Images of Ultra-Arcs

Paul Bankston, Marquette University

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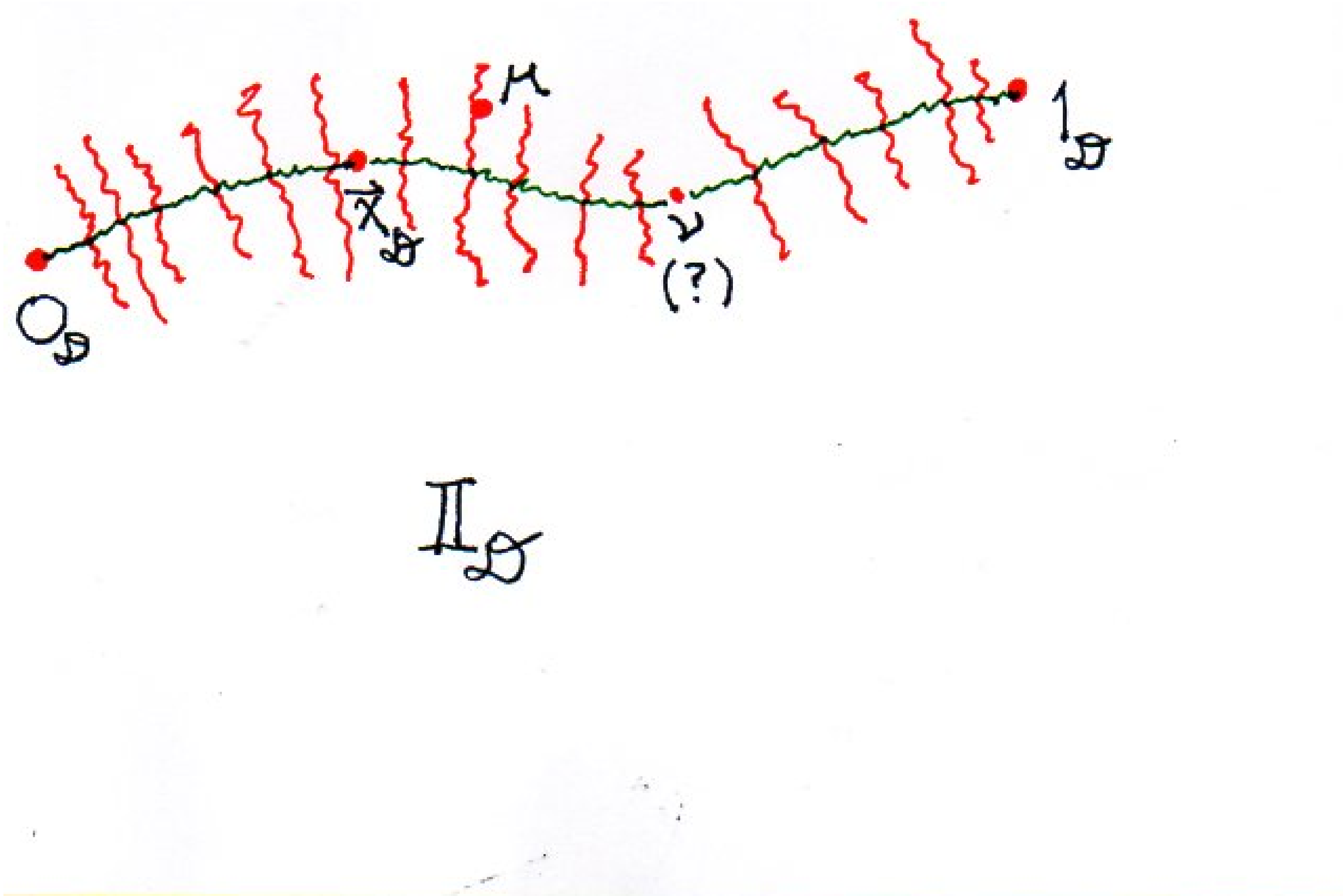
Introduction.

Ultra-arcs are the "standard subcontinua" of the Stone-Čech remainder $\mathbb{H}^*$ of the half-line $\mathbb{H} := [0, \infty)$.

Our interest in this talk is the problem of deciding when a continuum is the image of an ultra-arc under certain kinds of continuous map.

