Dendrites and their Kin, from a Betweenness Perspective

Paul Bankston, Marquette University

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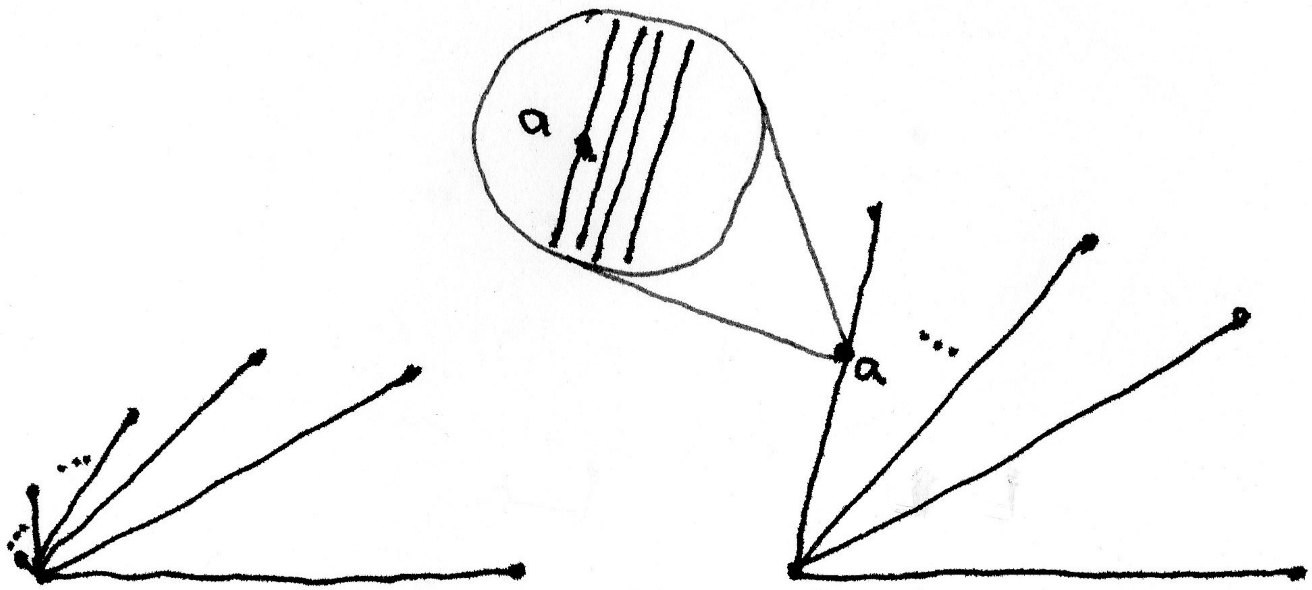
0. Background.

A **continuum** is a connected compact Hausdorff space.

Continua "come in one piece, fit in a box, have no loose ends, and their points don't stick to each other." We do not assume the existence of metrics, although a lot of continuum theorists pack metrizability into the definition of "continuum."

A topological space is **locally connected** if it has an open base consisting of connected sets; a **dendrite** is a locally connected metrizable continuum containing no simple closed curves. Here's a picture:

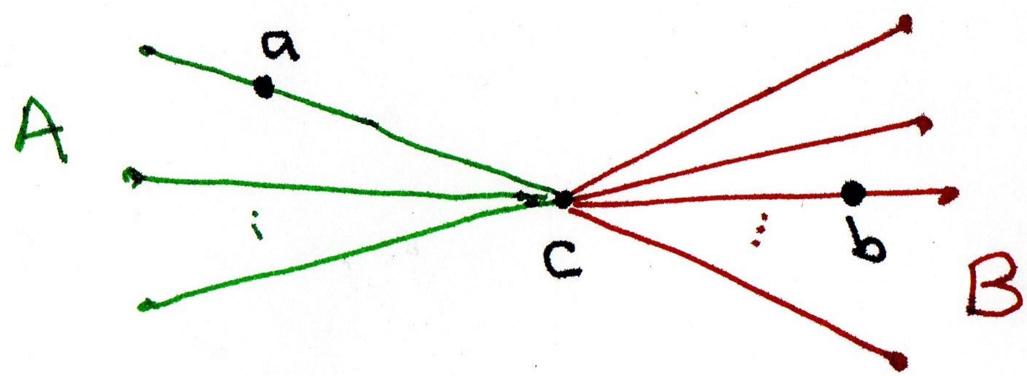


yes dendrite

no dendrite

Dendrites are "spindly," with "no circuits." Furthermore, each point has arbitrarily small connected neighborhoods.

