When Semi-Monotone Implies Monotone

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Introduction.

The basic theme of this talk is the extrinsic description of objects by means of morphisms. One way to do this is to say that all monomorphisms from the object are "special" in some way; the dual of this is to single out epimorphisms to the object.

Injectivity and projectivity are the most familiar to algebraists:

- An abelian group is *injective* (resp., *projective*): every monomorphism from it (resp., every epimorphism to it) is a coretraction (resp., retraction).

- A normal topological space is an *absolute retract*: it is a retract of every normal space in which it is embedded as a closed subset. (Every closed embedding from it is a coretraction).

And in model theory, we have:

- A model of a universal theory is *existentially closed*: whenever it is embedded in another model of the theory, existential statements (with parameters in the small model) which are true in the larger model are true in the smaller model as well. (Every embedding from it is an existential embedding.)

- A weak version of existential closedness for abelian groups, where the existential statements have the form $\exists\, x\ (nx = a)$, is called *absolute purity*.

In this talk our objects are continua (= connected compact Hausdorff spaces), and our morphisms are epimorphisms (a.k.a., continuous surjections).

By a *mapping characterization theorem* we mean a proposition that takes the form:

- *Every epimorphism in mapping class $\mathfrak{F}$ onto Y is also in mapping class $\mathfrak{G}$ iff Y is in continuum class $\mathcal{K}$.*

For convenience, we refer to the "if" and the "only if" directions as the *universal half* and the *existential half*, respectively, of the characterization.

Continuum theory is rich in its capacity to describe interesting mapping classes—see [J. J. Charatonik and W. J. Charatonik, *Continua determined by mappings*, Pub. de L'Institut Math. **67** (2000), 133-144]. The ones we take up today are defined by what the pre-images of subcontinua look like.

Definition. Let $f : X \to Y$ be an epimorphism between continua. Then f is:

- *monotone*: if $f^{-1}[K]$ is a subcontinuum of X for every subcontinuum K of Y; and

- *semi-monotone* if for each subcontinuum K of Y, there is a component C of $f^{-1}[K]$ such that $f[C] = K$ and $f^{-1}[\text{Int}(K)] \subseteq C$.

(Note: The coinage *semi-monotone* first appears in the topological literature in [P. B., *Defining topological properties via interactive mapping classes*, Top. Proc. **34** (2009), 39-45].)

Two more important properties related to monotonicity are given in the following

Definition. An epimorphism $f : X \to Y$ between continua is:

- *confluent* if *each* component of the pre-image of a subcontinuum K of Y maps onto K via f; and

- *weakly confluent* if *some* component of the pre-image of a subontinuum K of Y maps onto K via f.

Here is a schematic of how these properties are implicationally related:

$$\begin{tikzcd} & \text{monotone} \arrow[dl] \arrow[dr] & \\ \text{semi-monotone} \arrow[dr] & & \text{confluent} \arrow[dl] \\ & \text{weakly confluent} \arrow[d] & \\ & \text{epimorphic} & \end{tikzcd}$$

Fourteen of the 25 possible “mapping characterization theorems” merely state the obvious: $\mathcal{K} = \{\text{all continua}\}$ whenever $\mathfrak{F} \subseteq \mathfrak{G}$.

Of the eleven remaining, all but one of those where $\mathfrak{G} = \{\text{monotone maps}\}$ yield $\mathcal{K} = \{\text{degenerate continua}\}$; the lone exception is where $\mathfrak{F} = \{\text{semi-monotone maps}\}$.

Semi-Monotone $\Rightarrow$ Monotone.

Our main result is the following.

Theorem 1. *Every semi-monotone mapping onto Y is also monotone iff Y is locally connected.*

The universal half of every mapping characterization theorem I know of uses standard topological arguments, and—with the exception of this theorem—every existential half takes a $Y \notin \mathcal{K}$ and conjures up a continuum X of the form

$$((Y \times \{0\}) \cup (K \times \{1\}))/\sim,$$

where K is a subcontinuum of Y and $\sim$ identifies a suitably chosen $\langle y, 0\rangle$ with its companion $\langle y, 1\rangle$. The map $f \in \mathfrak{F} \setminus \mathfrak{G}$ is the induced first-coordinate projection.

