Co-Existential Metric Images of Ultra-Arcs

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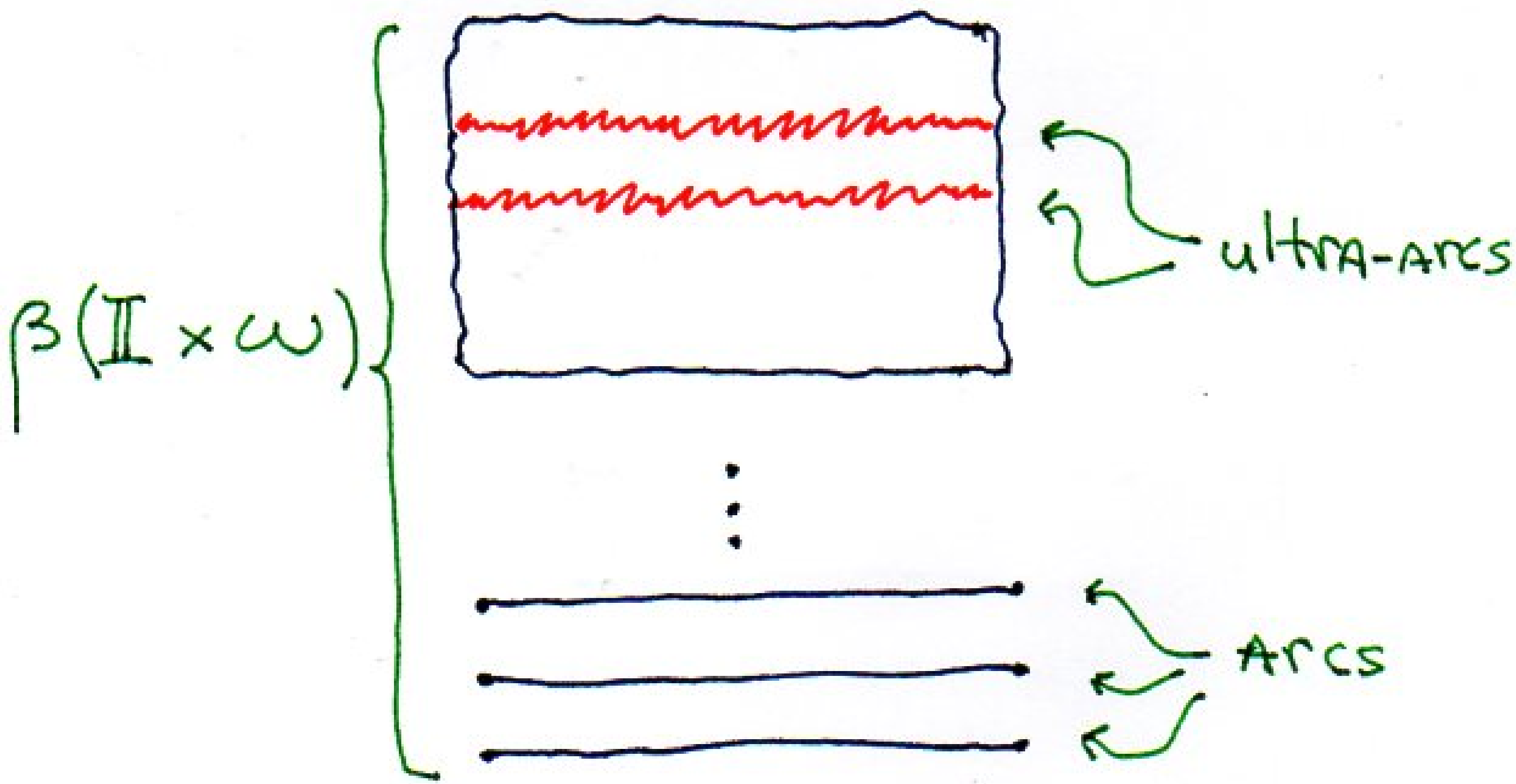
Introduction.

