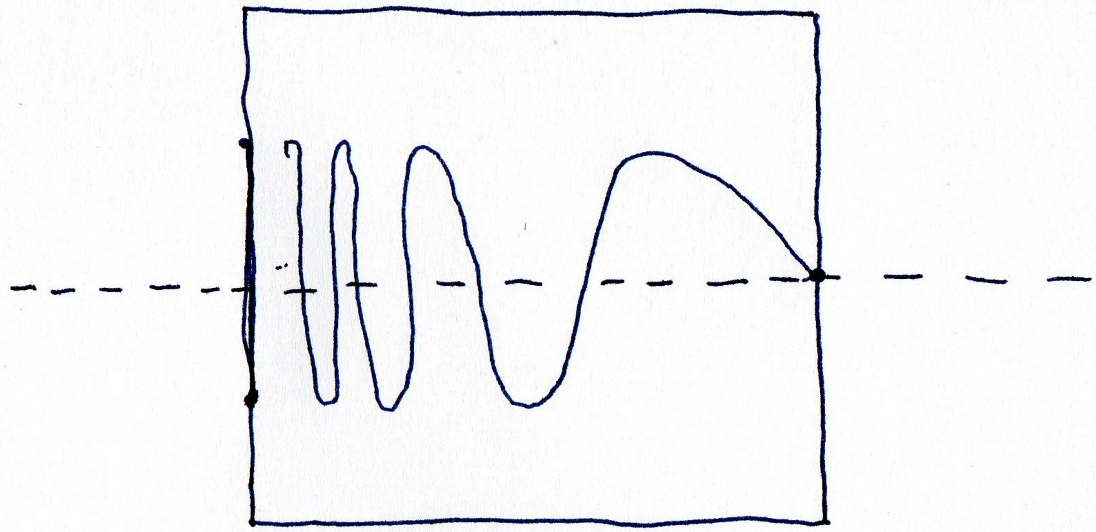
Proof (part 2). Assume X is hereditarily unicoherent, as well as K-closed, with a and b distinct points of X. Then $[a,b,a]_K$ does not hold; and by K-closedness, there are open sets U_a and U_b, with $a \in U_a$ and $b \in U_b$, such that if $\langle x,z,y\rangle \in U_a \times U_b \times U_a$, then $[x,z,y]_K$ does not hold either. In particular, for each $\langle x,z\rangle \in U_a \times U_b$, there is a subcontinuum of X that contains both a and x, but not z. Thus, for each $x \in U_a$ we have $[a,x]_K \cap U_b = \emptyset$; and so

$$M = \overline{\bigcup_{x \in U_a} [a,x]_K}$$

contains U_a and misses U_b. By hereditary unicoherence (Proposition 5.1), each $[a,x]_K$ is a subcontinuum of X. Hence M is a subcontinuum of X that contains a in its interior and excludes b, thereby establishing aposyndesis for X. That X is C-gap free now follows from Theorem 2.1 (aposyndetic $\Rightarrow$ C=K), since hereditary unicoherence trivially implies K-gap freeness. $\square$


M
U_a
x
a
U_b
b

K-closedness is not enough by itself to imply aposyndesis: indeed, consider the "topologist's oscilloscope" X in the plane, described as the union of the two horizontal segments $[0,1] \times \{i\}$, $i = \pm 1$, the two vertical segments $\{i\} \times [-1,1]$, $i = 0,1$, and the curve $\{\langle x, \frac{1}{2}\sin(\frac{\pi}{x})\rangle : 0 < x \leq 1\}$. X is non-aposyndetic, but its betweenness relation, being trivial, is just $(\Delta_X \times X) \cup (X \times \Delta_X) \subseteq X^3$. Thus X is K-closed.



$$[\cdot,\cdot,\cdot]_K = (\Delta_X \times X) \cup (X \times \Delta_X)$$

So, returning to the question of whether C-gap free continua are aposyndetic: an affirmative answer would give us

C-gap free $\Leftrightarrow$ K-closed + hereditarily unicoherent.

[“K-closed + hereditarily unicoherent $\Rightarrow$ C-gap free” and “aposyndetic $\Rightarrow$ K-closed” come from Theorem 8.1; so we would have “C-gap free $\Rightarrow$ K-closed.” Aposyndesis gives us C=K, hence K-antisymmetry and thus strong K-gap freeness. Now apply Theorem 6.1 to obtain hereditary unicoherence.]

Even if we were able to show C-gap free continua are K-antisymmetric, we could conclude that

C-gap free $\Rightarrow$ arboroid.

9. C-Gap Freeness and Strong K-Gap Freeness.

The modest result we can prove now is that C-gap free continua are λ-arboroids, and hence strongly K-gap free, by Theorem 6.1. (This is definitely not a characterization because the sin($\frac{1}{x}$)-continuum is a λ-dendroid that is not C-gap free. Indeed, the comb space is a dendroid that is not C-gap free.)