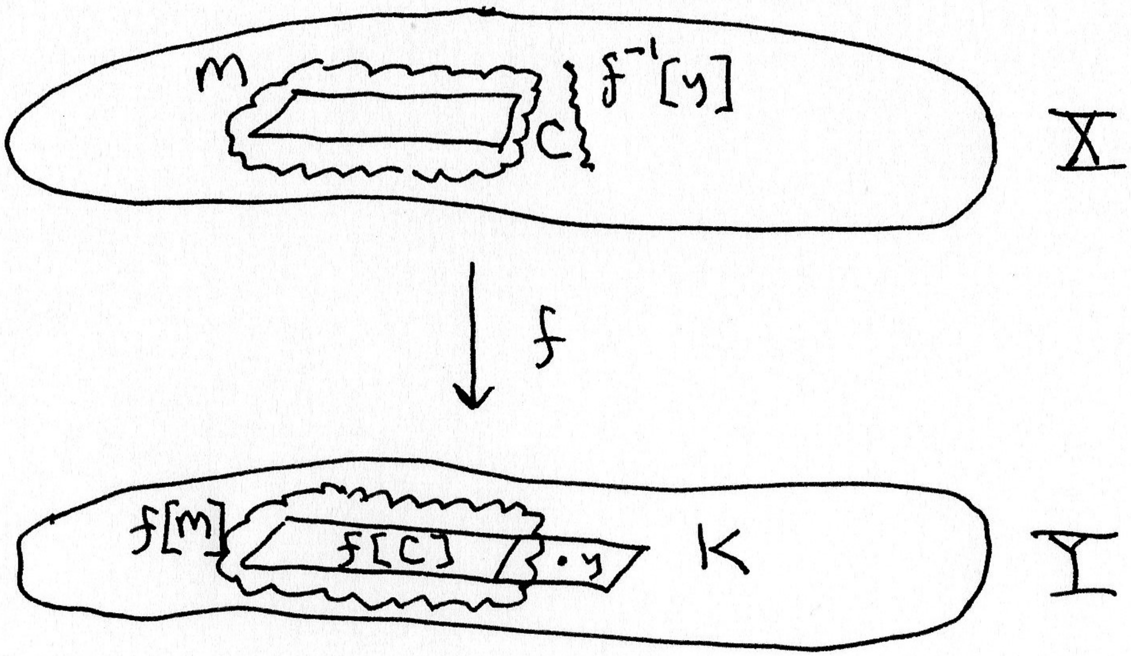
Before proving Theorem 1, let us illustrate a classical mapping characterization theorem.

Theorem 2. (H. Cook, A. Lelek and D. R. Read). *Every weakly confluent mapping onto* Y *is confluent iff* Y *is hereditarily indecomposable.*

Proof of Theorem 2 (Universal Half). Fix continuum Y, and assume there is an epimorphism $f : X \to Y$ that is not confluent. Then there is a subcontinuum K of Y and a component C of $f^{-1}[K]$ such that $f[C]$ is properly contained in K.

Let $y \in K \setminus f[C]$ be fixed. Then C is disjoint from $f^{-1}[y]$; and, by "boundary bumping," there is a subcontinuum M of X such that $C \subseteq M$, $C \neq M$, and $M \cap f^{-1}[y] = \emptyset$.

Thus $f[M]$ is a subcontinuum of Y that intersects K because it contains C. We have $y \in K \setminus f[M]$ because $M \cap f^{-1}[y] = \emptyset$, and we have $f[M] \setminus K \neq \emptyset$ because C is maximally connected in $f^{-1}[K]$. This says that Y is not hereditarily indecomposable. $\square$