0.1 Theorem (classical). *A metrizable continuum X is a dendrite iff each two points of X can be separated by a third point; i.e., given $a, b \in X$, there is a point $c \in X \setminus \{a, b\}$ such that $X \setminus \{c\} = A \cup B$, where A and B are disjoint clopen subsets of $X \setminus \{c\}$, $a \in A$, and $b \in B$.*



c separates a from b

Notice this is a kind of "betweenness" condition, and there's no mention of local connectedness or of simple closed curves.

And, with no simple closed curves, there is no essential use made of the metrizability restriction.

We define a continuum to be a **dendron** if each two points can be *separated* by a third point in this sense.

So "dendron" is a nice way of saying "not-necessarily-metrizable dendrite."

Dendrons have been studied from various points of view for decades; in particular, they are the "simplest" continua from the point of view of G. Whyburn's cyclic element theory.

Today's point of view is betweenness.

1. The Gap Free Axiom.

This brings us to interpretations of betweenness in continua.

For us "betweenness" is a pre-theoretical term, which may be given a precise meaning in a variety of ways.

The first-order language of betweenness has a single ternary predicate symbol $[\cdot,\cdot,\cdot]$, and we read $[a,c,b]$ as saying: "c lies between a and b" (with $c \in \{a,b\}$ permitted).

A two-point set $\{a,b\}$ is a **gap** if $[a,c,b]$ implies $c \in \{a,b\}$; **gap freeness** says that there are no gaps. This is expressed formally as

- $\forall\, ab\, (a \neq b \rightarrow \exists\, x\, ([a,x,b] \wedge x \neq a \wedge x \neq b))$

For example, gap freeness holds for the obvious betweenness relation associated with a linear ordering iff the ordering is dense in the usual sense.

The theme of this talk is gap freeness in three natural betweenness interpretations for continua, and various comparisons that can be made.

2. Three Topological Interpretations.

Let X be a continuum, and fix points $a, b, c \in X$. Assuming $c \notin \{a, b\}$, we define:

- $[a, c, b]_{\mathsf{Q}}$ if there's a disconnection $\langle A, B\rangle$ of $X \setminus \{c\}$ such that $a \in A$ and $b \in B$ (i.e., a and b lie in different quasi-components of $X \setminus \{c\}$). (Note: this is the betweenness condition used in defining "dendron.")

- $[a, c, b]_{\mathsf{C}}$ if no connected subset of $X \setminus \{c\}$ contains $\{a, b\}$ (i.e., a and b lie in different components of $X \setminus \{c\}$); and

- $[a, c, b]_{\mathsf{K}}$ if no subcontinuum of $X \setminus \{c\}$ contains $\{a, b\}$ (i.e., a and b lie in different continuum components of $X \setminus \{c\}$).

Clearly $[a,c,b]_{\mathsf{Q}} \Rightarrow [a,c,b]_{\mathsf{C}} \Rightarrow [a,c,b]_{\mathsf{K}}$ always; hence

Q-gap free $\Rightarrow$ C-gap free $\Rightarrow$ K-gap free.

