Large Continua via Ultracoproducts

Paul Bankston, Marquette University

Analytic Topology Seminar,
Mathematical Institute, Oxford University
06 March, 2019.

A **continuum** is a connected compactum (=compact Hausdorff space).

In this talk we concentrate on constructing "large" continua; in particular we consider results of the following form:

For each continuum X there is a "special" continuum Y and a "special" surjective mapping $f : Y \to X$.

Here are some classic examples.

Theorem 1A (G. R. Gordh, Jr., 1972). *The map* $f : Y \to X$ *may be chosen so that* Y *is indecomposable, with* ≥ 2 *composants, and* f *is a retraction.*

Theorem 1B (Michel Smith, 1976). *For any cardinal* α, $f : Y \to X$ *may be chosen so that* Y *is indecomposable, with* $\geq \alpha$ *composants, and* f *is a retraction.*

Theorem 1C (Smith, 1980). *The map* $f : Y \to X$ *may be chosen so that* Y *is indecomposable, with exactly one composant (or exactly two composants), and* f *is a retraction.*

Some terminology re Theorems 1A, 1B, 1C:

- A continuum Y is **decomposable** if it is the union of two proper subcontinua; **indecomposable** otherwise.

- A **composant** of Y is the union of all proper subcontinua of Y that contain a given point of Y. (Any two composants of an indecomposable continuum are disjoint.)

- $f : Y \to X$ is a **retraction** if there is a continuous $g : X \to Y$ such that $f \circ g : X \to X$ is the identity map.

In a somewhat different vein, we also have the following.

Theorem 2 (K. P. Hart, Jan van Mill, Roman Pol, 2000). *The map* $f : Y \to X$ *may be chosen so that* Y *is hereditarily indecomposable, of covering dimension one, of weight* $= w(X)$*, and* f *is weakly confluent.*

(Note: $w(X)$ is the least *infinite* cardinal γ such that X has an open base of cardinality $\leq \gamma$. Thus even degenerate continua have weight $\aleph_0$.)

In this talk we concentrate on possible variants of Theorem 2.

Some terminology re Theorem 2:

- Y is **hereditarily indecomposable** if no nondegenerate subcontinuum is decomposable.

- Y is of **covering dimension one** if Y is nondegenerate, and every open cover of Y has a finite open refinement so that no point of Y lies in more than two members of the refinement.

- $f : Y \to X$ is **weakly confluent** if each subcontinuum of X is the image, under f, of a subcontinuum of Y. (Retraction maps are weakly confluent.)

Theorem 1A uses a clever *ad hoc* construction, but both its successors (Theorems 1B and 1C) use the limit of an inverse system of uncountably many indecomposable continua.

Theorem 2 uses a mix of topological and model-theoretic methods.

Notice we've got a stronger condition on Y, but a weaker one on $f : Y \to X$. We can't strengthen *weakly confluent* to *retraction* or *confluent* because these function properties preserve hereditary indecomposability.

In this talk we use the ultracoproduct construction to tweak Theorem 2 so that it looks more like Theorem 1B.

This method is apparantly useless for placing reasonable upper bounds on the composant number, however, and so is not appropriate for obtaining an analogue of Theorem 1C.

We do not know whether you can place a reasonable upper bound on the number of composants of Y in Theorem 2. Indeed, it is unknown whether there exists a nondegenerate hereditarily indecomposable continuum with just one composant (although M. Smith has shown it can have exactly two).

Theorem 3. *For any cardinal* α, $f : Y \to X$ *may be chosen so that* Y *is hereditarily indecomposable, of covering dimension one, with* $\geq \alpha$ *composants, and* f *is weakly confluent.*

The proof of Theorem 3 uses Theorem 2 as a lemma. The first step is to fix $g : Z \to X$ so that Z is hereditarily indecomposable, of covering dimension one, and g is weakly confluent. (We don't really care about $w(Z)$ at this point.) Our continuum Y is going to be an ultracopower of Z.

And now for a quick review of ultracopwers.

Recall that if Z is any compactum, I is a discrete infinite set, and $\mathcal{D}$ is an ultrafilter on I (i.e., $\mathcal{D} \in \beta(I)$), then the $\mathcal{D}$-ultracopower $Z_{\mathcal{D}}$ is obtained as follows:

• Let $p : Z \times I \to Z$ and $q : Z \times I \to I$ be the coordinate projections.