An ultra-arc looks something like an "arc with hair":



μ
1_ϑ
$\vec{x}_\vartheta$
ν
(?)
0_ϑ
Π_ϑ

As usual, **generalized arcs** are continua with exactly two noncut points.

Arcs are metrizable generalized arcs, and hence homeomorphic to $\mathbb{I} := [0, 1]$.

Ultra-arcs arise in $\mathbb{H}^*$ as follows:

Given a discrete unbounded sequence

$$a_0 < b_0 < a_1 < b_1 < \dots$$

from $\mathbb{H}$ and a nonprincipal ultrafilter $\mathcal{D} \in \omega^*$, a typical standard subcontinuum takes the form

$$\bigcap_{J \in \mathcal{D}} \mathrm{Cl}_{\beta(\mathbb{H})}\Big(\bigcup_{n \in J} [a_n, b_n]\Big).$$

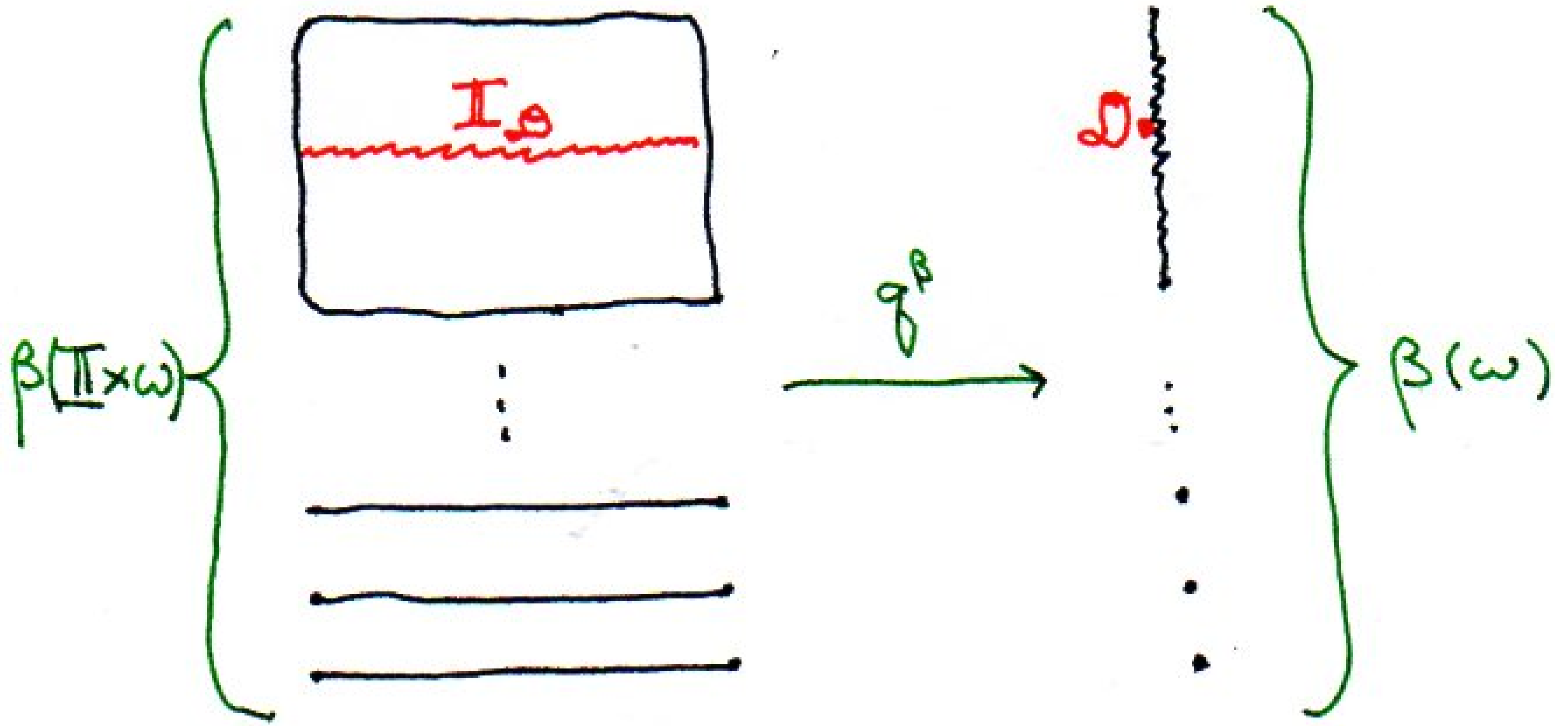
Alternatively, ultra-arcs are the components of $(\mathbb{I}\times\omega)^*$, and can be indexed using ω^* as follows:

Let $q : \mathbb{I} \times \omega \to \omega$ be the second coordinate projection map. Each component of $(\mathbb{I} \times \omega)^*$ can be written uniquely as a point-inverse image

$$\mathbb{I}_{\mathcal{D}} := (q^{\beta})^{-1}[\mathcal{D}],$$

where $\mathcal{D} \in \omega^*$.

Every standard subcontinuum of $\mathbb{H}^*$ is homeomorphic to some $\mathbb{I}_{\mathcal{D}}$.



β(𝕀×ω)
$\mathbb{I}_\theta$
q^β
θ
β(ω)

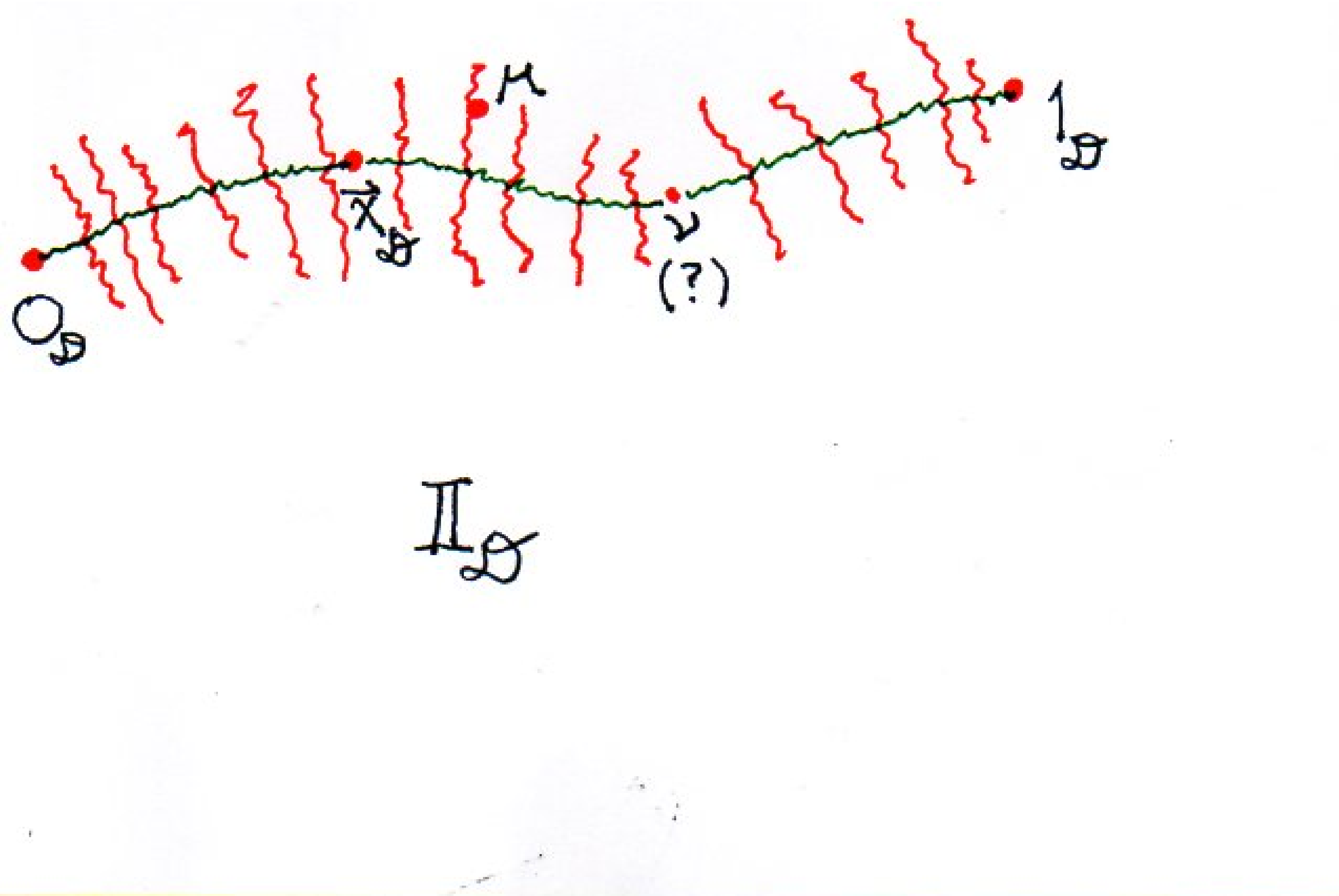
Informally speaking, the ultra-arc $\mathbb{I}_{\mathcal{D}}$ carries a natural pre-order, dictated by the ultrapower ordering $\leq^{\mathcal{D}}$ on the corresponding "nonstandard arc."