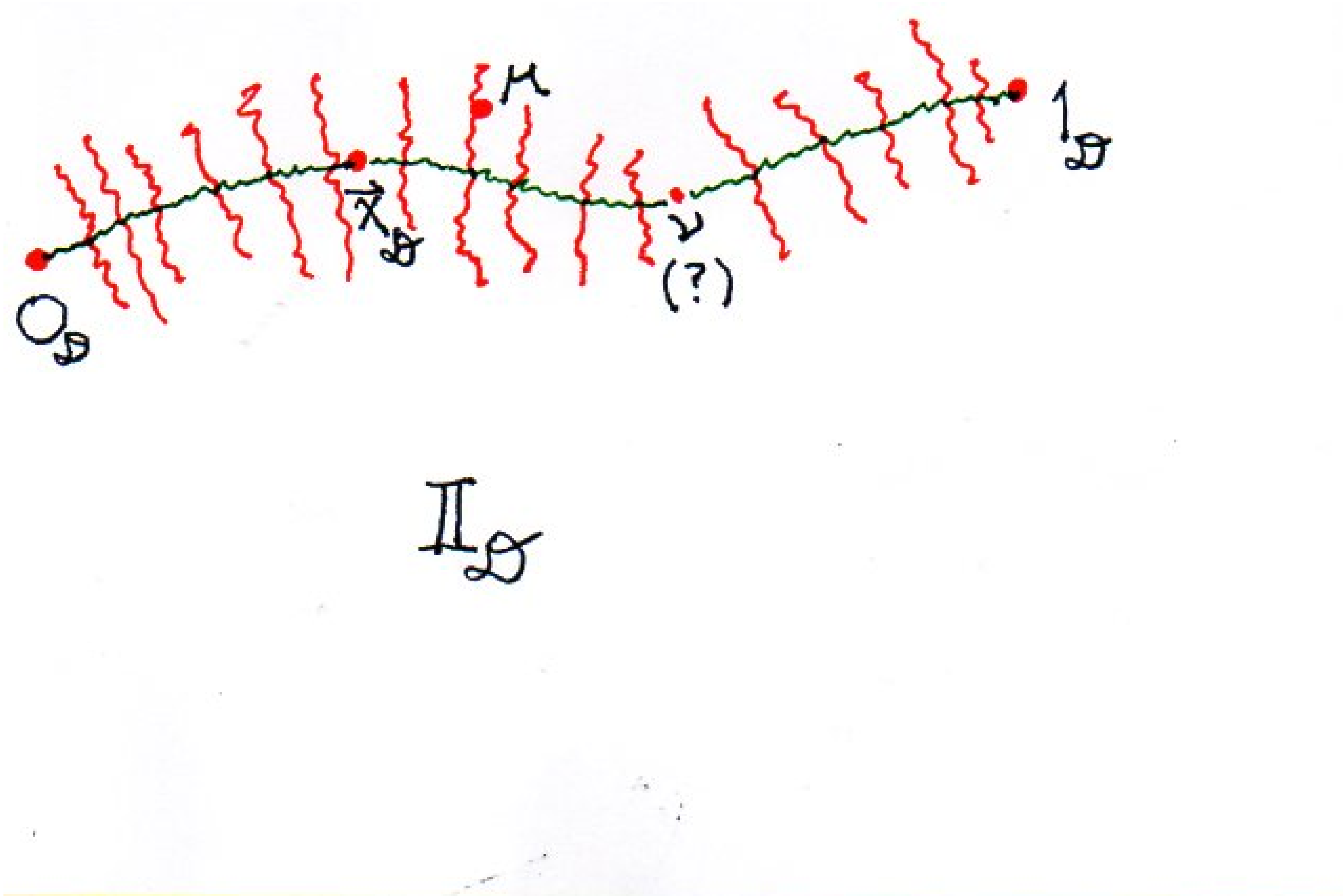
A **generalized arc** is any continuum (= connected compact Hausdorff space) with exactly two noncut points; an **arc** is metric generalized arc, well known to be homeomorphic to the real unit interval $\mathbb{I} := [0,1]$.

By an **ultra-arc** we mean an ultracopower $\mathbb{I}_{\mathcal{D}}$ of $\mathbb{I}$ via a nonprincipal ultrafilter $\mathcal{D}$ on the set $\omega := \{0,1,\ldots\}$ of natural numbers.

Alternatively, the ultra-arcs are the components of the Stone-Čech remainder $(\mathbb{I} \times \omega)^* := \beta(\mathbb{I} \times \omega) \setminus (\mathbb{I} \times \omega)$.



β(𝕀 × ω)
ultra-arcs
Arcs



μ
1_ϑ
$\vec{x}_\vartheta$
ν
(?)
0_ϑ
Π_ϑ

Ultra-arcs were first introduced by Mioduszewski in the mid 1970s in order to study $\mathbb{H}^*$, where $\mathbb{H}$ is the real half-line $[0, \infty)$. They naturally embed as the "standard subcontinua" of $\mathbb{H}^*$, but that is not our focus here.

What we are interested in is the problem of deciding when a continuum is a continuous image of an ultra-arc under various kinds of mapping.

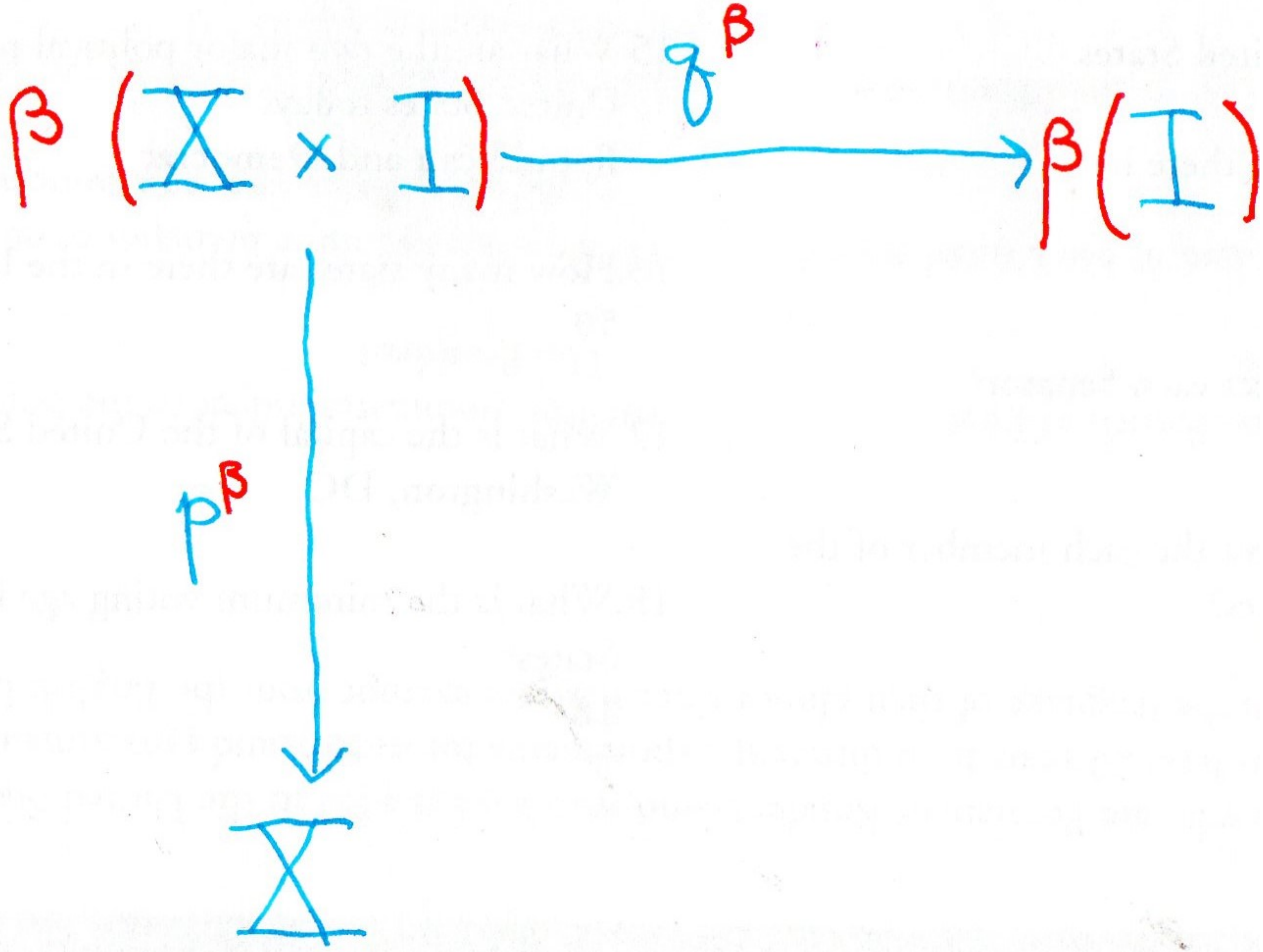
If we make no restrictions on the mapping, we have the following important result, due to Dow-Hart (extending a much older theorem of Bellamy beyond the metric context).