• Apply the Stone-Čech functor to obtain $p^{\beta} : \beta(Z \times I) \to Z$ and $q^{\beta} : \beta(Z \times I) \to \beta(I)$.

• $Z_{\mathcal{D}}$ is defined to be the inverse image, under q^{β}, of the point $\mathcal{D} \in \beta(I)$. The mapping $p_{\mathcal{D}} : Z_{\mathcal{D}} \to Z$ is the restriction of p^{β} to $Z_{\mathcal{D}} \subseteq \beta(Z \times I)$; it is a continuous surjection known as the *$\mathcal{D}$-codiagonal map*, and is well known to be weakly confluent.

It is a basic fact that $Z_{\mathcal{D}}$ is a continuum (totally disconnected compactum) iff the same is true of Z.

Lemma 1. *Let Z be a continuum, $\mathcal{D}$ an ultrafilter on discrete infinite set I.*
(1) $Z_{\mathcal{D}}$ and Z share the same covering dimension.
(2) $Z_{\mathcal{D}}$ is (hereditarily) indecomposable iff the same is true of Z.

The following uses some model theory; i.e., the Löwenheim-Skolem theorem and the Keisler-Shelah isomorphism theorem.

Lemma 2. *Let Z be a continuum. Then there is a metrizable continuum M and an ultrafilter $\mathcal{D}$ such that the ultracopowers $Z_{\mathcal{D}}$ and $M_{\mathcal{D}}$ are homeomorphic.*

Remark re Lemma 2. In the original GCH-fueled version, due to H.J. Keisler, any good ultrafilter on an index set of cardinality $\geq w(X)$ will do. In the GCH-free version, due to S. Shelah, $\mathcal{D}$ may be constructed by transfinite induction on any index set of cardinality $\geq 2^{w(X)}$.

Lemma 3. *Let Z be an indecomposable continuum. Then there is an ultrafilter $\mathcal{D}$ such that $Z_{\mathcal{D}}$ is an indecomposable continuum with infinitely many (pairwise disjoint) composants.*

By Lemmas 1 and 2, $Z_{\mathcal{D}} \simeq M_{\mathcal{D}}$ for some metrizable indecomposable continuum M. It is a well-known fact that any nondegenerate indecomposable metrizable continuum has infinitely many (indeed $2^{\aleph_0}$) composants. If $T \subseteq M$ is an infinite *transversal* for the "being-in-the-same-composant" equivalence relation for M (i.e., T intersects each composant of M in at most one point), then the $\mathcal{D}$-ultrapower $T^{\mathcal{D}}$ is infinite, and with a little work, can be shown to be a transversal for $M_{\mathcal{D}}$. Thus $Z_{\mathcal{D}}$ has infinitely many composants.

Recall that an ultrafilter $\mathcal{D} \in \beta(I)$ is **regular** if there is a subfamily $\mathcal{S} \subseteq \mathcal{D}$, of cardinality $|I|$, such that each $i \in I$ lies in just finitely many sets in $\mathcal{S}$.

The following is a classic result, much used in model theory.

Lemma 4. *Suppose $\mathcal{D}$ is a regular ultrafilter on a set of infinite cardinality λ. If S is an infinite set, then the cardinality of the ultrapower $S^{\mathcal{D}}$ is $|S|^{\lambda}$.*

So recapping: Let X be a nondegenerate continuum. Using Theorem 2, fix $g : Z \to X$ so that Z is hereditarily indecomposable, of covering dimension one, and g is weakly confluent.

From Lemmas 1, 3, we may find an ultrafilter $\mathcal{E}$ so that $Z_{\mathcal{E}}$ is hereditarily indecomposable, of covering dimension one, with infinitely many composants. Let $T \subseteq Z_{\mathcal{E}}$ be a countably infinite transversal. Given our infinite cardinal α, we let $\mathcal{D}$ be a regular ultrafilter on a set of cardinality α, and set $Y = (Z_{\mathcal{E}})_{\mathcal{D}}$. Then $T^{\mathcal{D}} \subseteq Y$ has cardinality 2^{α}, by Lemma 4, and is also a transversal. (BTW, Y is an ultracopower of X via the *Fubini product* $\mathcal{D} \cdot \mathcal{E}$.) Letting $p_{\mathcal{D}} : Y \to Z_{\mathcal{E}}$ and $p_{\mathcal{E}} : Z_{\mathcal{E}} \to Z$ be the respective codiagonal maps, the map $f = g \circ p_{\mathcal{E}} \circ p_{\mathcal{D}} : Y \to X$ is weakly confluent.