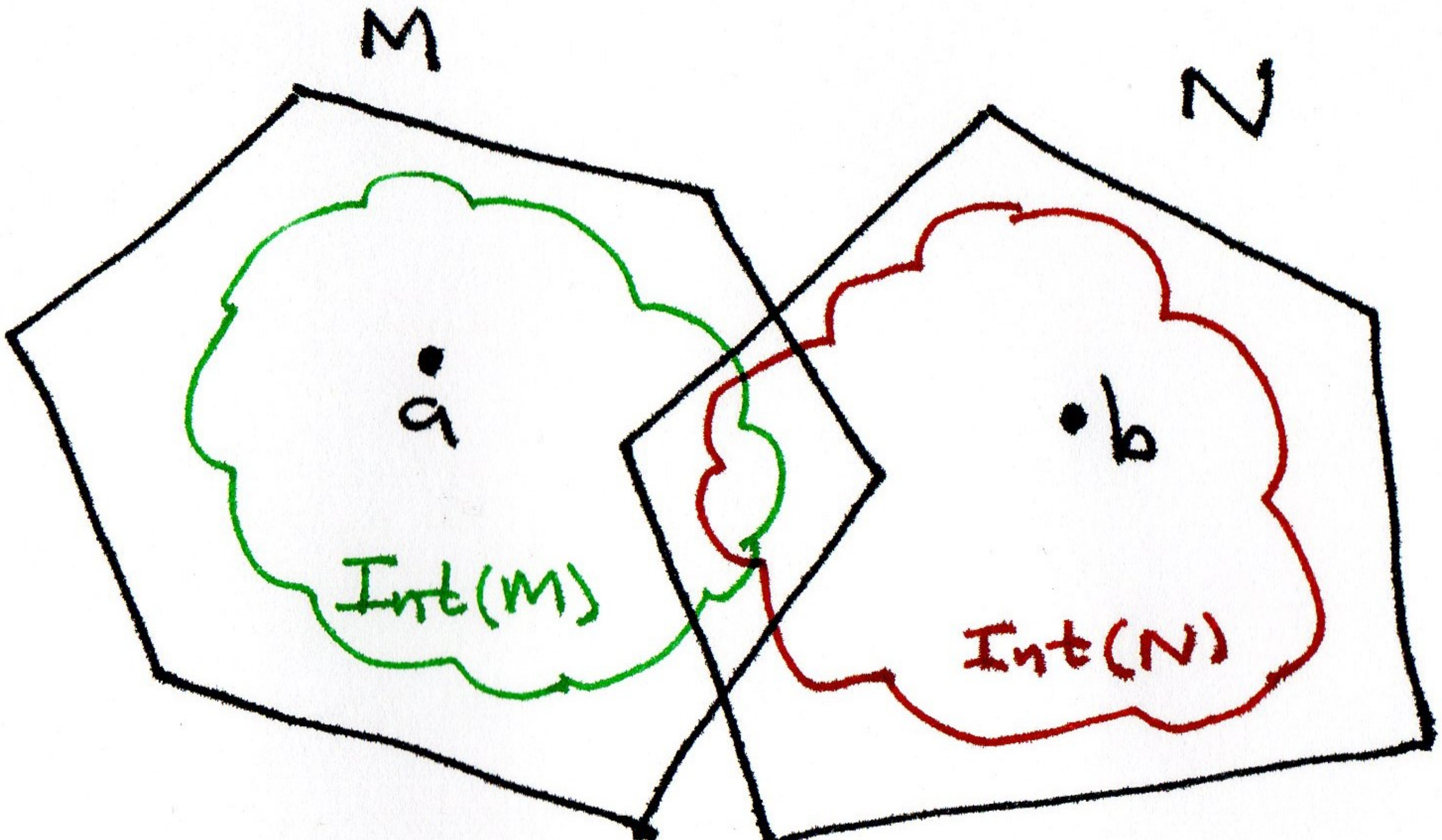
A continuum is a **Q-dendron** (resp., **C-dendron**, **K-dendron**) if it is Q-gap free (resp., C-gap free, K-gap free).

So Q-dendron = dendron.

3. When Interpretations Agree.

A continuum is **aposyndetic** (after F. B. Jones, 1941: *apo-* denoting "being away from;" *syndesis* denoting "being bound together") if for each two of its points, one lies in the interior of a subcontinuum that excludes the other.

(This looks like a souped-up T_1 axiom, but is really a weak form of local connectedness.)



Aposyndesis

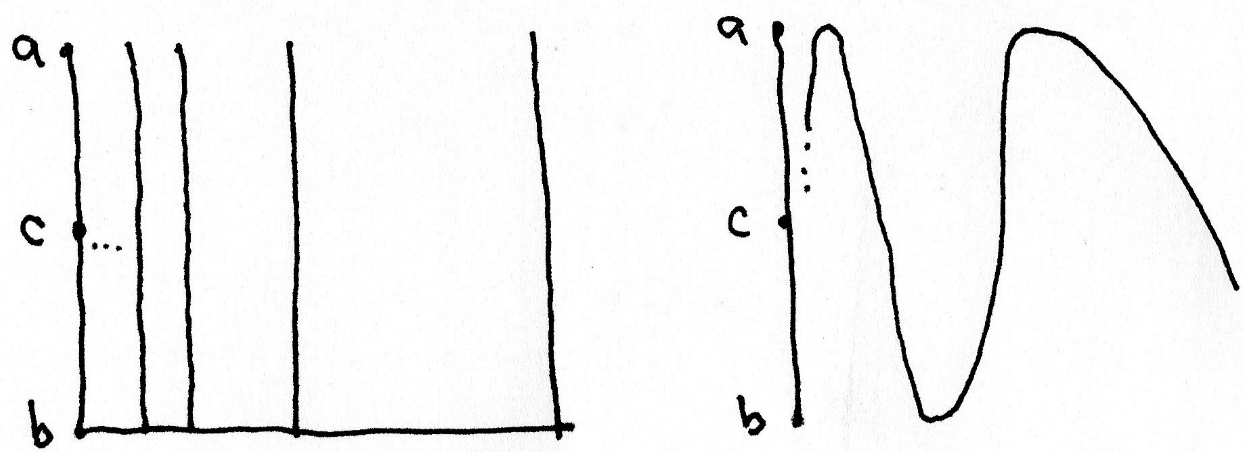
3.1 Theorem (PB, 2014). *If X is an aposyndetic continuum, then $[\cdot,\cdot,\cdot]_{\mathsf{K}} = [\cdot,\cdot,\cdot]_{\mathsf{C}}$. If X is also locally connected, then $[\cdot,\cdot,\cdot]_{\mathsf{K}} = [\cdot,\cdot,\cdot]_{\mathsf{Q}}$.* $\square$