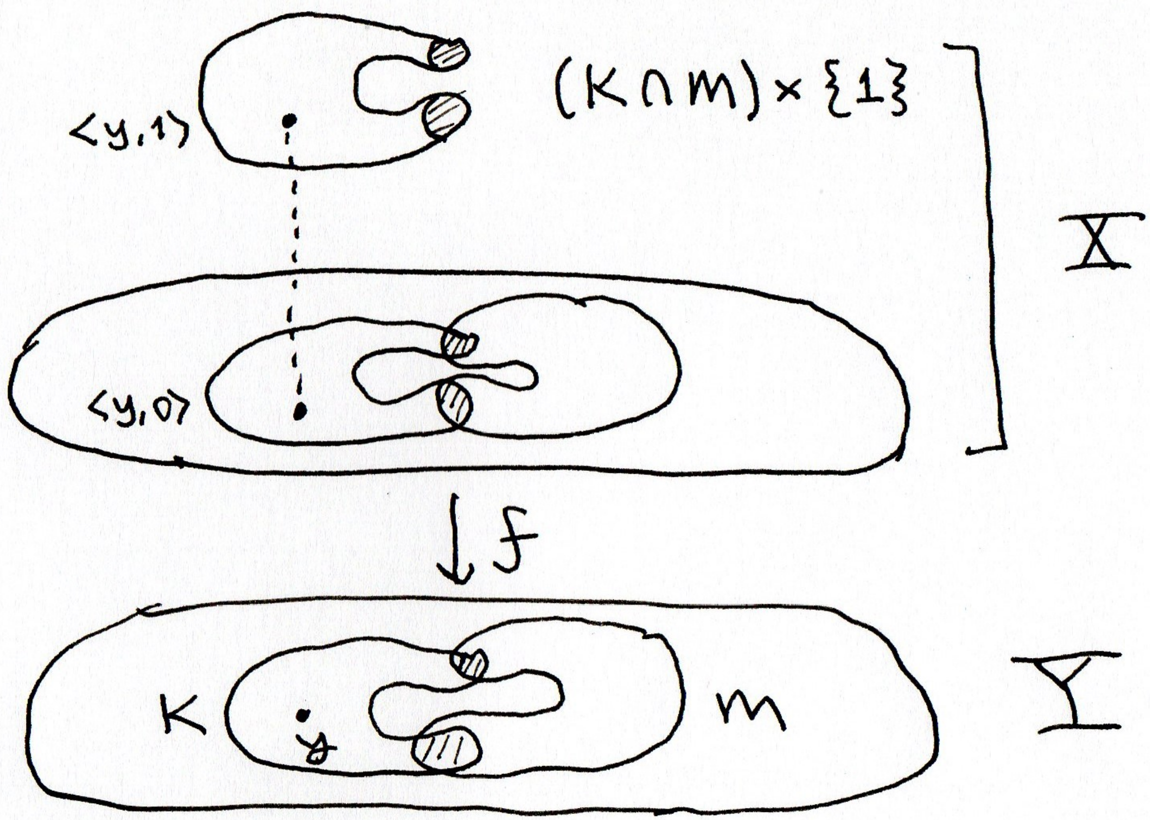
Y has decomposable subcontinuum

$f[M] \cup K$

Proof of Theorem 2 (Existential Half). If continuum Y is not hereditarily indecomposable, then there are subcontinua K and M with $K \cap M$, $K \setminus M$, and $M \setminus K$ all nonempty. Fix $y \in K \setminus M$, and let

$$X = ((Y \times \{0\}) \cup (K \times \{1\}))/\sim,$$

where $\langle y, 0\rangle \sim \langle y, 1\rangle$. With $f : X \to Y$ induced by the standard first-coordinate projection, we see that f is weakly confluent. However, $(K \cap M) \times \{1\}$ contains components of $f^{-1}[M]$ that f does not send onto M; hence f is not confluent. $\square$

