On Sorting the Points of a Continuum

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0. Overview.

• Strong noncut $\Rightarrow$ distal $\Rightarrow$ shore $\Rightarrow$ noncut, for Hausdorff continua; all implications are proper.

• They're all the same for aposyndetic Hausdorff continua.

• Noncut points exist for all T_1 continua.

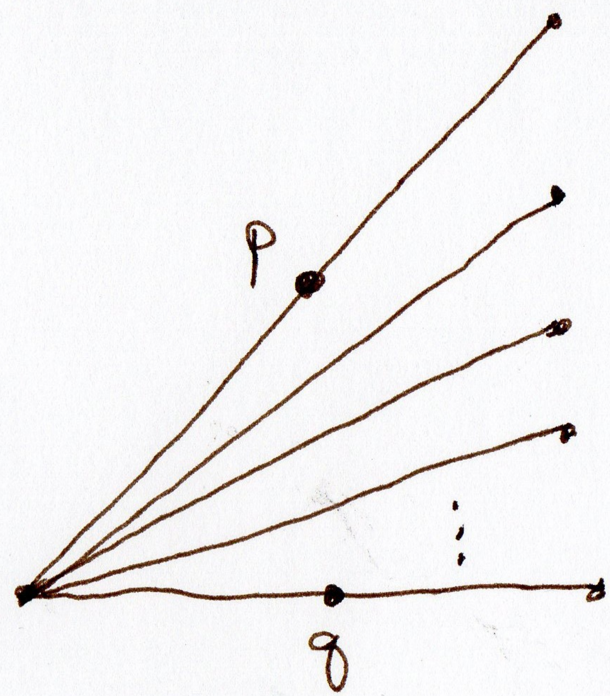
• Shore points exist for all separable Hausdorff continua.

• Strong noncut points don't exist at all for nondegenerate metrizable continua that are indecomposable.

1. Moore's Theorem.

• A **continuum** is a topological space that is both connected and compact. A point c of a continuum X is a **non-cut point** if $X \setminus \{c\}$ is connected; otherwise c is a **cut point** of X.

• A subset of a space is a **subcontinuum** if it is a continuum in its subspace topology; a continuum is **nondegenerate** if it has more than one point.



p is A cutpoint ; q is non-cut

• Theorem 1.1 (R. L. Moore, 1920). *Every nondegenerate metrizable continuum has at least two non-cut points.*

• Historical Aside. Moore's 1920 result is actually a bit stronger than Theorem 1.1: *If a nondegenerate metrizable continuum contains no more than two non-cut points, then it must be an arc.* In 1921 S. Mazurkiewicz published a somewhat weaker version of Theorem 1.1, and in 1923 Moore presented Theorem 1.1 more in its present form. This, no doubt, firmily established his own priority.

The purpose of this talk is to survey some of the work that has grown out of Moore's theorem in the 95 years since its publication. The first major upgrade took almost a half-century to appear.

2. Whyburn's Theorem.

• Theorem 2.1 (G. T. Whyburn, 1968). *Every nondegenerate* T_1 *continuum has at least two non-cut points.*

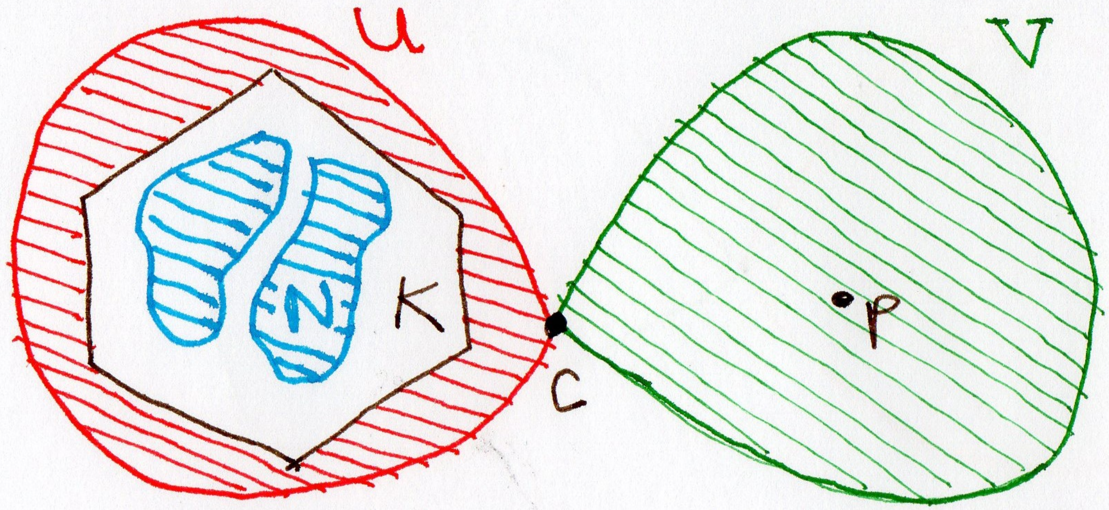
Note that any infinite set with the cofinite topology is a T_1 continuum. In this case it's easy to see that every point is non-cut.