4. When Interpretations Disagree.

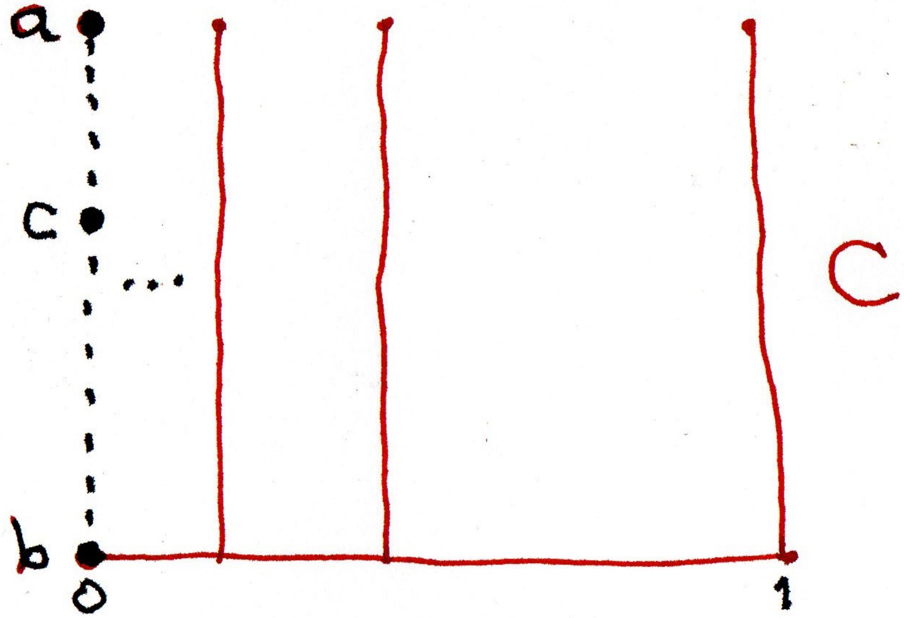
Any comb space or $\sin(1/x)$-continuum serves to show that $[\cdot,\cdot,\cdot]_C$ needn't coincide with $[\cdot,\cdot,\cdot]_K$.

What's more, these continua are K-dendrons but not C-dendrons.

Thus K-gap free $\not\Rightarrow$ C-gap free (for continua).



$[a,c,b]_K$, but $\neg[a,c,b]_C$



C is a connected set containing a and b, but no c in $\{0\} \times (0,1)$. $\{a,b\}$ is a C-gap.

We know that $[\cdot,\cdot,\cdot]_K$ can disagree with $[\cdot,\cdot,\cdot]_C$.

We even have the stronger statement that being K-gap free does not imply being C-gap free for a continuum.

Now we want to compare C and Q.

Our first big question is whether these two interpretations of betweenness necessarily agree for continua, and the answer is NO.

4.1 Example. *Hairball: a plane continuum for which* $[\cdot,\cdot,\cdot]_{\mathsf{C}} \neq [\cdot,\cdot,\cdot]_{\mathsf{Q}}$.