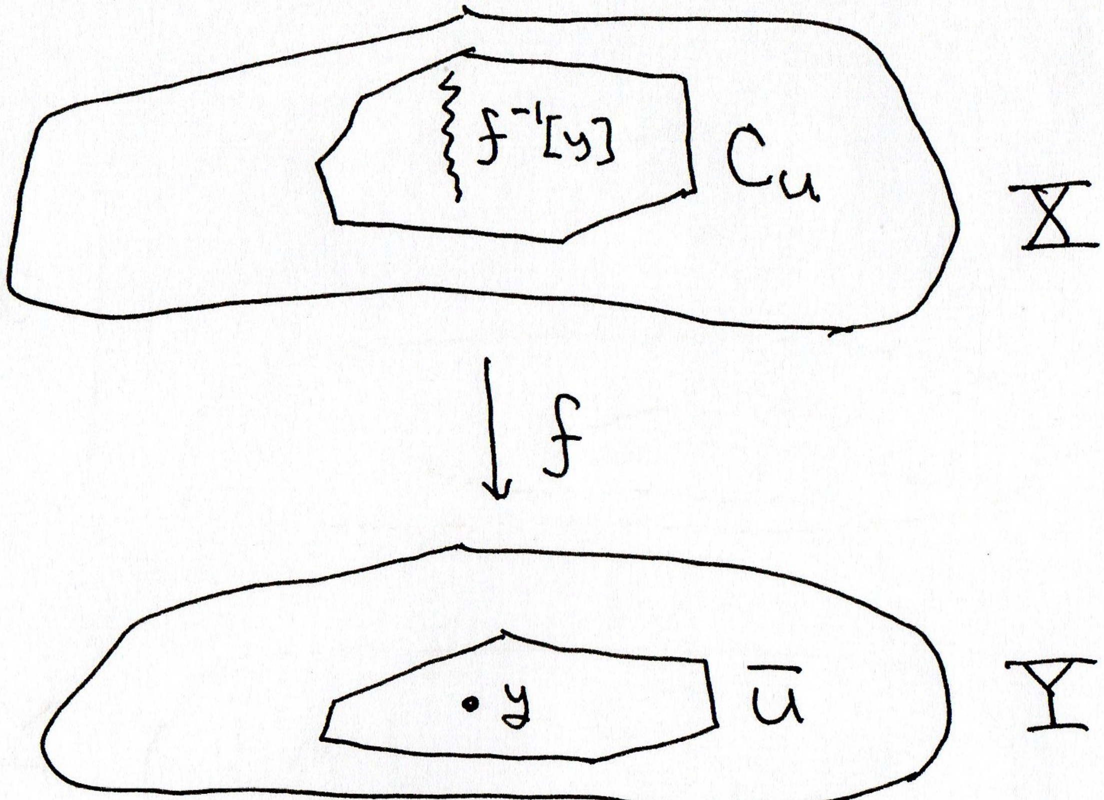


$f[(K \cap M) \times \{1\}] \neq M$

Proof of Theorem 1 (Universal Half). Suppose Y is a locally connected continuum, with $f : X \to Y$ a semi-monotone map. To show f to be monotone, it suffices to prove that $f^{-1}[y]$ is connected for each $y \in Y$.

Indeed, let $\mathcal{B}$ be a base at y, consisting of connected open sets. For each $U \in \mathcal{B}$, semi-monotonicity guarantees a (necessarily unique) subcontinuum C_U of X such that $f[C_U] = \overline{U}$ and $f^{-1}[U] \subseteq C_U$.

Then, because $\bigcap \mathcal{B} = \{y\}$, $\mathcal{C} = \{C_U : U \in \mathcal{B}\}$ is a family of subcontinua of X whose intersection is $f^{-1}[y]$. Moreover, it is easy to show that $\mathcal{C}$ is directed downwards: for if $U, V \in \mathcal{B}$, there is some $W \in \mathcal{B}$ with $\overline{W} \subseteq U \cap V$. Then $C_W \subseteq C_U \cap C_V$. By elementary continuum theory, $f^{-1}[y] = \bigcap \mathcal{C}$ is connected. $\square$



$$f^{-1}[y] = \bigcap \{ C_U : U \in \mathcal{B} \}$$

Proof of Theorem 1 (Existential Half). Suppose Y is a continuum that is not locally connected. Our plan is to create an ultracopower $Y_{\mathcal{D}}$, dually analogous with the ultrapowers from model theory, whose canonical codiagonal epimorphism $p_{\mathcal{D}} : Y_{\mathcal{D}} \to Y$ (dually analogous with the canonical ultrapower monomorphism) is not monotone. Codiagonal maps are always semi-monotone (and much more), so this will give us our result.