Proposition 1. *Every continuum of weight* $\leq \aleph_1$ *is a continuous image of any ultra-arc.*

In this talk, we focus down on the classes of monotone and of co-existential maps. The first is familiar, the second somewhat less so.

Ultracopowers and Co-Existential Maps.

Given a compactum X and (discrete) set I, first form the cartesian product $X \times I$, with coordinate maps $p : X \times I \to X$ and $q : X \times I \to I$. Next apply the Stone-Čech functor, obtaining the following diagram.


β(X × I)
q^β
β(I)
p^β
X

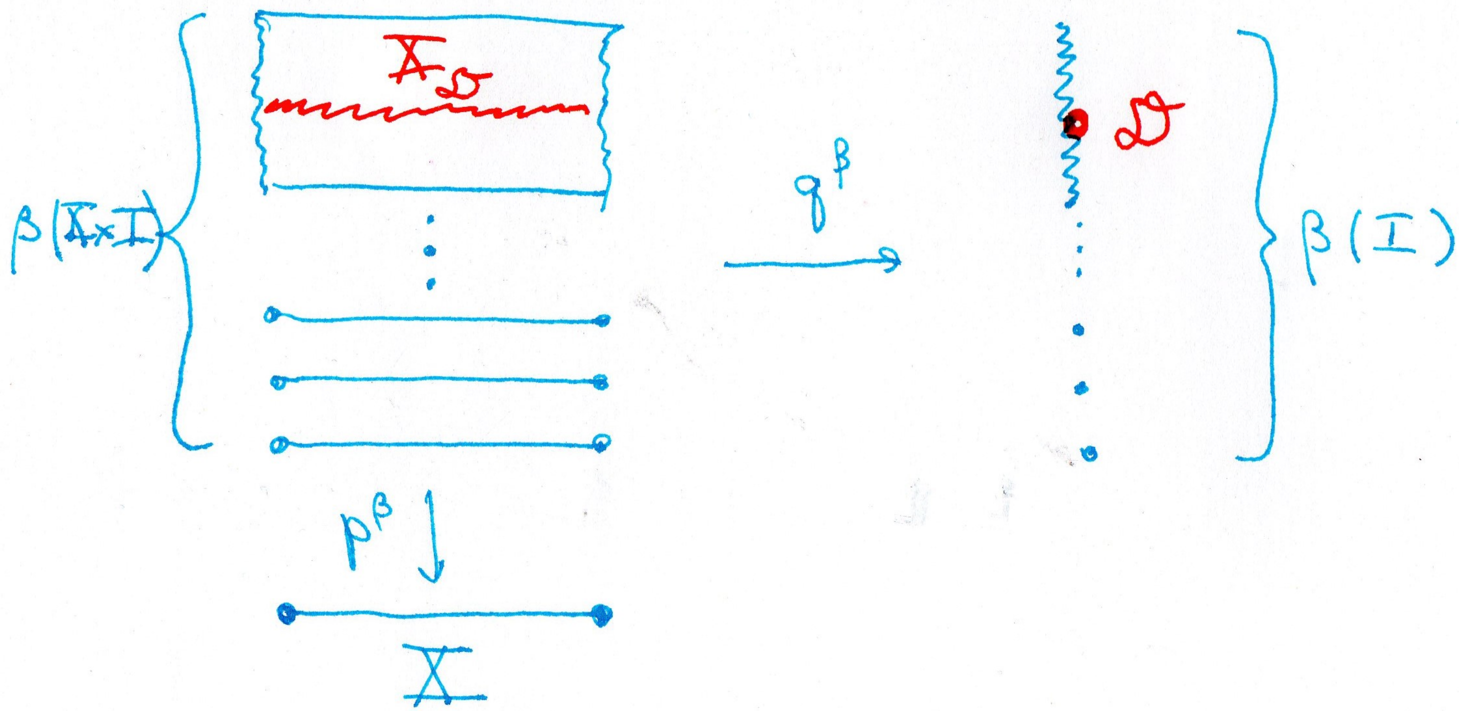
If $\mathcal{D}$ is an ultrafilter on I, then it may be viewed as a point in $\beta(I)$. Denote by $X_{\mathcal{D}}$ the pre-image of $\{\mathcal{D}\}$ under q^{β}. This is the $\mathcal{D}$-**ultracopower** of X.