Construction. For $n = 1, 2, \ldots,$ let

$$A_n = ([0,1] \times \{1/n\})$$

$$\cup\{\langle x, y\rangle : x \geq 1 \text{ and } (x-1)^2 + y^2 = (1/n)^2\}$$

$$\cup((1/2, 1] \times \{-1/n\}),$$

let

$$A = [0, 1/2) \times \{0\},$$

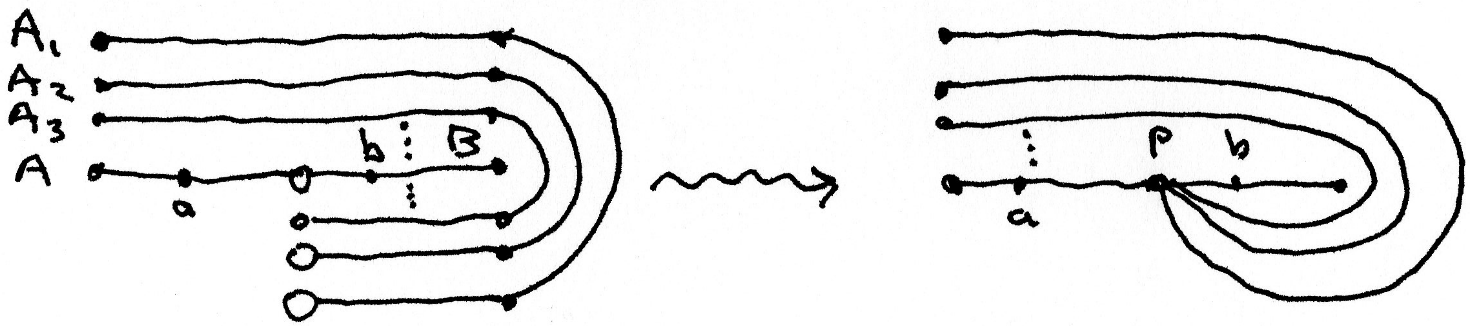
$$B = (1/2, 1] \times \{0\},$$

and put

$$X = (\bigcup_{n=1}^{\infty} A_n) \cup (A \cup B).$$

X is a locally compact topological space, no component of which is compact. Hence the Alexandroff one-point compactification $Y = X \cup \{p\}$ is a continuum. And if $a \in A$, $b \in B$, we have $[a, p, b]_{\mathsf{C}}$, but not $[a, p, b]_{\mathsf{Q}}$. Here's the picture of our hairball:

Hairball



X $\qquad\qquad$ $Y = X \cup \{p\}$

In Y: $[a,p,b]_C$ holds, but $[a,p,b]_Q$ doesn't.

Y is a K-dendron but not a C-dendron.

5. Are C-Dendrons Actually Dendrons Anyway?

Our second big question is whether C-gap free continua are necessarily Q-gap free–i.e., are C-dendrons just dendrons traveling incognito–and the answer is a surprising YES.

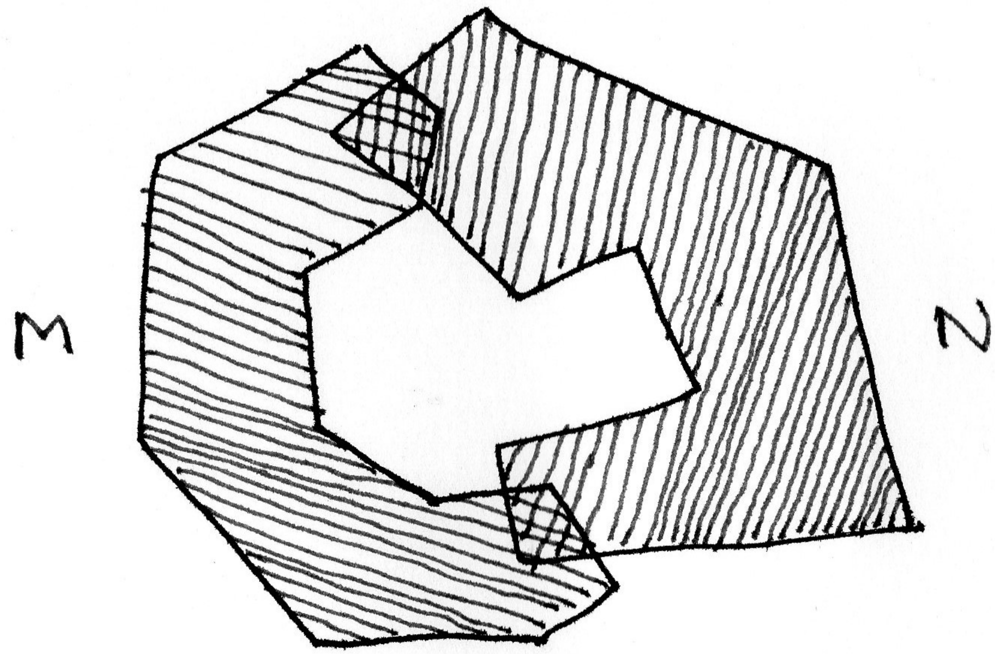
First some nomenclature: A continuum X is:

- **decomposable** if X is the union of two proper subcontinua;