A continuum X is **irreducible** about a subset S if no proper subcontinuum of X contains S.

Whyburn, by judicious use of Zorn's lemma, proved that if c is a cut point of X and $\langle U, V\rangle$ is a disconnection of $X \setminus \{c\}$ into disjoint nonempty open sets (open in X because of T_1), then each of U and V contains a non-cut point of X. As a consequence, we have

• Corollary 2.2. *A* T_1 *continuum is irreducible about its set of non-cut points.*

Proof. Supposing, WLOG, X to be nondegenerate, let N be the set of non-cut points of X, with K a proper subcontinuum of X containing N. Let $c \in X \setminus K$. Then c is a cut point of X; hence there is a disconnection $\langle U, V \rangle$ of $X \setminus \{c\}$. But $K \subseteq U \cup V$ can't intersect both U and V because of being connected; say $K \cap V = \emptyset$. Whyburn's theorem tells us there is a non-cut point in V, a contradiction. Hence no proper subcontinuum of X can contain all the non-cut points of X $\square$



p is a non-cut point lying outside N, a contradiction.

3. Further Developments I: Shore Points.

In their study of dendroids and dendrites, I. Puga-Espinosa et al (1990s) introduced the notion of *shore point*; and in 2014, R. Leonel took the study of shore points into the broader realm of metrizable continua.

• Let X be a metric continuum (i.e., a particular metric is specified). $p \in X$ is a **shore point** if for any $\epsilon > 0$ there is a subcontinuum $K \subseteq X \setminus \{p\}$ which is ϵ-close to X, relative to the Hausdorff metric on the hyperspace of subcontinua of X.

But we can broaden further: Because, for metric continua, the Hausdorff metric gives rise to the Vietoris topology, we quickly have the following, which allows the definition of *shore point* to make sense for any topological space.

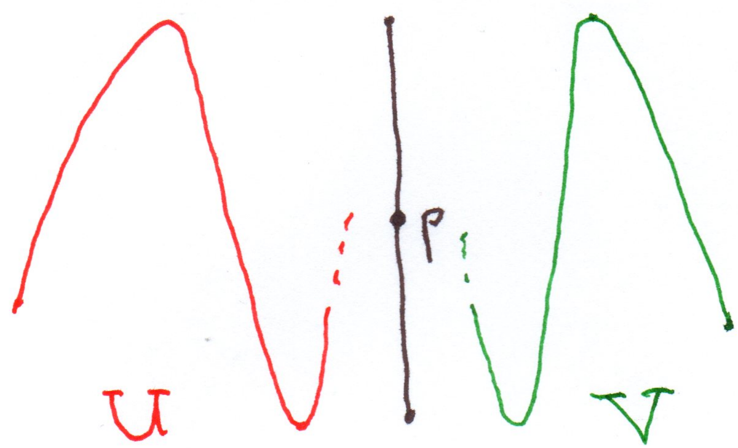
• Proposition 3.1. *In a metric continuum* X, p *is a shore point of* X *iff whenever* $\mathcal{U}$ *is a finite family of nonempty open subsets of* X, *there is a subcontinuum* $K \subseteq X \setminus \{p\}$ *intersecting each set in* $\mathcal{U}$.

• Proposition 3.2. *A shore point of a* T_1 *continuum is a non-cut point; the converse fails in general for metrizable continua.*

Proof of Proposition 3.2. If $c \in X$ is a cut point, we have a disconnection $\langle U, V \rangle$ of $X \setminus \{c\}$. U and V are both open in X because $\{c\}$ is closed. No connected subset of $X \setminus \{c\}$ can intersect both U and V; hence c cannot be a shore point of X.

An example of a metrizable continuum with a a non-cut point that is not a shore point is depicted on the next slide.