Since Y is not locally connected, there is a point $x \in Y$ at which Y is not connected *im kleinen*; i.e., there is an open neighborhood U of x such that for any open neighborhood V of x contained in U, there is some $y \in V$ such that no subcontinuum of U contains both x and y.

Fix neighborhood W of x such that $\overline{W} \subseteq U$, and let $\{V_i : i \in I\}$ be an indexed open neighborhood base for x, consisting of sets in W. By the failure of connectedness *im kleinen* at x, we may pick $y_i \in V_i$ such that y_i and x are not in the same component of $\overline{W}$. Since $\overline{W}$ is a compactum, there is a set H_i that is clopen on $\overline{W}$, contains y_i, and doesn't contain x.

For each $i \in I$, let $i^+ := \{j \in I : V_j \subseteq V_i\}$. Then the collection $\{i^+ : i \in I\}$ satisfies the finite intersection property and is hence contained in an ultrafilter $\mathcal{D}$ on I.

The next step is to form the topological ultracopower $p_{\mathcal{D}} : Y_{\mathcal{D}} \to Y$, and show that $p_{\mathcal{D}}$ is a semi-monotone mapping that is not monotone.

The ultracopower, along with its canonical ("co-elementary") codiagonal epimorphism, is exactly dual to the model-theoretic ultrapower, along with its canonical elementary monomorphism. Ultracopowers of Y may be obtained as Stone spaces of ultrapowers of lattice bases for Y; however a purely topological version of $Y_{\mathcal{D}}$ arises as follows:

Given the diagram

$$\begin{array}{ccc} Y \times I & \xrightarrow{q} & I \\ \downarrow p & & \\ Y & & \end{array}$$

where p and q are the standard projection maps, we apply the Stone-Čech functor to obtain the diagram

$$
\begin{array}{ccc}
\beta(Y \times I) & \xrightarrow{q^\beta} & \beta(I) \\
\downarrow p^\beta & & \\
Y & &
\end{array}
$$

Now the ultrafilter $\mathcal{D}$ is a point in $\beta(I)$, and it turns out that $Y_\mathcal{D}$ is canonically homeomorphic to the pre-image of $\mathcal{D}$ under q^β.

What's more, the restriction $p_\mathcal{D} := p^\beta | Y_\mathcal{D}$ is a semi-monotone map from $Y_\mathcal{D}$ onto Y. (Indeed, for any subcontinuum K of Y, the signal component of $p_\mathcal{D}^{-1}[K]$ is a canonical copy of $K_\mathcal{D}$ in $Y_\mathcal{D}$.)

The idea at this juncture—details omitted—is to use the sets $\{H_i : i \in I\}$ and the points $\{y_i : i \in I\}$ to form a subcompactum $\sum_{\mathcal{D}} H_i$ of $\overline{W}_{\mathcal{D}}$ and a point $\sum_{\mathcal{D}} y_i \in \sum_{\mathcal{D}} H_i$ such that:

- $\sum_{\mathcal{D}} H_i$ is clopen in $\overline{W}_{\mathcal{D}}$;
- $p_{\mathcal{D}}^{-1}[x] \subseteq \overline{W}_{\mathcal{D}}$;
- $x_{\mathcal{D}} \in p_{\mathcal{D}}^{-1}[x] \setminus \sum_{\mathcal{D}} H_i$; and
- $\sum_{\mathcal{D}} y_i \in p_{\mathcal{D}}^{-1}[x] \cap \sum_{\mathcal{D}} H_i$.

These four assertions immediately imply that $p_{\mathcal{D}}^{-1}[x]$ is disconnected; witnessing the fact that $p_{\mathcal{D}}$ is a non-monotone, semi-monotone mapping onto Y. $\square$

Loose Ends.

1. As mentioned before, all three mapping classes $\{$confluent $\Rightarrow$ monotone$\}$,
   $\{$weakly confluent $\Rightarrow$ monotone$\}$, and $\{$epimorphic $\Rightarrow$ monotone$\}$ comprise the class of degenerate continua.