We now want to obtain a version of Theorem 2 where the number of composants of Y is "large" in two different ways: not only does it have an arbitrarily large number of composants, but it has as many composants as possible, for its size.

Let us call a continuum X **replete** if it has as many composants as points; i.e., it has a transversal T such that $|T| = |X|$.

Since decomposable continua contain either one or three composants, any replete continuum must be indecomposable (including the degenerate case).

It is well known (S. Mazurkiewicz, 1927) that all indecomposable metrizable continua are replete, but (D. Bellamy, 1978) nonmetrizable indecomposable continua can have exactly one composant (or two composants).

As for axiom-sensitivity, the Stone-Čech remainder $[0, \infty)^*$ of the half-line—well known to be indecomposable—is replete when the continuum hypothesis (CH) is assumed (M. E. Rudin, 1970) and has just one composant under the near coherence of filters axiom (J. Mioduszewski, 1974). (Given two ultrafilters $\mathcal{D}, \mathcal{E} \in \beta(\omega)$, there is a nondecreasing $f : \omega \to \omega$ such that $f^\beta(\mathcal{D}) = f^\beta(\mathcal{E})$.)

Our argument for Theorem 3 doesn't ensure that Y has enough composants to be replete: indeed, even assuming that $\alpha \geq w(Z_{\mathcal{E}})$, we obtain $w(Y) \leq 2^{\alpha}$; so the smallest upper bound we can get for $|Y|$ is $2^{(2^{\alpha})}$ (while Y is only guaranteed to have 2^{α} composants).

This brings us to a totally different approach, where the generalized continuum hypothesis (GCH) seems to play an essential role. (It's useless in the previous argument, as it stands.)

For an infinite cardinal α, let GCH$_\alpha$ be the statement, $2^\alpha = \alpha^+$; so that the CH is GCH$_{\aleph_0}$ and the GCH is $\forall\alpha$GCH$_\alpha$. Each assertion "GCH$_\alpha$" is called an *instance* of the GCH.

Theorem 4. (GCH) *(1) The map $f : Y \to X$ may be chosen so that Y is hereditarily indecomposable, of covering dimension one, replete, of weight $\leq 2^{w(X)}$, and f is weakly confluent.*
(GCH) *(2) For any cardinal α, $f : Y \to X$ may be chosen so that Y is hereditarily indecomposable, of covering dimension one, replete, with $\geq \alpha$ composants, and f is weakly confluent.*

Our proof sketch requires a discussion of ultracopowers using good ultrafilters.

An ultrafilter $\mathcal{D}$ on an infinite set I is **good** if: (1) $\mathcal{D}$ is countably incomplete; and (2) if $f : \wp_\omega(I) \to \mathcal{D}$ is *monotone* (i.e., $s \subseteq t \Rightarrow f(s) \supseteq f(t)$), there is a $g : \wp_\omega(I) \to \mathcal{D}$ such that $g(s) \subseteq f(s)$, for $s \in \wp_\omega(I)$, and g is *multiplicative* (i.e., $g(s \cup t) = g(s) \cap g(t)$, for $s, t \in \wp_\omega(I)$).

Good ultrafilters are regular, which are in turn countably incomplete, hence free. When the index set is countable, all four concepts coincide. Keisler originally conceived of the notion of goodness in order to achieve high degrees of saturatedness in ultraproducts, and proved the existence of good ultrafilters using the GCH. K. Kunen later proved that, in ZFC, each set of α elements supports $2^{(2^\alpha)}$ distinct good ultrafilters.

Now for some key cardinality properties of spaces.

For an infinite cardinal α, a point a in a topological space X is a **P$_\alpha$-point** if whenever $\mathcal{U}$ is a family of $\leq \alpha$ open neighborhoods of a, there is an open set V with
$a \in V \subseteq \bigcap \mathcal{U}$.

X is a **P$_\alpha$-space** if each point of X is a P$_\alpha$-point. This is equivalent to saying that intersections of $\leq \alpha$ open sets are open.

The space X is an **almost P$_\alpha$-space** if whenever $\mathcal{U}$ is a family of $\leq \alpha$ open sets and $\bigcap \mathcal{U} \neq \emptyset$, it follows that $\bigcap \mathcal{U}$ has nonempty interior.