$\square$

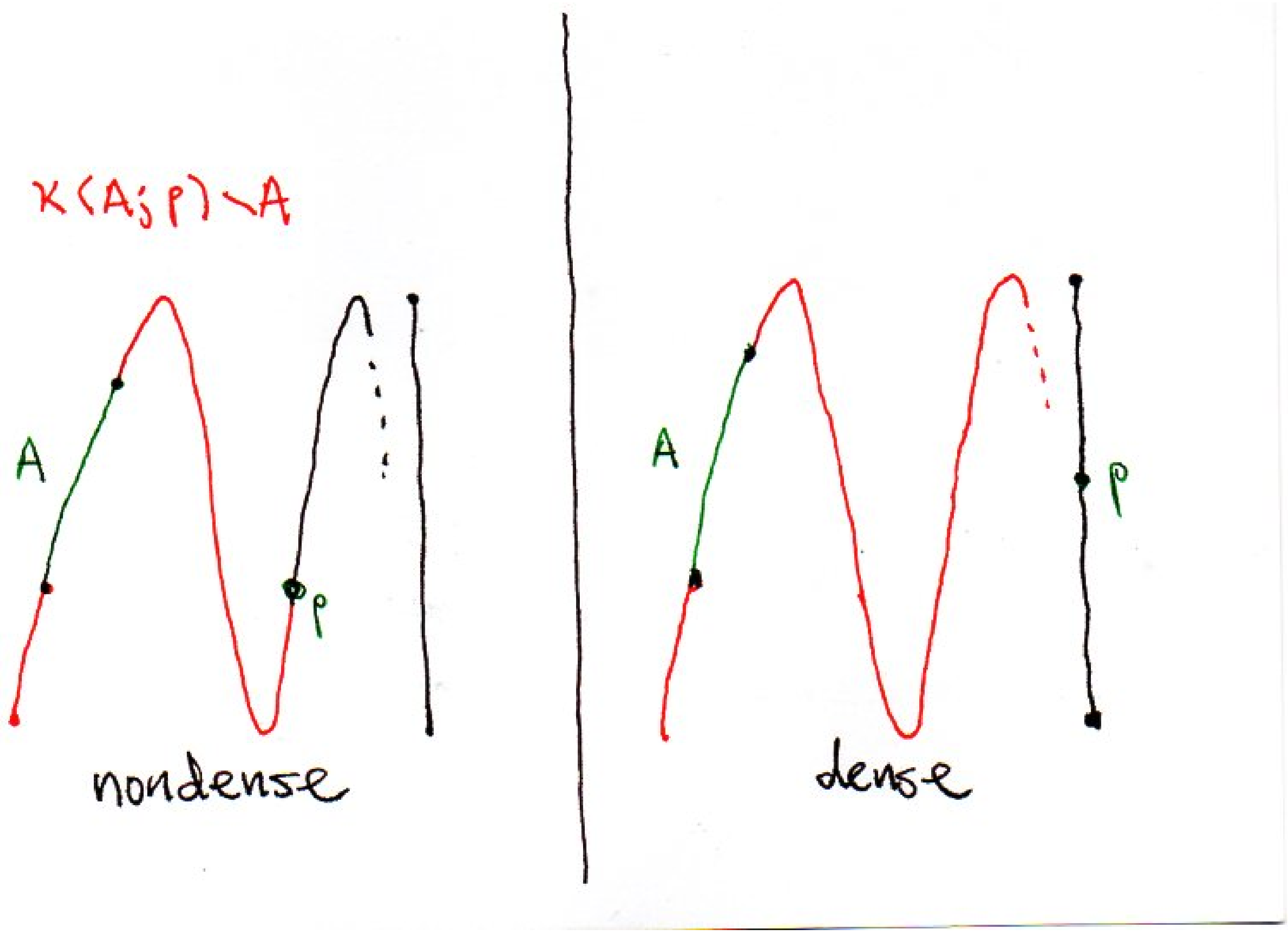


p is a noncut point that isn't a shore point.

Leonel improved on Moore's Theorem 1.1 by showing that every nondegenerate metrizable continuum has at least two shore points. To do this, she employed an interesting 1948 result of R. H. Bing. First some notation:

- If A and P are subsets of X, denote by $\kappa(A; P)$ the **relative composant**, consisting of the union of all proper subcontinua of X that contain A and are disjoint from P.

Note that if $A = \{a\}$, then $\kappa(\{a\}; \emptyset)$ is the (usual) **composant** $\kappa(a)$ of X containing the point a. If X is a Hausdorff continuum and $a \in X$, $\kappa(a)$ is well known to be dense in X; the same argument (i.e., "boundary bumping") shows $\kappa(A) := \kappa(A; \emptyset)$ to be dense whenever A is a proper subcontinuum of X. $\kappa(A; P)$ is frequently nondense, however.