When X is a continuum, these ultracopowers partition $\beta(X \times I)$ into its components.

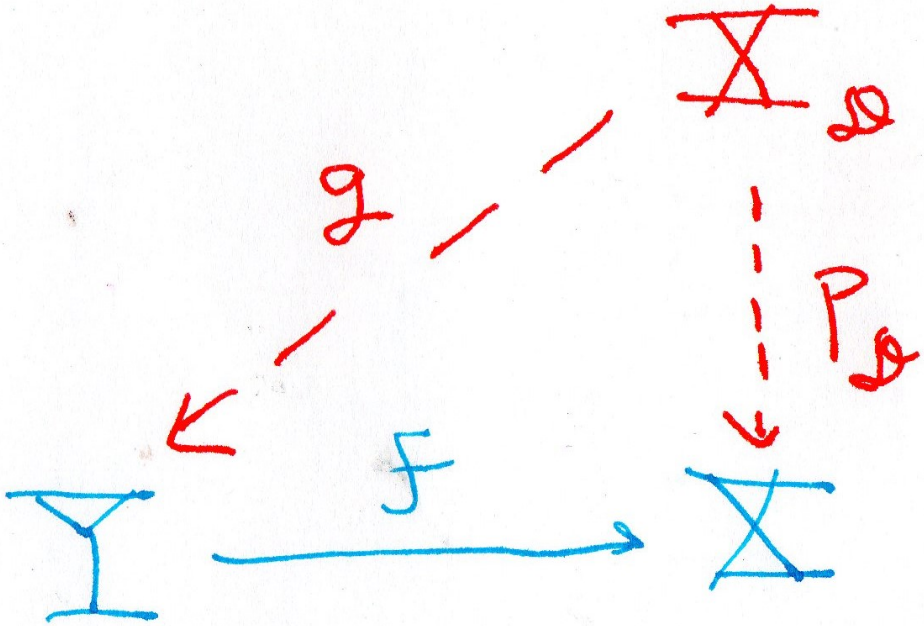
The map

$$p_{\mathcal{D}} := p^{\beta}|X_{\mathcal{D}} : X_{\mathcal{D}} \to X$$

is a continuous surjection, called the **ultracopower codiagonal map**.


$\beta(X \times I)$
X_ϑ
q^β
ϑ
$\beta(I)$
p^β
X

A mapping $f : Y \to X$ between compacta is **co-existential** if there is an ultracopower $X_{\mathcal{D}}$ and a surjective map $g : X_{\mathcal{D}} \to Y$ such that $f \circ g = p_{\mathcal{D}}$.



g
F

Co-existential maps play a category-theoretic rôle dual to that played by existential embeddings in model theory.

The classes of monotone and of co-existential maps are not directly related; however we can make the following assertion.

Proposition 2. *Every co-existential map with locally connected range is monotone. And if a compactum fails to be locally connected, there is an ultracopower of it whose associated codiagonal map is not monotone.*

Some Preservation Results.

(i) (Hereditary) unicoherence [both co-existential and monotone].

(ii) Hereditary decomposability (but not decomposability) [both].

(iii) (Hereditary) indecomposability [both].

(iv) Chainability [both].

(v) Irreducibility [monotone only].

(vi) Covering dimension one [co-existential only].

Regularized Ultracopowers.

The usual $\mathcal{D}$-*ultraproduct* of an I-sequence $\vec{F} = \langle F_i : i \in I \rangle$ of sets is denoted $\vec{F}^{\mathcal{D}}$ or $\prod_{\mathcal{D}} F_i$.

If X is a compactum the collection $\mathfrak{F}(X, \mathcal{D})$ of $\mathcal{D}$-ultraproducts of closed subsets of X constitutes an atomic lattice under $\cup$ and $\cap$.