This is a total ordering and induces a total ordering on the associated equivalence classes (the *layers*) of the pre-order.

The layers are indecomposable continua, and many are nondegenerate. Each indecomposable subcontinuum of an ultra-arc is contained in a layer.

The partition of an ultra-arc into layers is upper semicontinuous, and the resulting quotient space is a generalized arc of weight $2^{\aleph_0}$.

Here's the picture again.



μ
$1_{\mathfrak{D}}$
$\vec{x}_{\mathfrak{D}}$
ν
(?)
$0_{\mathfrak{D}}$
$\Pi_{\mathfrak{D}}$

Ultra-arcs were first introduced by Mioduszewski in the mid 1970s in order to study the subcontinuum structure of $\mathbb{H}^*$, but here we are interested in their continuous images.

(Unrestricted) Continuous Images.

Proposition 0. (D. Bellamy) *Every metrizable continuum is a continuous image of any ultra-arc (in fact, of any nondegenerate subcontinuum of* $\mathbb{H}^*$*).*

Proposition 1. (Dow-Hart) *Every continuum of weight* $\leq \aleph_1$ *is a continuous image of any ultra-arc.*

In this talk, we focus down on the classes of monotone and of co-existential maps. The first is familiar, the second somewhat less so.

Monotone Images.

Proposition 2. *A nondegenerate monotone image of an ultra-arc is hereditarily unicoherent, irreducible, and decomposable.*

All three properties hold for ultra-arcs, and the first two are preserved by monotone maps.

As for decomposability, we show that any monotone mapping from an ultra-arc is irreducible on a decomposable subcontinuum. (Monotone maps do not preserve decomposability in general.)

It is true in general that monotone maps may raise covering dimension. This then raises the question:

Question 1. *Is a nondegenerate monotone image of an ultra-arc necessarily of covering dimension one?*

One can easily show that arcs are monotone images of ultra-arcs; but when we raise the weight, we get only a conditional result.

Proposition 3. (CH) *Every generalized arc of weight* $\leq \aleph_1$ *is a monotone image of every ultra-arc.*

This is a corollary of another result which we will mention later, and whose proof makes essential use of both the Löwenheim-Skolem Theorem and the CH-version of Keisler's Ultrapower Theorem.

[Note added after talk: Propositions 3 and 7 are true in ZFC.]

Ultra-arcs are far from being generalized arcs, and each ultra-arc is a monotone image of itself. But Proposition 3 does have a partial converse.

Proposition 4. *Every nondegenerate hereditarily decomposable monotone image of an ultra-arc is a generalized arc.*