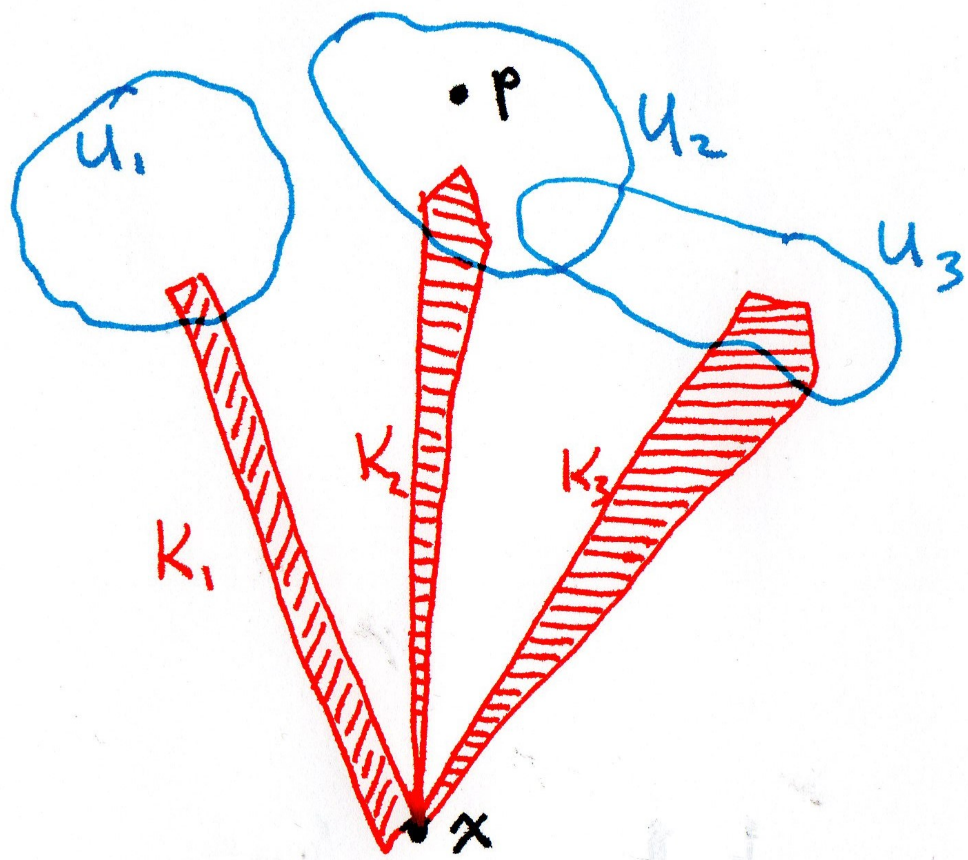

K(A;p) \ A
A
p
nondense
A
p
dense

• Theorem 3.3 (R. H. Bing, 1948). *If X is a metrizable continuum and A is a proper subcontinuum of X, then there exists a point $p \in X$ with $\kappa(A;p) := \kappa(A;\{p\})$ dense in X.*

• Corollary 3.4 (R. Leonel, 2014). *Each nondegenerate metrizable continuum contains at least two shore points.*

Proof. Pick $x \in X$ arbitrary; by Bing's theorem 3.3, pick $p \in X$ with $\kappa(x;p)$ dense in X. If $U_1, \ldots, U_n$ are nonempty open sets, use density to find subcontinua $K_1, \ldots, K_n$ such that: for each $1 \leq i \leq n$, K_i is a subcontinuum that contains x, doesn't contain p, and intersects U_i. Then $K = K_1 \cup \cdots \cup K_n$ is a subcontinuum that doesn't contain p, and which intersects each U_i, $1 \leq i \leq n$.

Hence we have one shore point $p \in X$. Now use Bing's theorem again to find $q \in X$ such that $\kappa(p;q)$ is dense in X. $\square$



$K_1 \cup K_2 \cup K_3$ intersects each of U_1, U_2, U_3.

Bing's theorem 3.3 actually shows more.

• Corollary 3.5. *A metrizable continuum is irreducible about its set of shore points.*

Proof. WLOG, let A be a proper subcontinuum of X. By Bing's theorem, there is a point $p \in X$ with $\kappa(A;p)$ dense in X. By a simple argument similar to the above, we see that p is a shore point; hence no proper subcontinuum can contain all shore points of a metrizable continuum. $\square$

What is significant in its absence is an analogue of Whyburn's Theorem 2.1. We restrict ourselves to the Hausdorff case and state the following.

• Open Problem 3.6. *Does Bing's Theorem 3.3 work for Hausdorff continua? (I.e., if X is a not-necessarily-metrizable Hausdorff continuum and A is a proper subcontinuum, is there a point $p \in X$ with $\kappa(A; p)$ dense in X?)*