Lemma 5. *Let Z be a compactum, with $\mathcal{D}$ a regular ultrafilter on a set of infinite cardinality α. Then $Z_{\mathcal{D}}$ has a dense set of P_α-points. If $\mathcal{D}$ is also good, then $Z_{\mathcal{D}}$ is an almost P_α-space.*

A space is α-**Baire** if the intersection of any family of $\leq \alpha$ dense open sets is dense. By the Baire category theorem, all compacta are $\aleph_0$-Baire.

Lemma 6. *Every almost P_α-compactum is α^+-Baire.*

The proof of this is quite like that of the classic Baire category theorem.

The following is well known for metrizable continua; the proof in that case readily generalizes.

Lemma 7. *Let X be a continuum of weight $\leq \alpha$. Then each composant of X is a union of $\leq \alpha$ proper subcontinua of X.*

To see this, let $\kappa(a)$ be the composant of $a \in X$, and let $\mathcal{U}$ be an open base for $X \setminus \{a\}$, consisting of $\leq \alpha$ sets. For each $U \in \mathcal{U}$, let C_U be the component of $X \setminus U$ containing a. Then each C_U is a proper subcontinuum of X containing a; so $\bigcup\{C_U : U \in \mathcal{U}\} \subseteq \kappa(a)$. On the other hand, if $x \in \kappa(a)$, let K be a proper subcontinuum of X containing $\{a, x\}$. Since $\mathcal{U}$ is an open base for $X \setminus \{a\}$, there is some $U \in \mathcal{U}$ with $U \subseteq X \setminus K$. Both K and C_U are subcontinua of $X \setminus U$ containing a; hence $K \subseteq C_U$. Thus

$$\kappa(a) = \bigcup\{C_U : U \in \mathcal{U}\},$$

a union of $\leq \alpha$ proper subcontinua of X.

To complete the proof of Theorem 4, let X be an arbitrary continuum, As with the proof of Theorem 3, use Theorem 2 to obtain a weakly confluent $g : Z \to X$, where Z is hereditarily indecomposable, of covering dimension one, and of weight $\gamma = w(X)$.

Let $\mathcal{D}$ be a good ultrafilter on a set of cardinality γ, and let $Y = Z_{\mathcal{D}}$, with

$$f = g \circ p_{\mathcal{D}} : Y \to X.$$

Then Y is hereditarily indecomposable, of covering dimension one, f is weakly confluent, and $w(Y) \leq 2^\gamma$. (Actually equality holds because $\mathcal{D}$ is regular.)

Since the weight of Y is at most 2^γ, we have $|Y| \leq 2^{(2^\gamma)}$.

Now assume GCH_γ, and—for the sake of contradiction—that Y has $\leq \gamma^+ = 2^\gamma$ composants. Because Y has weight $\leq \gamma^+$, we apply Lemma 7 and conclude that Y is a union of $\leq \gamma^+$ proper subcontinua. Each proper subcontinuum of an indecomposable continuum is nowhere dense; and, by Lemma 6, Y is γ^+-Baire. This is our contradiction; hence we infer that Y has at least γ^{++} composants.

At this point we know $|Y| \leq 2^{(\gamma^+)}$ only; so we seem to need another instance of the GCH, namely GCH_{γ^+}. Then $|Y| \leq \gamma^{++}$, and is therefore replete.

This proves Theorem 4 (1). To obtain (2), suppose we're given α. WLOG, we may assume $\alpha \geq w(X)$ and take our index set to have cardinality α. Then invoke $\mathsf{GCH}_\alpha \wedge \mathsf{GCH}_{\alpha^+}$.

Parting remark: If Z is an indecomposable metrizable continuum and $\mathcal{D}$ is a free ultrafilter on a countable set, then $\mathcal{D}$ is also good; hence $Y = Z_{\mathcal{D}}$ is $\aleph_1$-Baire. Thus, assuming the CH, Y has at least $\aleph_2$ composants, while having cardinality $\leq 2^{\aleph_1}$. So in order to get Y to be replete, we seem to need $2^{\aleph_1} = \aleph_2$ also; i.e., more than just the CH. In 1970, M. E. Rudin proved that $[0,1)^*$ has $2^{\aleph_1}$ composants using the CH alone; so perhaps we're missing an argument that possibly gets us $2^{(\gamma^+)}$ composants in the proof of Theorem 4, under the assumption GCH$_\gamma$.

THANK YOU!