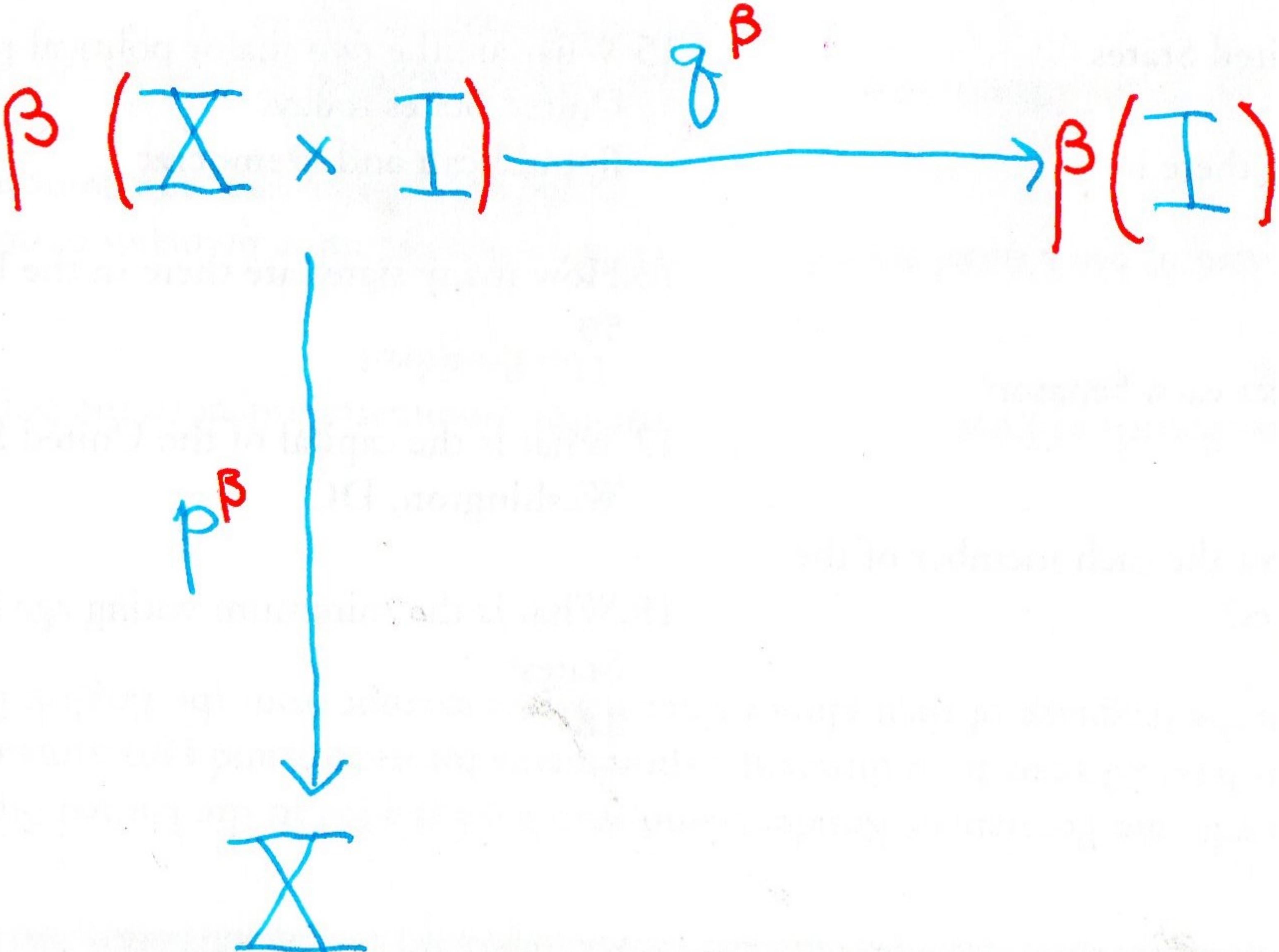
The proof of this relies on the fact that layers of ultra-arcs are indecomposable continua.

Question 2. *Is a nondegenerate metrizable monotone image of an ultra-arc necessarily an arc?*

(If not, it would still have to be some kind of "arc with indecomposable hair.")

Ultracopowers and Co-Existential Maps.

Given a compactum X and (discrete) set I, first form the cartesian product $X \times I$, with coordinate maps $p : X \times I \rightarrow X$ and $q : X \times I \rightarrow I$. Next apply the Stone-Čech functor, obtaining the following diagram.