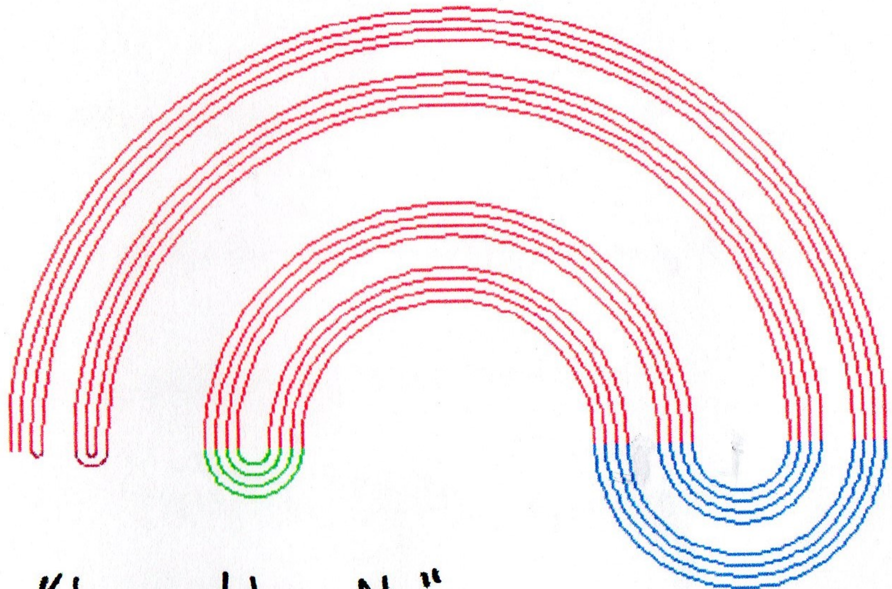
If so, then one can show that every nondegenarate Hausdorff continuum is irreducible about its set of shore points.

4. Further Developments II: A Reduction Process.

• Define a Hausdorff continuum X to be **coastal** at a proper subcontinuum $A \subseteq X$ if $\kappa(A; p)$ is dense in X for some $p \in X$.

Bing showed each metrizable continuum to be coastal at each of its proper subcontinua; we do not know whether this is still true for an arbitrary Hausdorff continuum.

• Remark. Two points of a Hausdorff continuum X lie in the same composant of X iff there is a proper subcontinuum containing both points. A continuum is **indecomposable** if it is not the union of two proper subcontinua; i.e., iff the relation "being in the same composant" is an equivalence relation. Nondegenerate indecomposable *metrizable* continua are well known to possess $\mathfrak{c} := 2^{\aleph_0}$ (pairwise disjoint) composants, but D. Bellamy (1978) has constructed an indecomposable Hausdorff continuum–of weight $\aleph_1$–with just one composant. Let us call such a continuum a **Bellamy continuum**. Bellamy showed every metrizable continuum may be embedded as a retract of such a continuum, and M. Smith (1992) extended this result to all Hausdorff continua.



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not decomposable.

• Theorem 4.1 (D. Anderson, 2015). *If X is a Hausdorff continuum that fails to be coastal at proper subcontinuum A, then there is a continuous surjection $f : X \to Y$ where: (i) Y is a Bellamy continuum; and (ii) Y fails to be coastal at the proper subcontinuum $f[A]$, which may be taken to be a single point.*

• Corollary 4.2. *If every Bellamy continuum is coastal at each of its points, then every Hausdorff continuum is coastal at each of its proper subcontinua.*

• Remark. Since composants are dense, and different composants of indecomposable continua are disjoint, it is clear that any indecomposable continuum with more than one composant is coastal at each of its proper subcontinua.

• Further Remark. Bellamy's construction starts with an ω_1-indexed inverse system of metrizable indecomposable continua, with retractions for bonding maps. The result is a continuum X, which has weight $\aleph_1$ and exactly two composants. The final touch is to identify a point of one composant of X with a point of the other; the result is a Bellamy continuum. However, any Bellamy continuum resulting from such an identification is easily shown to be coastal at proper subcontinua.

For any space X, let $d(X)$ be its **density**; i.e., the minimal cardinality of a dense subset. A space X is d-**Baire** if intersections of at most $d(X)$ dense open subsets are dense. The Baire category theorem says that separable compact Hausdorff spaces are d-Baire.