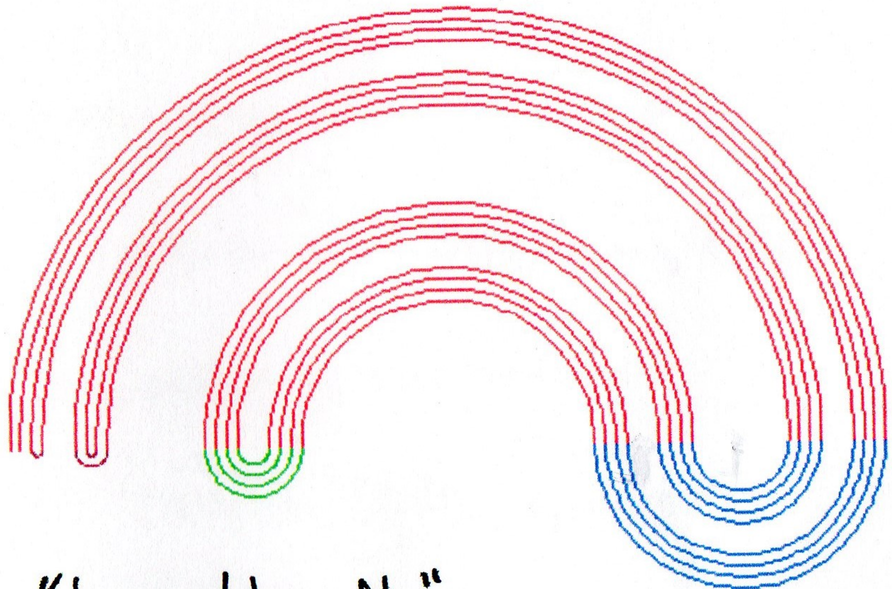
- **indecomposable** otherwise; and

- **unicoherent** if X is not the union of two subcontinua whose intersection is disconnected.

The addition if "hereditarily" in front of any of these adjectives confers the associated property to all nondegenerate subcontinua.



$X = M \cup N$ is decomposable, but not unicoherent.



Knaster "buckethandle":
not decomposable.

A **cut point** in a space is one whose removal disconnects the space.

5.1 Lemma. *Suppose X is a C-dendron. Then for each nondegenerate connected subset K of X, there is a point $c \in K$ such that c is a cut point of every connected subset of X containing K.*

Proof. Given K a nondegenerate connected subset of X, pick $a, b \in K$ distinct. By C-gap freeness, we have a point $c \in X \setminus \{a, b\}$ with $[a, c, b]_{\mathrm{C}}$ holding. Every connected subset of X must contain c if it contains both a and b; so if M is one such, with $K \subseteq M$, then $M \setminus \{c\}$ cannot be connected. Hence c is a cut point of M. $\square$

In 1988, L. E. Ward dubbed as “Property T” the condition that *for nondegenerate subcontinua* $K \subseteq M$, K *contains a cut point of* M. Clearly the conclusion of Lemma 5.1 implies Property T, but it is actually no stronger. Indeed, Ward proved that Property T is equivalent to being a dendron. (He states in his paper that W. Bula and E. D. Tymchatyn had together proved the same result.)

Thus we immediately have

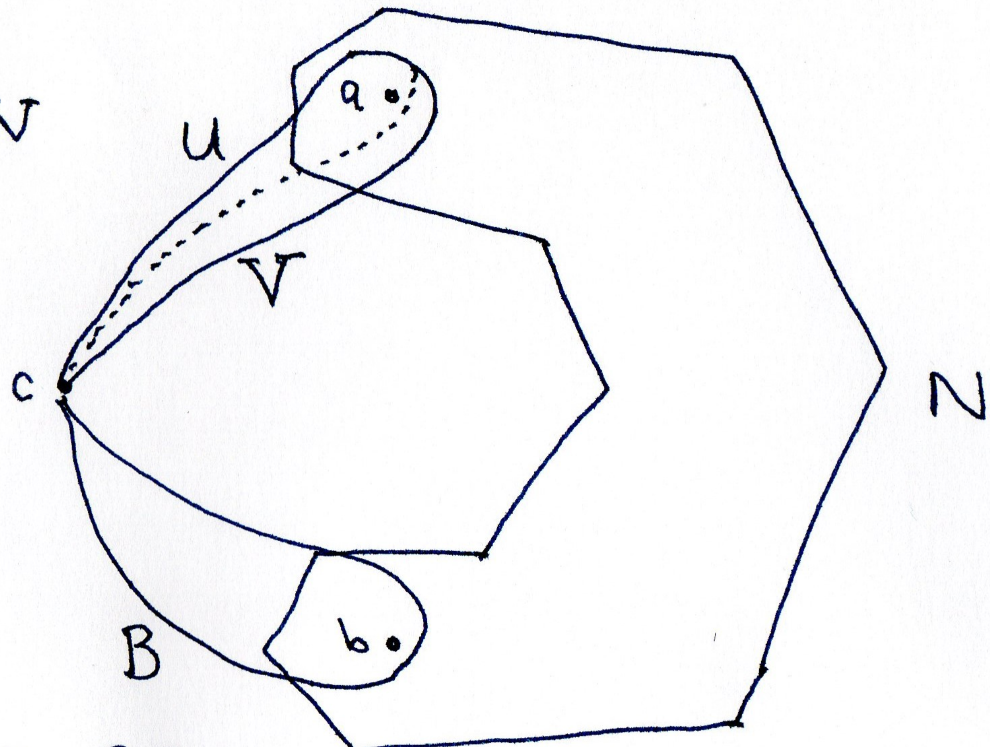
5.2 Theorem (Ward, 1988 (essentially)). *C-dendrons are dendrons; i.e., C-gap free* $\Rightarrow$ *Q-gap free (for continua).*