9.1 Lemma *Suppose X is a C-gap free continuum. Then for each two points $a, b \in X$, there is a third point c such that c is a cut point of every connected subset of X containing $\{a, b\}$. In particular, each nondegenerate connected subset C of X has a point which is a cut point of every connected subset of X containing C.*

Proof. Let C be a connected subset of X containing the doubleton $\{a, b\}$. By C-gap freeness, we have a third point c with $[a, c, b]_C$ holding. Thus, in particular, $c \in C$. No connected subset of $X \setminus \{c\}$ can contain both a and b; hence $C \setminus \{c\}$ is disconnected, and so c is a cut point of C. The second sentence follows immediately. $\square$

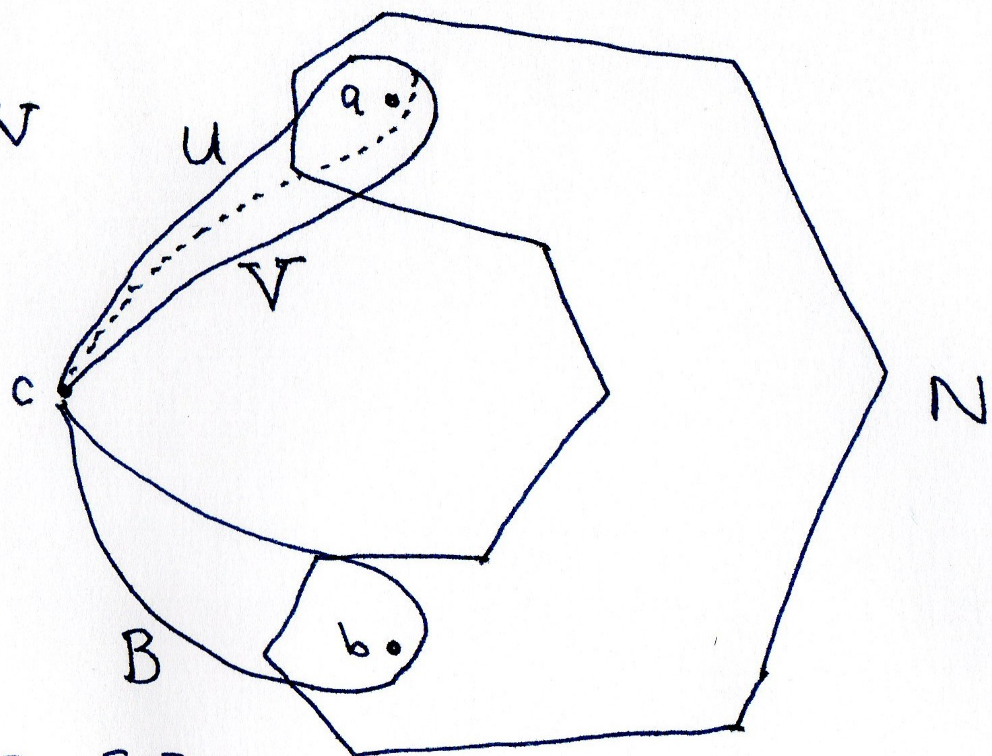
9.2 Theorem (PB, unpublished). *C-gap free continua are λ-arboroids.*

Proof (Hereditary Decomposability). By Lemma 9.1, every nondegenerate subcontinuum of a C-gap free continuum has a cut point, and hence must be decomposable.

(Hereditary Unicoherence). If C-gap free continuum X fails to be hereditarily unicoherent, then there exist points $a, b \in X$ and two subcontinua M and N, both irreducible about $\{a, b\}$, with neither contained in the other. Since $M \setminus N$ is a nonempty open subset of M, boundary bumping ensures that $M \setminus N$ contains a nondegenerate subcontinuum of X. Hence, by Lemma 9.1, there is a point $c \in M \setminus N$ that is a cut point of both M and $M \cup N$.

Let $\langle A, B\rangle$ be a disconnection of $M\backslash\{c\}$. If, say, A contained both a and b, then $A \cup \{c\}$ would be proper subcontinuum of M containing $\{a, b\}$, contradicting irreducibility. Hence we may assume $a \in A$ and $b \in B$. Suppose A had a disconnection $\langle U, V\rangle$, say, with $a \in U$. Then U is clopen in A and A is clopen in $M \setminus \{c\}$; hence U is clopen in $M \setminus \{c\}$. But then we have both a and b contained in the subcontinuum $(U \cup \{c\}) \cup (B \cup \{c\})$, properly contained in M. Again we contradict irreducibility, and conclude that both A and B are connected. But then we have $(M \cup N)\backslash\{c\} = A \cup N \cup B$, a connected set; so c is not a cut point of $M \cup N$, contradicting Lemma 9.1. $\square$

$A = U \cup V$



$M = A \cup B \cup \{c\}$

THANK YOU!

Slides available at
http://www.mscs.mu.edu/~paulb/talks.html