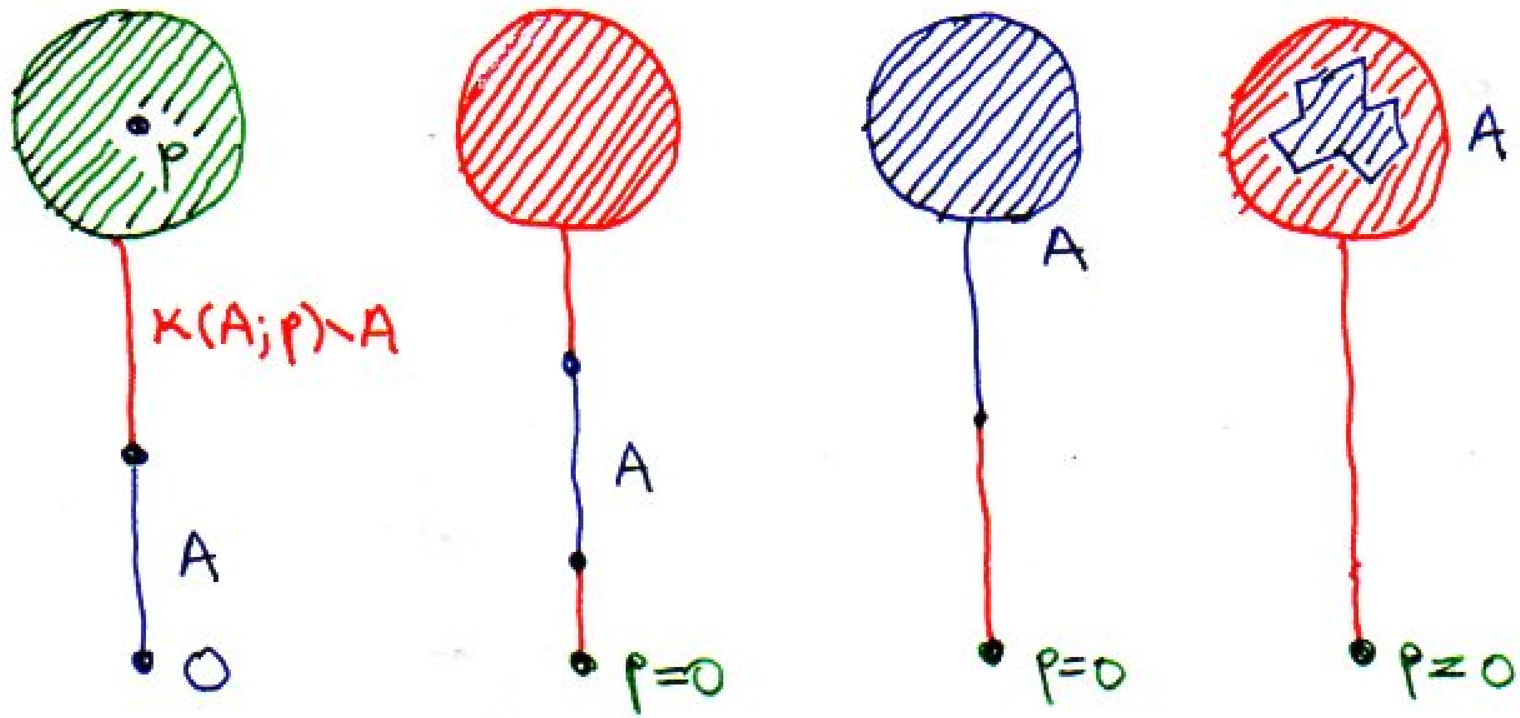
Using Theorem 4.1, Anderson has been able to show that separable Hausdorff continua are coastal at their proper subcontinua, and hence are irreducible about their sets of shore points. Indeed, he proves the more general result:

• Theorem 4.3 (D. Anderson, 2015). *Each d-Baire Hausdorff continuum is coastal at its proper subcontinua.*

An interesting example of a separable continuum that isn't metrizable is the Stone-Čech compactification $\beta\mathbb{H}$ of the real half-line $\mathbb{H} = [0, \infty)$. Theorem 4.3 implies $\beta\mathbb{H}$ is coastal at its proper subcontinua; however we can see this directly:

If $A \subseteq \beta\mathbb{H}$ is a proper subcontinuum that contains 0, then $A \subseteq \mathbb{H}$ and thus $\kappa(A; p) = \mathbb{H}$ is dense for any $p \in \mathbb{H}^* := \beta\mathbb{H} \setminus \mathbb{H}$.

If A doesn't contain 0, then $\kappa(A; 0) = \beta\mathbb{H} \setminus \{0\}$ is dense.



The four representative scenarios for a proper subcontinuum A of $\beta\mathbb{H}$.

What is even more interesting is the question of whether $\mathbb{H}^*$ is coastal at proper subcontinua: R. G. Woods (1968) (also Bellamy (1971)) showed that $\mathbb{H}^*$ is indecomposable; hence the question has an easy *yes* answer if there is more than one composant. M. E. Rudin (1970) proved that, under CH, $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ composants.

On the other hand, J. Mioduszewski (1974) proved that, under NCF, $\mathbb{H}^*$ is a Bellamy continuum. (See the 1987 survey paper by A. Blass on the Near Coherence of Filters axiom.)

Recently Anderson has announced he has a proof that $\mathbb{H}^*$ is coastal at its points. It also has a dense set of shore points.