β(X × I)
q^β
β(I)
p^β
X

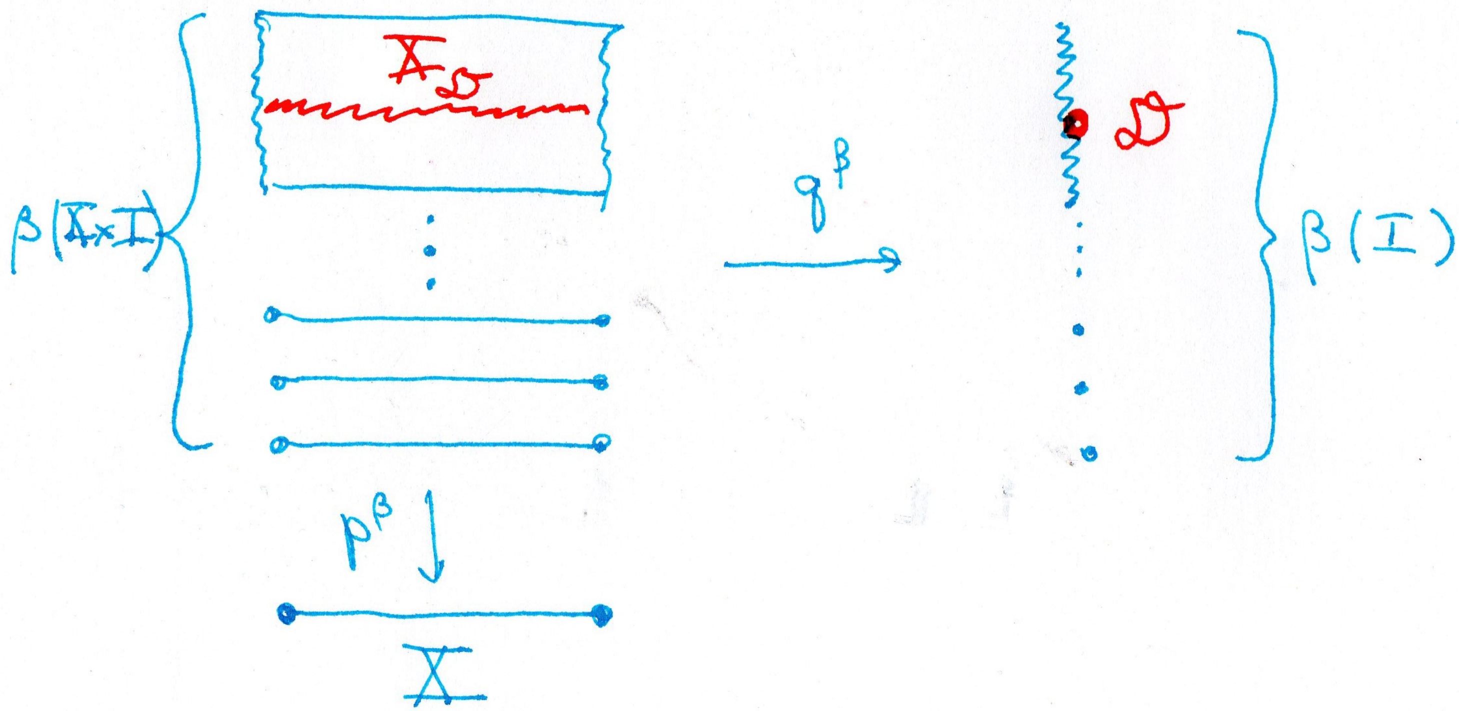
If $\mathcal{D}$ is an ultrafilter on I, then it may be viewed as a point in $\beta(I)$. Denote by $X_{\mathcal{D}}$ the pre-image of $\{\mathcal{D}\}$ under q^{β}. This is the $\mathcal{D}$-**ultracopower** of X.

When X is a continuum, these ultracopowers partition $\beta(X \times I)$ into its components.