Proof Outline. Assuming X to be a C-dendron, first use Lemma 5.1 to show that X is hereditarily unicoherent. Then for each two points $a, b \in X$, the set of all $c \in X$ with $[a, c, b]_{\mathsf{K}}$ holding is the intersection of all subcontinua containing both a and b, and is a subcontinuum of $X-$ which we denote by $[a, b] = [a, b]_{\mathsf{K}}$. Call $[a, b]$ the **K-interval** determined by a and b.

Next show that each nondegenerate K-interval $[a, b]$ has only a and b as non-cut points, and is hence an arc. (Metrizable arcs are all homeomorphic to a closed bounded interval in $\mathbb{R}$.)

The following picture illustrates how you show hereditary unicoherence.

$A = U \cup V$



$M = A \cup B \cup \{c\}$

The final step is to pick $a, b \in X$ distinct, and to let $c \in [a, b] \setminus \{a, b\}$. Then c is a cut point of $[a, b]$, and hence a cut point of X, by Lemma 5.1.

We then proceed to show that $[a, c, b]_{\mathsf{Q}}$ holds, by showing $X \setminus \{c\} = A_a \cup A_b$, where, for $x \in X \setminus \{c\}$, A_x is the union of all subarcs in $X \setminus \{c\}$ containing x. That $A_a \cap A_b = \emptyset$ follows since c is a cut point of X. The only tricky–but not too tricky–part is to show each A_x is open in X. $\square$

5.3 Theorem (PB, unpublished) *Let X be a continuum; the following are equivalent:*

(i) X is a dendron.

(ii) X is hereditarily unicoherent and locally connected.

(iii) X is hereditarily unicoherent and aposyndetic.

(iv) X is an aposyndetic K-dendron.

(v) X is a C-dendron.

(The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are well known in the metric case.)

6. K-Dendrons.

We recall the following fact, used in the proof of Theorem 5.2.

6.1 Proposition. *A continuum is hereditarily unicoherent iff each of its K-intervals is a subcontinuum.* $\square$

Hereditary unicoherent continua are clearly K-dendrons, and it is natural to ask whether the converse is true.

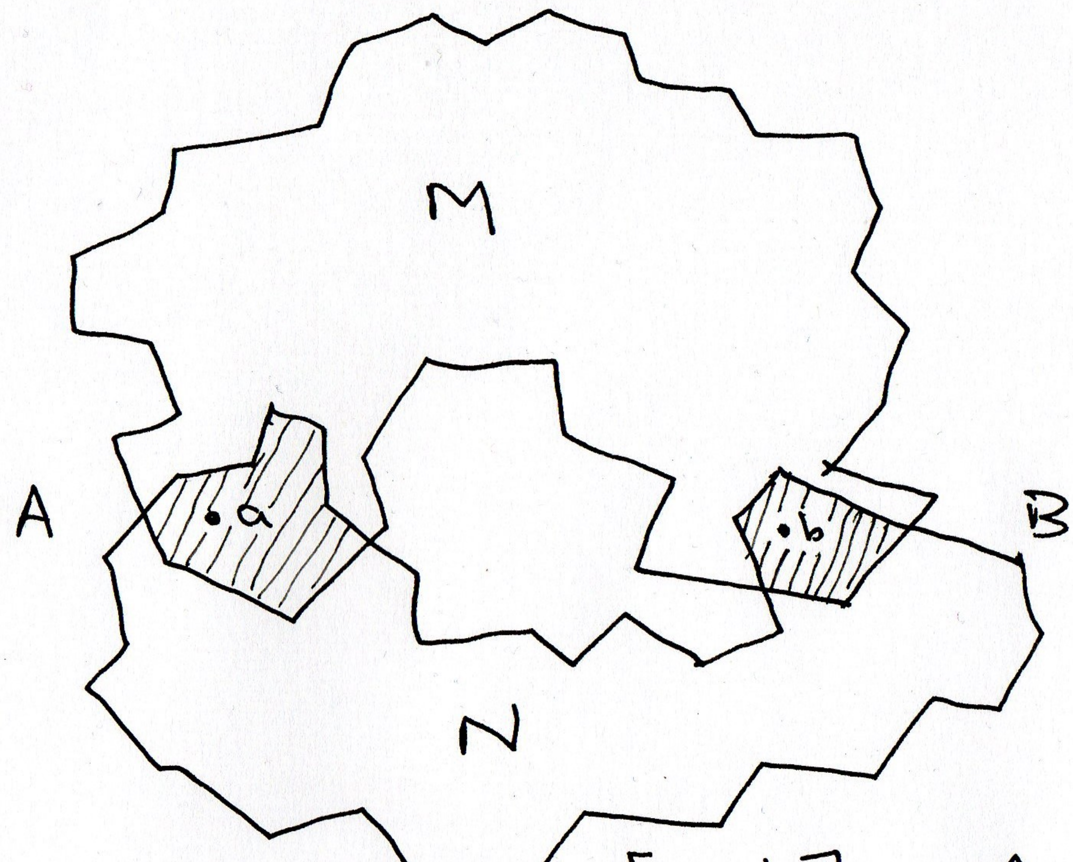
The answer turns out to be NO.