5. Further Developments III: Distal Points.

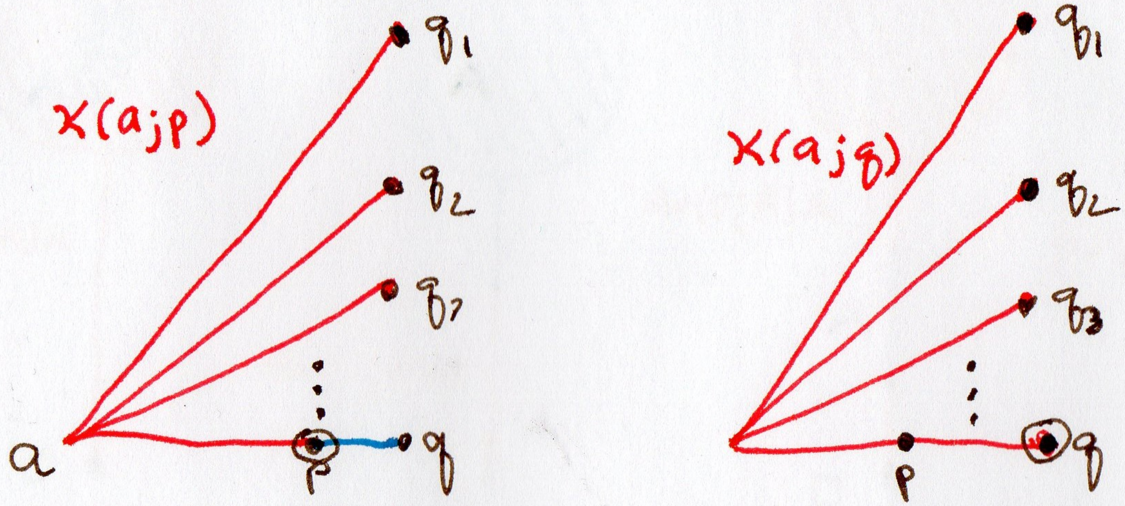
• Given continuum X and proper subcontinuum A, let $\mathcal{K}(A)$ be the family $\{\kappa(A;q) : q \in X\}$, partially ordered by set inclusion. Define $p \in X$ to be A-**distal** if $\kappa(A;p)$ is maximal in $\mathcal{K}(A)$. A point is **distal** if it is A-distal for some proper subcontinuum A.

• Proposition 5.1. *If X is a Hausdorff continuum, A is a proper subcontinuum and $p \in X$ is A-distal, then $\kappa(A;p)$ is dense in X. Thus distal points are shore points; the converse fails in general for metrizable continua.*

Proof of Proposition 5.1. For any $p \in X \setminus A$, $\kappa(A;p)$ is a connected set that contains A but does not contain p. $\overline{\kappa(A;p)}$ is therefore a subcontinuum of X that contains A. If it did not contain p, then boundary bumping would allow a subcontinuum $M \subseteq X \setminus \{p\}$ that properly contains $\overline{\kappa(A;p)}$, a contradiction. Hence $p \in \overline{\kappa(A;p)}$.

Suppose $\kappa(A;p)$ is not dense in X and let $q \in X \setminus \overline{\kappa(A;p)}$. Then $\kappa(A;p) \subseteq \overline{\kappa(A;p)} \subseteq \kappa(A;q)$. The two relative composants can't be equal; otherwise we would have $q \notin \overline{\kappa(A;q)}$, a contradiction. Hence $\kappa(A;p)$ is not maximal in $\mathcal{K}(A)$.

That distal points are shore points immediately follows. A depiction of a shore point that is not distal is left to the next slide. $\square$



p is a shore point which is not distal.
The points $q_1, q_2, q_3, \ldots, q$ are the distal points of $\mathbb{X}$.