2. $\{$weakly confluent $\Rightarrow$ semi-monotone$\}$ and $\{$confluent $\Rightarrow$ semi-monotone$\}$ both comprise the class of indecomposable continua. (This is easy, given the characterization of indecomposability as the condition that every proper subcontinuum has empty interior.)

3. Of the remaining instances of $\mathfrak{F} \Rightarrow \mathfrak{G}$, $\{$epimorphic $\Rightarrow$ weakly confluent$\}$ has received the most attention in the literature; but the known results concern metrizable continua only. Lelek's designation Class(W) refers to the metrizable members of $\{$epimorphic $\Rightarrow$ weakly confluent$\}$, and Grispolakis-Tymchatyn (1978) have provided interesting characterizations in terms of hyperspace notions.

4. Once $\{$epimorphic $\Rightarrow$ weakly confluent$\}$ is known, $\{$epimorphic $\Rightarrow$ semi-monotone$\}$ simply consists of the members of $\{$epimorphic $\Rightarrow$ weakly confluent$\}$ that are indecomposable (easy, given (2) above). The two classes are distinct because all arc-like continua are well known to be in Class(W). In particular, arcs are in $\{$epimorphic $\Rightarrow$ weakly confluent$\}$, but not in $\{$epimorphic $\Rightarrow$ semi-monotone$\}$.

5. By the proof of Theorem 2, we see that {epimorphic $\Rightarrow$ confluent} also comprises the hereditarily indecomposable continua. From this, and (4) above, we have {epimorphic $\Rightarrow$ confluent} contained in {epimorphic $\Rightarrow$ semi-monotone}. The two classes are distinct because the Knaster buckethandle is arc-like and indecomposable, and so is in the latter. It is not hereditarily indecomposable, hence it is not in the former.

6. This leaves the mapping class {semi-monotone $\Rightarrow$ confluent}, which clearly contains both {semi-monotone $\Rightarrow$ monotone} and {weakly confluent $\Rightarrow$ confluent}. So a continuum is in {semi-monotone $\Rightarrow$ confluent} if it is either locally connected or hereditarily indecomposable. We don't know whether the containment is proper.

7. The question naturally arises whether a non-locally connected metrizable continuum is the image of a *metrizable* continuum under a semi-monotone mapping that is not monotone. The answer is yes: more generally, if Y is not locally connected and $p_{\mathcal{D}} : Y_{\mathcal{D}} \to Y$ is constructed as in the proof of Theorem 1, then—by means of the Löwenheim-Skolem theorem from model theory—one may obtain a commutative diagram

$$
\begin{array}{ccc}
Y_{\mathcal{D}} & \xrightarrow{g} & X \\
p_{\mathcal{D}} \downarrow & \swarrow & f \\
Y & &
\end{array}
$$

where f is "enough like" $p_{\mathcal{D}}$ to be semi-monotone but not monotone, and X has the same weight as Y.

8. In the diagram above we start with Y and $\mathcal{D}$, and construct X, f, and g. If, on the other hand, we're given $f : X \to Y$ and are able to find $\mathcal{D}$ and g making the diagram commute, this situation is exactly dual to the ultrapower characterization of existential embeddings in model theory. In this situation, we call f a *co-existential map*. Co-existential maps are semi-monotone, but not necessarily confluent [P. B., *Not every co-existential map is confluent*, Houston J. Math. **36** (4) (2010), 1233-1242]. And in view of the characterization of {semi-monotone $\Rightarrow$ monotone} in Theorem 1, it is clear that every co-existential map onto Y is monotone iff Y is locally connected.

So we end with the following question:

What is a suitable ("intrinsically defined") $\mathcal{K}$ to characterize $\{$epimorphic $\Rightarrow$ co-existential$\}$, the class of *co-existentially closed* continua?

So far, all we know is that:

- $\mathcal{K}$ is contained within the class of hereditarily indecomposable continua of covering dimension 1; and

- If X is any continuum, then X is a continuous image of a co-existentially closed continuum of the same weight as X.

THANK YOU!