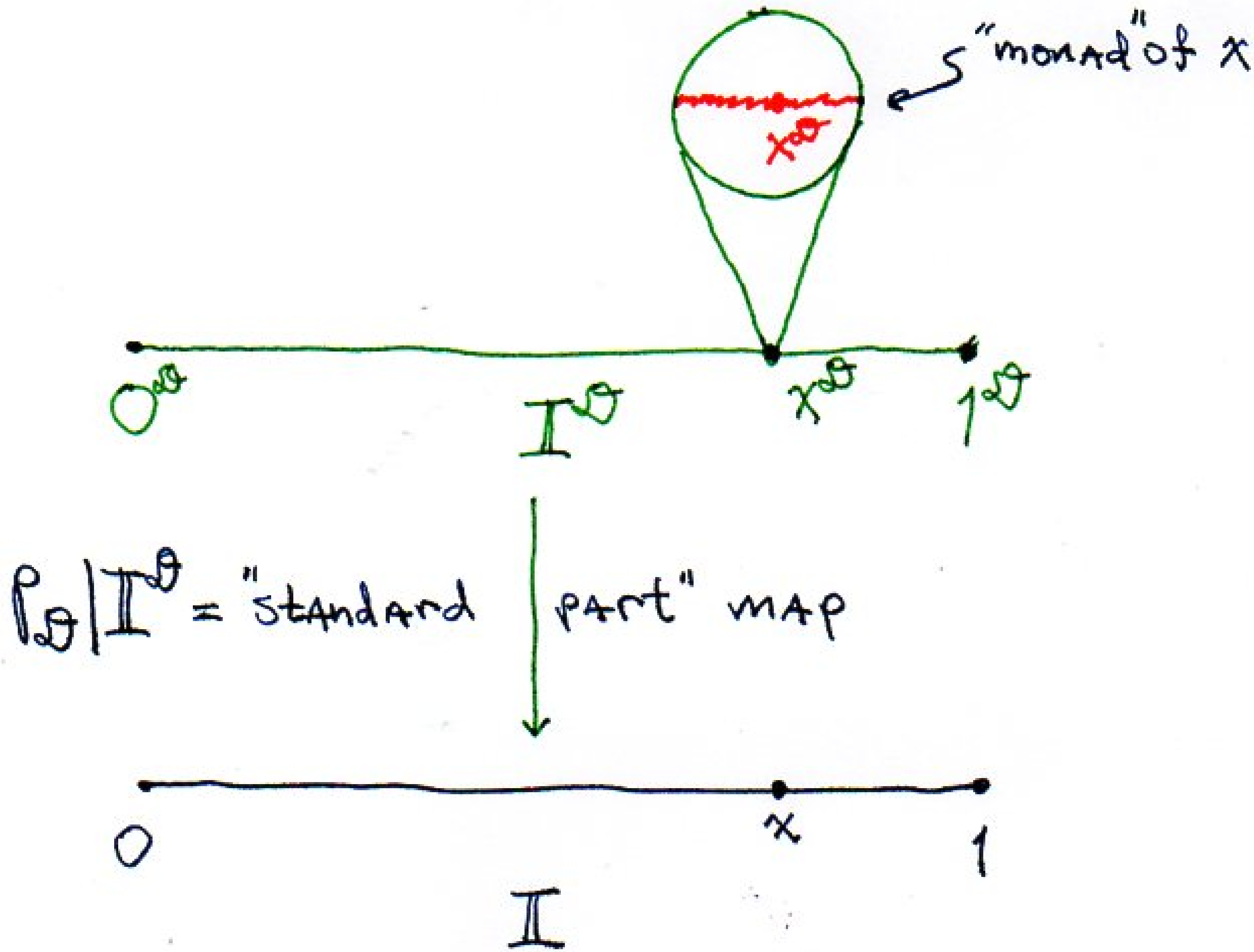
The $\mathcal{D}$-ultracopower $X_{\mathcal{D}}$ is the Wallman construction applied to $\mathfrak{F}(X, \mathcal{D})$; the points of $X^{\mathcal{D}}$ may be viewed as the maximal filters of this lattice.

This makes it possible to view $X^{\mathcal{D}}$ as a dense subset of $X_{\mathcal{D}}$; its points are the **regular points** of the ultracopower.

Moreover...

If the topology on X is induced by a total ordering $<$ (say, when X is a generalized arc), then the ultrapower ordering $<^{\mathcal{D}}$–also total–induces the subspace topology on $X^{\mathcal{D}}$.

The regular points of an ultra-arc may be viewed as "non-standard reals":


"monad" of x
x^{ϑ}
0^{ϑ}
$\mathbb{I}^{\vartheta}$
x^{ϑ}
1^{ϑ}
$\rho_{\vartheta}|\mathbb{I}^{\vartheta}$ = "standard part" map
0
x
1
$\mathbb{I}$

Given continuum X, define $\mu, \nu \in X_{\mathcal{D}}$ to be $\mathcal{R}$-**equivalent** if they contain the same closed-set ultraproducts $\vec{K}^{\mathcal{D}}$, whose factors are subcontinua of X.

Clearly the $\mathcal{R}$-class of any regular point of $X_{\mathcal{D}}$ is degenerate, so there are lots of equivalence classes in general.

The topological quotient $X^{\mathcal{R}}_{\mathcal{D}}$ is called the **regularized** $\mathcal{D}$-ultracopower of X.

$X^{\mathcal{R}}_{\mathcal{D}}$ is a compact connected T_0 space, which may fail to be T_1.

However $X^{\mathcal{R}}_{\mathcal{D}}$ is Hausdorff in certain "nice" cases.

One such case is when X is locally connected, as well as n-SLC; i.e., each subcontinuum has arbitrarily small neighborhoods whose complements have $\leq n$ components.

$\mathbb{I}$ is locally connected and 2-SLC, so $\mathbb{I}^{\mathcal{R}}_{\mathcal{D}}$ is a continuum.

Ultra-Arcs in More Detail.

Here are some useful facts about the ultra-arc $\mathbb{I}_{\mathcal{D}}$.