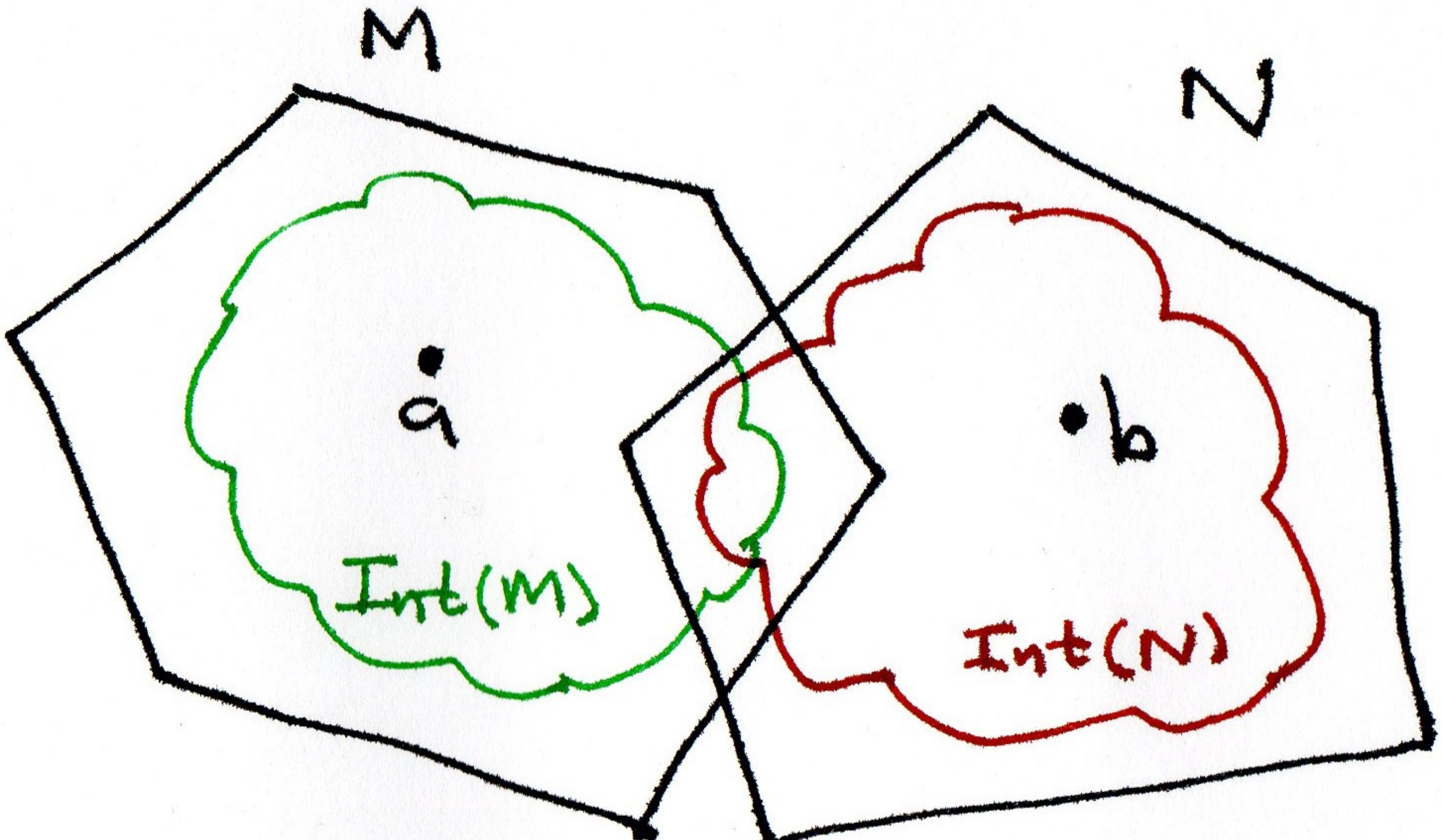
• A point c of a connected topological space X is a **strong non-cut point** if $X \setminus \{c\}$ is continuumwise connected; otherwise c is a **weak cut point.**

• Proposition 5.2. *Every strong non-cut point in a connected topological space is distal; the converse fails in general for metrizable continua.*

Recall that a continuum is **aposyndetic** if for any two of its points, each is contained in the interior of a subcontinuum that does not contain the other.

• Theorem 5.3 (F. B. Jones, 1952). *In an aposyndetic continuum, every non-cut point is strong non-cut.*

In particular, in aposyndetic continua, the strong non-cut points, the distal points, the shore points, and the non-cut points are the same.



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Proof of Proposition 5.2. Assume p is not a distal point of X and fix $a \in X \setminus \{p\}$. Then p is not $\{a\}$-distal; thus there is a point b with $\kappa(a;p) \subseteq \kappa(a;b)$, but $\kappa(a;b) \nsubseteq \kappa(a;p)$. Immediately we have from the second condition that $b \in X \setminus \{a,p\}$, so a,b,p are three distinct points. The first condition says that, since $b \notin \kappa(a;b)$, we know $b \notin \kappa(a;p)$. Hence any subcontinuum containing both a and b must also contain p. Thus $X \setminus \{p\}$ is not continuumwise connected, and p is therefore a weak cut point.

In a nondegenerate indecomposable metrizable continuum, each point is both a weak cut point and distal. This is because all composants are proper subsets. $\square$

Any nondegenerate indecomposable metrizable continuum stands in the way of a "strong non-cut point existence theorem." So we end with two problems.

• Open Problem 5.4: *Identify an interesting class* $\mathfrak{K}$ *of Hausdorff continua such that: (i) weak cut points are not necessarily cut points for members of* $\mathfrak{K}$*; and (ii) each continuum in* $\mathfrak{K}$ *is irreducible about its set of strong non-cut points.*

• Open Problem 5.5. *If* A *is a proper subcontinuum of a Hausdorff (or even metrizable) continuum* X*, does there always exist an* A*-distal point?* (Clearly *yes*, unless $\kappa(A) = X$.)

THANK YOU!