The map

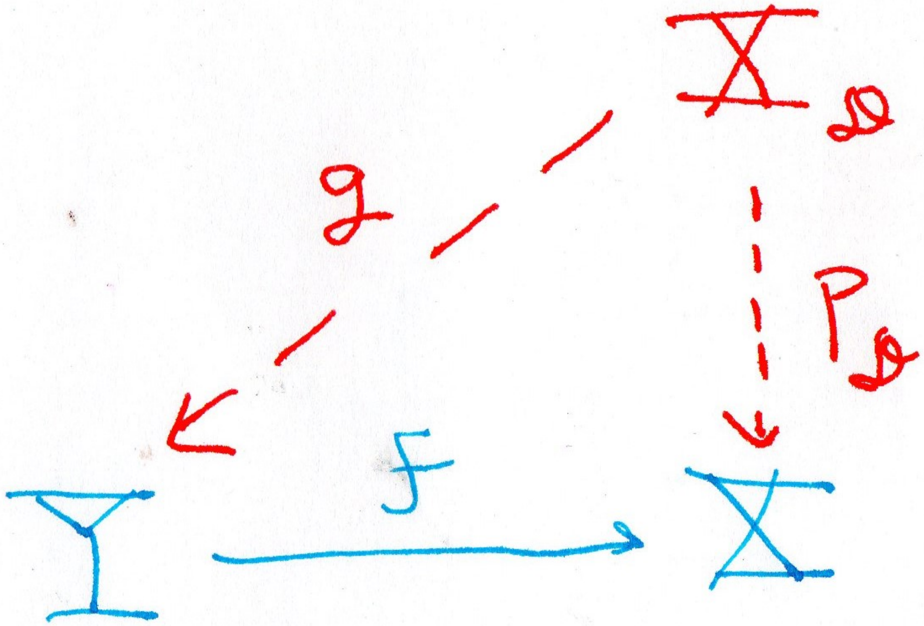
$$p_{\mathcal{D}} := p^{\beta}|X_{\mathcal{D}} : X_{\mathcal{D}} \to X$$

is a continuous surjection, called the **ultracopower codiagonal map**.



β(X×I)
X_θ
q^β
θ
β(I)
p^β
X

A mapping $f : Y \to X$ between compacta is **co-existential** if there is an ultracopower $X_{\mathcal{D}}$ and a surjective map $g : X_{\mathcal{D}} \to Y$ such that $f \circ g = p_{\mathcal{D}}$.



g
F

Co-existential maps play a category-theoretic role dual to that played by existential embeddings in model theory.

The classes of monotone and of co-existential maps are not directly related; however we can make the following assertion.

Proposition 5. *Every co-existential map with locally connected range is monotone. And if a compactum fails to be locally connected, there is an ultracopower of it whose associated codiagonal map is not monotone.*

So in the case $X = \mathbb{I}$ and $\mathcal{D} \in \omega^*$, $p_{\mathcal{D}}$ turns out to be a monotone map from $\mathbb{I}_{\mathcal{D}}$ onto $\mathbb{I}$.

Co-Existential Images.

In comparison with Proposition 2, we have:

Proposition 6. *A co-existential image of an ultra-arc is hereditarily unicoherent and of covering dimension one. A metrizable co-existential image is irreducible as well.*

This is because: (1) the first two properties hold for ultra-arcs and are also preserved by co-existential maps; (2) co-existential maps preserve *NOT* being a weak triod; (3) an ultra-arc is never a weak triod; and (4) Sorgenfrey's theorem: *A unicoherent metrizable continuum is irreducible if it is not a (weak) triod.*

While nondegenerate monotone images of ultra-arcs must be decomposable, co-existential images need not be (as we shall see).

Co-existential maps need not preserve irreducibility in general, so we ask the following.

Question 3. *Is a co-existential image of an ultra-arc necessarily irreducible?*

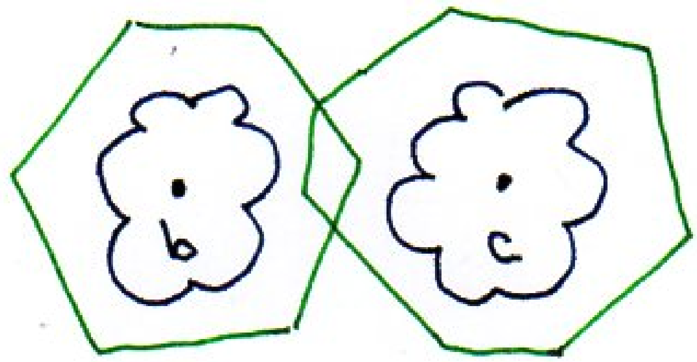
Because generalized arcs are locally connected, Proposition 3 is now a corollary of the following.