A continuum X is a **crooked annulus** if it has a decomposition $X = M \cup N$ into subcontinua such that:

- Both M and N are hereditarily indecomposable; and

- $M \cap N = A \cup B$, where A and B are disjoint nondegenerate subcontinua.

6.2 Theorem (PB, 2013). *Every crooked annulus is a K-dendron, which fails to be unicoherent.* $\Box$



$$[a,b]_K = A \cup B$$

We know dendrons are hereditarily decomposable; by virtue of this, they're **curves** (i.e., of covering dimension 1)–as they *should* be. But hereditarily unicoherent continua–and hence K-dendrons–can be of any dimension at all. Even a crooked annulus, being the union of two hereditarily unicoherent continua, can have any old dimension, by a 1954 result of R. H. Bing.

Our third big question is whether K-dendrons are anything in particular.

For example, are they hereditarily unicoherent if assumed to be hereditarily decomposable?

Are there *any* interesting properties that K-dendrons enjoy?

7. Strong K-Dendrons.

Recall the first-order statement of gap freeness from above.

- Gap Freeness:
  $\forall\, ab\, (a \neq b \rightarrow \exists\, x\, ([a, x, b] \wedge x \neq a \wedge x \neq b))$

If we replace negations of equality in the conclusion with negations of betweenness, we obtain a stronger property (when betweenness is interpreted so that $[x, x, y]$ and $[x, z, y] \leftrightarrow [y, z, x]$ always hold).

- Strong Gap Freeness:
  $\forall\, ab\, (a \neq b \rightarrow \exists\, x\, ([a, x, b] \wedge \neg[x, a, b] \wedge \neg[a, b, x]))$

With the Q- and the C-interpretations, strong gap freeness is equivalent to gap freeness.

But strong K-gap freeness *really is stronger*.

7.1 Theorem (PB, 2013). *A continuum is a strong K-dendron if and only if it is both hereditarily unicoherent and hereditarily decomposable.* $\square$

In particular, strong K-dendrons are curves.

8. Antisymmetric K-Dendrons.

The reason strong gap freeness is no stronger than gap freeness for Q and C is the following:

8.1 Proposition. *Both* $[\cdot,\cdot,\cdot]_{\mathsf{Q}}$ *and* $[\cdot,\cdot,\cdot]_{\mathsf{C}}$ *satisfy the* **anti-symmetry** *condition: if* $[a,c,b]$ *and* $[a,b,c]$ *hold, then* $b = c$*.*
$\square$

A continuum X is **antisymmetric** if its K-interpretation of betweenness satisfies the antisymmetry condition. This is equivalent to saying that for any $a, b, c \in X$ with $b \neq c$, there is a subcontinuum of X containing a and exactly one of b and c.

Recall from above that an **arc** is any continuum with exactly two non-cut points. Metrizable arcs are all homeomorphic to $[0,1] \subseteq \mathbb{R}$, but nonmetrizable arcs can be quite exotic; e.g., the lexicographically ordered square.

X is **arcwise connected** if each two of its points are the non-cut points of an arc in X.

8.2 Theorem (PB, 2013). *A continuum is an antisymmetric K-dendron if and only if it is both hereditarily unicoherent and arcwise connected.* $\square$

It it interesting to note that lots of topological properties of the form

HEREDITARILY UNICOHERENT + ????

are expressible first-order gap freeness conditions for appropriate interpretations of betweenness in a continuum.

So our last big question is: how about hereditary unicoherence all by itself? Is there a first-order sentence φ–involving equality and one ternary predicate–such that a continuum is hereditarily unicoherent iff its associated K-betweenness structure satisfies φ?

THANK YOU!