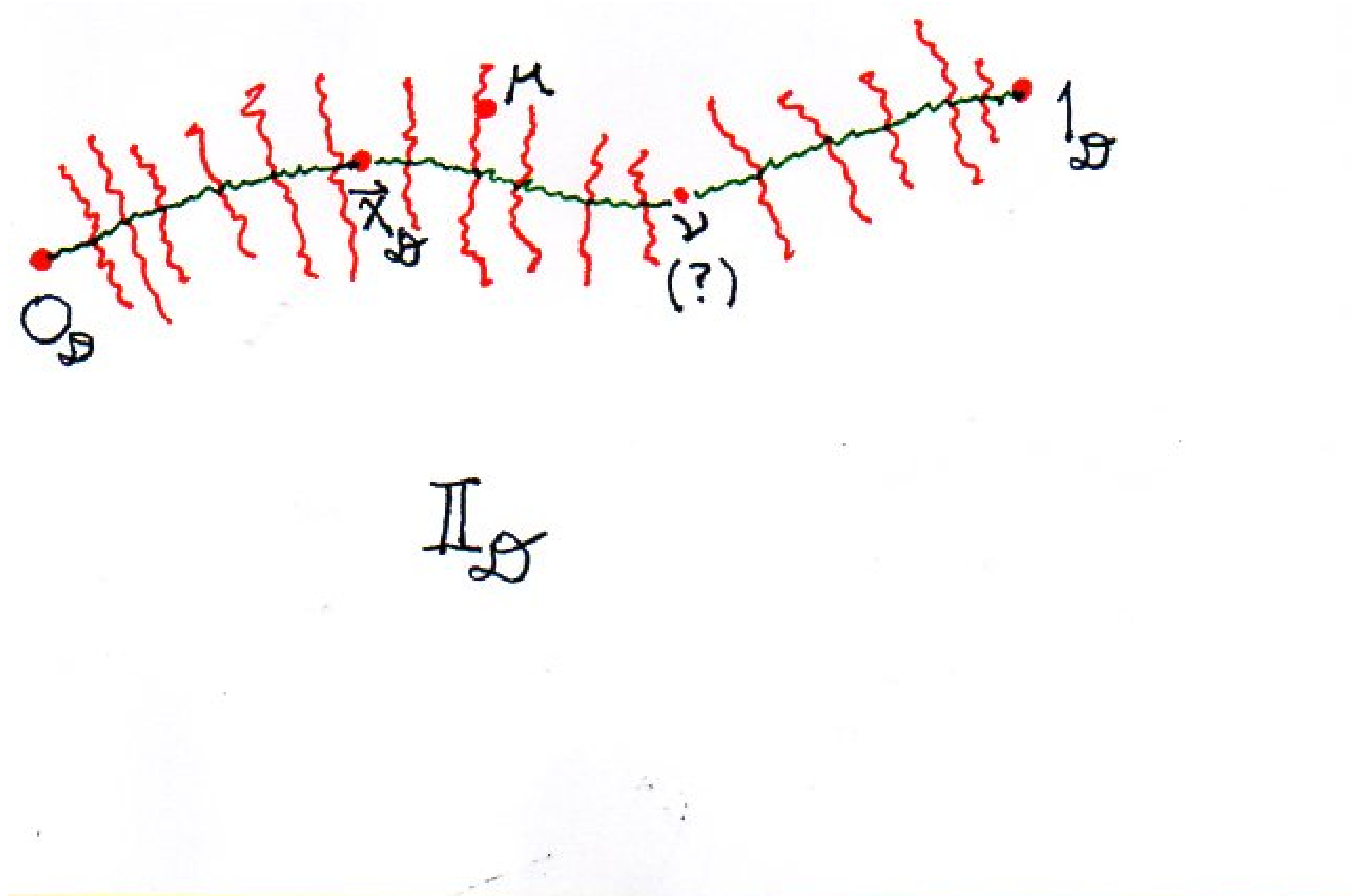
(i) It is hereditarily unicoherent, as well as irreducible about $\{0_{\mathcal{D}}, 1_{\mathcal{D}}\}$.

(ii) The $\mathcal{R}$-classes–also known as *layers*–form an upper semicontinuous partition into nowhere dense subcontinua. Moreover, these subcontinua are indecomposable.

(iii) The ultrapower ordering $<^{\mathcal{D}}$ (obtained from the usual ordering on $\mathbb{I}$) induces a total ordering of the layers of $\mathbb{I}_{\mathcal{D}}$, so that the regularized ultra-arc $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$ is a generalized arc. (Never metric.)

(iv) Every subcontinuum is either contained in a layer or is a union of layers.

This suggests imagining an ultra-arc as a "generalized arc with indecomposable hair"...



μ
$1_{\mathfrak{D}}$
$\vec{x}_{\mathfrak{D}}$
ν
(?)
$0_{\mathfrak{D}}$
$\Pi_{\mathfrak{D}}$

Necessary Conditions.

Proposition 3. *A nondegenerate monotone image of an ultra-arc is hereditarily unicoherent and irreducible.*

(We do not know whether it is of covering dimension one; monotone maps can raise dimension.)

Proposition 4. *A co-existential image of an ultra-arc is hereditarily unicoherent and of covering dimension one. Any metric image is irreducible as well.*

(We do not know whether nonmetric images of ultra-arcs are always irreducible; co-existential maps need not preserve irreducibility.)

Sufficient Conditions I: Generalized Arcs.

The codiagonal maps $p_{\mathcal{D}} : \mathbb{I}_{\mathcal{D}} \to \mathbb{I}$ witness that the arc is a co-existential (and monotone) image of any ultra-arc.

We have only a conditional answer to the question of which nonmetric generalized arcs are such images.

Proposition 5. (CH) *Every generalized arc of weight* $\leq \aleph_1$ *is a co-existential monotone image of every ultra-arc.*

Remark. The proof of Proposition 5 makes essential use of both the Löwenheim-Skolem Theorem and Keisler's Ultrapower Theorem (CH version).

Sufficient Conditions II: Chainabile + Metric.