Proposition 7. (CH) *Every generalized arc of weight* $\leq \aleph_1$ *is a co-existential (monotone) image of every ultra-arc.*

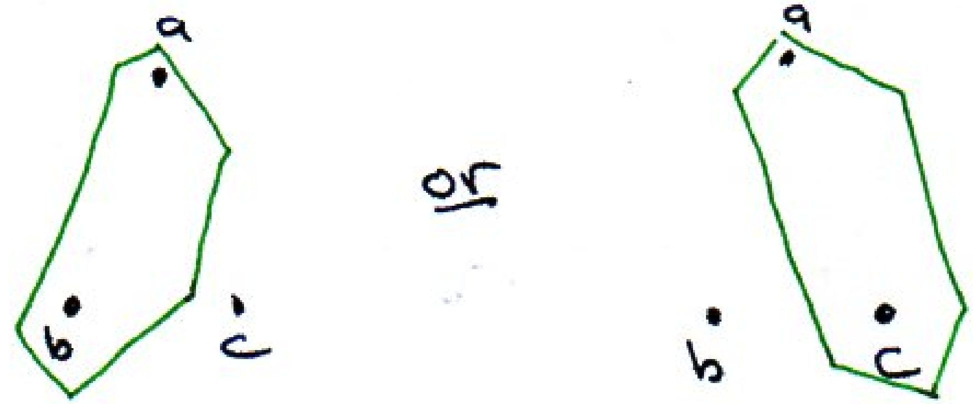
In contrast with Proposition 4, not every hereditarily decomposable co-existential image of an ultra-arc is a generalized arc: we will see that the $\sin(1/x)$-curve is a suitable example.

In Proposition 4, “hereditarily decomposable” may be replaced with “antisymmetric.” This means given any points a, b, c such that $b \neq c$, there is a subcontinuum containing a and exactly one of b, c. The terminology comes from the theory of pre-orders; antisymmetry is a consequence of aposyndesis.

Aposyndesis:



Antisymmetry:



Proposition 8. *Every nondegenerate antisymmetric monotone image of an ultra-arc is a generalized arc.*

When we consider co-existential images, it appears that antisymmetry needs to be strengthened.

Proposition 9. *Every aposyndetic co-existential image of an ultra-arc is a generalized arc.*

In this proposition we may replace “aposyndetic” with “antisymmetric and metrizable.” This uses the fact mentioned above that metrizable co-existential images of ultra-arcs are irreducible.

Question 4. *Is every antisymmetric co-existential image of an ultra-arc a generalized arc?*

Finding interesting images of ultra-arcs is a bit easier with co-existential maps than it is with monotone ones for two reasons.

Reason 1 (whose proof involves an inverse limit argument):

Proposition 10. *A nondegenerate chainable metrizable continuum is a co-existential image of any ultra-arc.*

This includes such continua as: the $\sin(1/x)$-curve; Knaster's bucket handle; and the pseudo-arc. None of these continua are monotone images of any ultra-arc.

Reason 2:

A continuum X is **co-existentially closed** if whenever Y is a continuum and $f : Y \to X$ is a continuous surjection, then f is co-existential.

Fact 1 (more inverse limits). *Every continuum is a continuous image of a co-existentially closed continuum of the same weight.*

Fact 2. *Every co-existentially closed continuum is hereditarily indecomposable, and of covering dimension one.*

Proposition 11. *A co-existentially closed continuum of weight* $\leq \aleph_1$ *is a co-existential image of any ultra-arc.*

This follows from the definition, coupled with the Dow-Hart result (Proposition 1) above. When we add in Facts 1 and 2 we get lots of hereditarily indecomposable metrizable continua which are not chainable (or even of zero span, thanks to recent work of Hoehn-Oversteegen).

So the pseudo-arc is a co-existential image of any ultra-arc for two quite different reasons:

(1) because it's chainable; and

(2) because it's co-existentially closed (Eagle-Goldbring-Vignati).

Some final questions:

Question 5. Is every monotone image of an ultra-arc necessarily a co-existential one?

Question 6. Is a solenoid a co-existential image of an ultra-arc?

Question 7. Is the number of co-existentially closed metrizable continua equal to $2^{\aleph_0}$? (We know it's uncountable.)

THANK YOU!