Proposition 6. *Every nondegenerate chainable metric continuum is a co-existential image of every ultra-arc.*

Proof Idea. Write the nondegenerate chainable metric continuum X as an inverse limit of arcs and surjective bonding maps:

$$\mathbb{I} \xleftarrow{f_0} \mathbb{I} \xleftarrow{f_1} \mathbb{I} \xleftarrow{f_2} \dots$$

Let $\vec{\pi} = \langle \pi_n : n \in \omega \rangle$ be the associated sequence of projection maps from X to $\mathbb{I}$, so that $\pi_n = f_n \circ \pi_{n+1}$ always holds.

Next form the **ultracoproduct map**

$$\vec{\pi}_{\mathcal{D}} : X_{\mathcal{D}} \longrightarrow \mathbb{I}_{\mathcal{D}}$$

via the rule:

$$\prod_{\mathcal{D}} F_n \in \vec{\pi}_{\mathcal{D}}(\mu) \text{ iff } \prod_{\mathcal{D}} \vec{\pi}_n^{-1}[F_n] \in \mu.$$

The next trick is to define maps $g_n : \mathbb{I}_{\mathcal{D}} \to \mathbb{I}$ so that the commutativities

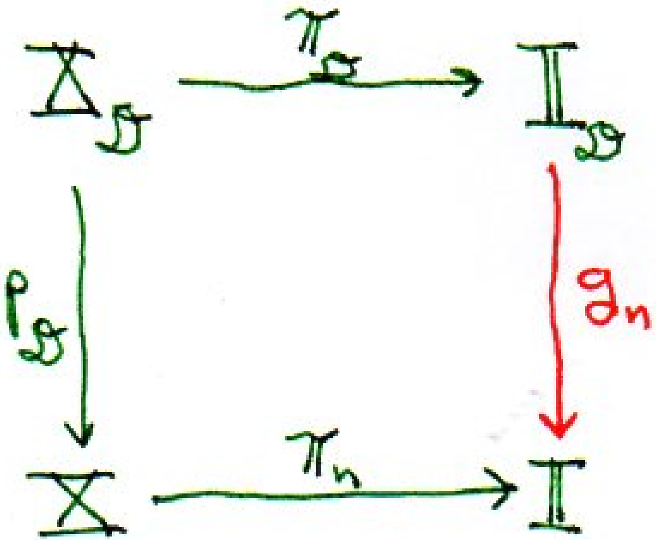
$$g_n \circ \vec{\pi}_{\mathcal{D}} = \pi_n \circ p_{\mathcal{D}}$$

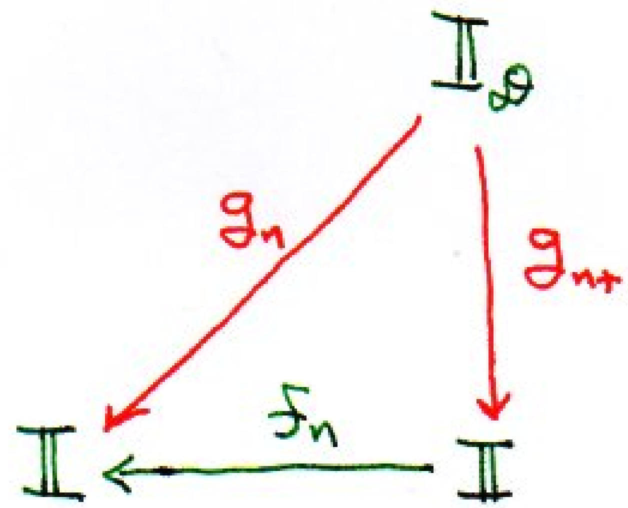
and

$$g_n = f_n \circ g_{n+1}$$

always hold.

(This is possible because every final segment of ω is in $\mathcal{D}$.)


$\mathbb{X}_\vartheta$
π_ϑ
$\mathbb{I}_\vartheta$
p_ϑ
g_n
$\mathbb{X}$
π_n
$\mathbb{I}$


$\mathbb{I}_\vartheta$
g_n
g_{n+1}
$\mathbb{I}$
f_n
$\mathbb{I}$

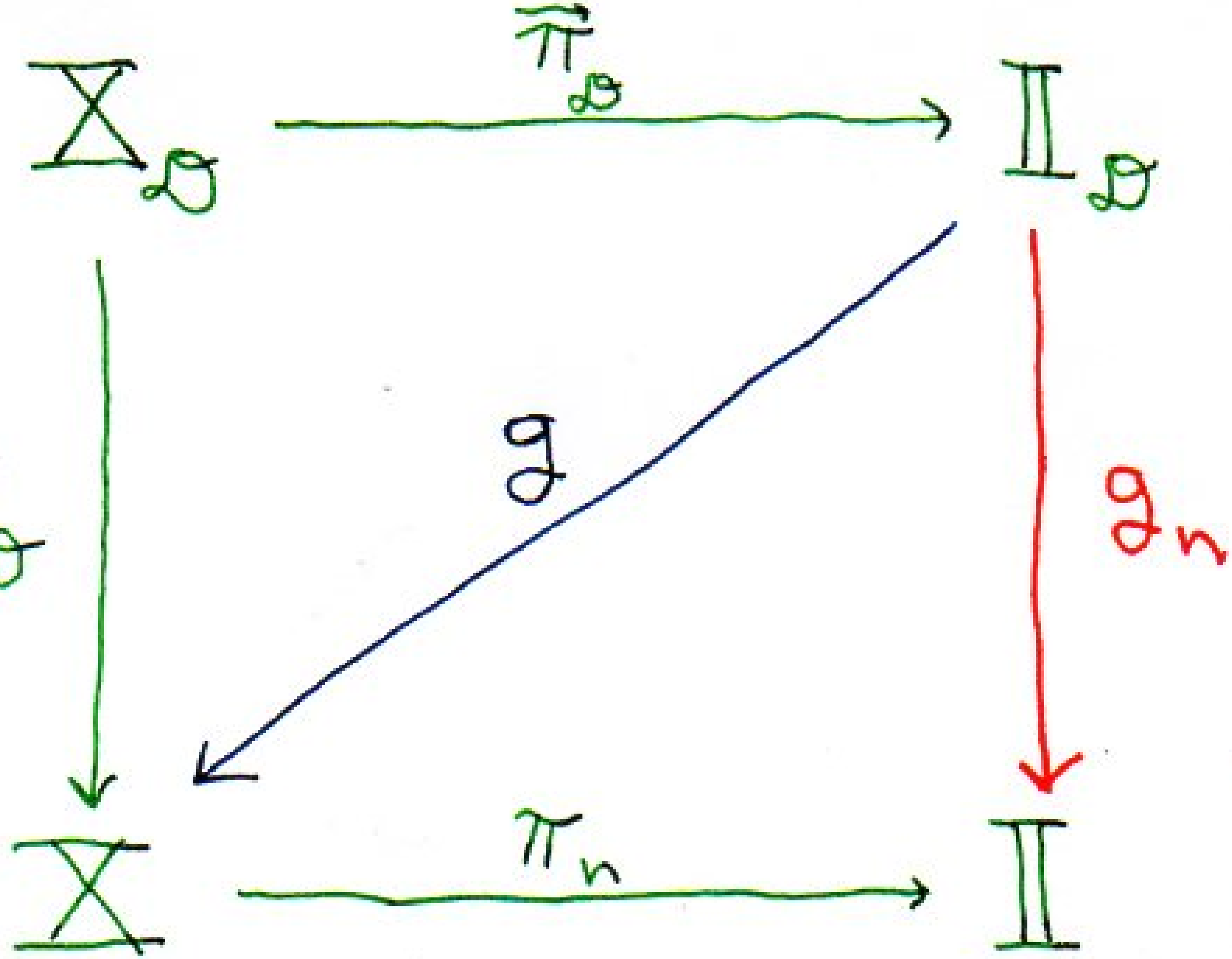
This gives rise to a unique map

$$g : \mathbb{I}_{\mathcal{D}} \to X$$

so that

$$g \circ \vec{\pi}_{\mathcal{D}} = p_{\mathcal{D}}$$

holds too.



$\vec{\pi}_{\mathcal{D}}$
$\mathbb{X}_{\mathcal{D}}$
$\mathbb{I}_{\mathcal{D}}$
$p_{\mathcal{D}}$
g
g_n
$\mathbb{X}$
π_n
$\mathbb{I}$

This makes g a co-existential map. $\square$

While co-existential maps preserve chainability, ultra-arcs are well known not to be chainable themselves.

Could it be that chainability is still a necessary condition for a metric continuum to be a co-existential image of an ultra-arc?

We next show the answer to be *no*.

Sufficient Conditions III: Co-Existentially Closed + Metric.

A **co-existentially closed continuum** is a continuum X such that every continuous map from a continuum onto X is co-existential.

(Compare this with classic continuum-theoretic notions, such as Class(C) and Class(W).)

Proposition 7.

(i) *Every co-existentially closed continuum is hereditarily indecomposable, as well as of covering dimension one. (In particular, it's nondegenerate.)*

(ii) *Every nondegenerate continuum is a continuous image of a co-existentially closed continuum, of the same weight.*

(iii) *There exists an uncountable family of pairwise non-homeomorphic metric co-existentially closed continua which are not chainable.*

(iv) (Eagle, Goldbring, Vignati) *The pseudo-arc is a co-existentially closed metric continuum (the only chainable one, up to homeomorphism).*

Harking back to the Dow-Hart Proposition 1, Bellamy originally showed that every metric continuum is a continuous image of any nondegenerate subcontinuum of $\mathbb{H}^*$. In particular:

Proposition 8. *Every co-existentially closed metric continuum is a co-existential image of every ultra-arc.*

Corollary 9. *There exists a nonchainable metric continuum which is a co-existential image of every ultra-arc.*

Proposition 10 (Hoehn, Oversteegen). *Every hereditarily indecomposable metric continuum of span zero is chainable.*

Corollary 11. *There exists a nonzero-span metric continuum which is a co-existential image of every ultra-arc.*

When Images are Generalized Arcs.

Proposition 12. *If X is a nondegenerate hereditarily decomposable monotone image of an ultra-arc, then X is a generalized arc.*

Remark. The proof of Proposition 12 makes essential use of the fact that layers of ultra-arcs are indecomposable continua. The assertion is no longer true with *monotone* replaced with *co-existential*: the sin($\frac{1}{x}$)-continuum is hereditarily decomposable, but is not an arc. Hence it is not a monotone image of any ultra-arc. On the other hand, it is chainable, and therefore a co-existential image of every ultra-arc, by Proposition 8.

(We do not know of a nondegenerate monotone image of an ultra-arc which is not a co-existential image.)

Proposition 13.

(i) *If X is an aposyndetic (resp., metric antisymmetric) co-existential image of an ultra-arc, then X is a generalized arc (resp., arc).*

(ii) *If X is a nondegenerate antisymmetric monotone image of an ultra-arc, then X is a generalized arc.*

Remark. Antisymmetry is strictly weaker than aposyndesis; the obstruction to being able to just assume *antisymmetric* in (i) above is that we do not know whether co-existential images of ultra-arcs are irreducible, except in the metric case. (Co-existential maps are known not to preserve this property in general.)

THANK YOU!