Fine Continua

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A **continuum** is a topological space that is compact, Hausdorff, and connected.

In this talk we concentrate on counting composants of a continum; in particular we are interested in when the number is as large as possible, relative to the size of the continuum itself.

The number of composants cannot exceed the number of points. When these two cardinalities match, we call the continuum *fine*.

For any topological space X, $|X|$ is the cardinality of (the underlying set of points of)X and $w(X)$ is the **weight** of X; i.e., the smallest infinite cardinal λ such that X has an open/closed base of cardinality $\leq \lambda$.

If X is a continuum, $a \in X$, the **composant at** a is

$\kappa(a) := \bigcup\{K : a \in K$ and K is a proper subcontinuum of $X\}$.

A **composant** of X is a composant at some point of X.

For example, if X is an arc with end points a, b, then $\kappa(a) = X \setminus \{b\}$, $\kappa(b) = X \setminus \{a\}$, and $\kappa(x) = X$ for any $x \in X \setminus \{a, b\}$.

Also, if X is a simple closed curve, then $\kappa(x) = X$ for any $x \in X$.

It is well known that

- Composants of continua are both connected and dense.

For $x, y \in X$, write $x \sim y$ to mean $x \in \kappa(y)$; equivalently, $y \in \kappa(x)$. Clearly this gives a relation that is both reflexive and symmetric.

When is it an equivalence relation?

Suppose $\sim$ is not transitive for X. Then we have $x, y, z \in X$ such that $x \sim y \sim z$ but $x \not\sim z$. This gives us subcontinua $K, L \subseteq X$ with $\{x, y\} \subseteq K \subseteq X \setminus \{z\}$ and $\{y, z\} \subseteq L \subseteq X \setminus \{x\}$. Since x and z are contained in the subcontinuum $K \cup L$, and both K and L are proper subcontinua of $K \cup L$, we have $X = K \cup L$.

This says that our continuum X is *decomposable*.

So for X an indecomposable continuum—e.g., a bucket handle, solenoid, or pseudo-arc—the composants of X form a partition of X into connected dense subsets.

For a continuum X, a **transversal** for X is a subset $T \subseteq X$ such that T hits each composant in at most one point.

Equivalently, for each two points $a, b \in T$, X is **irreducible about** $\{a, b\}$; i.e., no proper subcontinuum of X contains both a and b.

For a decomposable continuum X, the number of composants is either three or one, depending on whether or not X is irreducible about two of its points. Arcs are irreducible, simple closed curves are not.

In any case, no transversal for a decomposable continuum can have more than two points.

• A metrizable continuum is indecomposable iff it possesses a transversal of cardinality ≥ 3.

A continuum X is called **fine** (resp., w-**fine**) if it possesses a transversal of cardinality $|X|$ (resp., $w(X)$). w-fine continua−including the degenerate ones−are necessarily indecomposable.

In a 1920 Fundamenta article, Z. Janusczewski and C. Kuratowski proved—using a Baire category argument—that:

• *Nondegenerate indecomposable metrizable continua have uncountably many composants (and are* a fortiori *w-fine).*

Since nondegenerate metrizable continua have cardinality $\mathfrak{c} := 2^\omega$, this result does not show the existence of fine continua without the continuum hypothesis (CH), the statement that $\mathfrak{c} = \omega_1$.

A few years later, S. Mazurkiewicz (1927) improved on this result and showed:

• *Every nondegenerate indecomposable metrizable continuum has* $\mathfrak{c}$ *composants (and is hence fine).*

In this talk we show that w-fine continua of arbitrarily large size exist; we also provide a consistent (with ZFC) proof of the existence of arbitrarily large fine continua.

Here are some facts about composant numbers for non-metrizable indecomposable continua.

• Consider the Stone-Čech remainder $\mathbb{H}^*$ of the half-line $\mathbb{H} := [0, \infty)$. $\mathbb{H}^*$ is well known to be an indecomposable continuum of weight $\mathfrak{c}$ and cardinality $2^{\mathfrak{c}}$.

Is it (w-)fine?

- In 1970, M. E. Rudin used CH to show that $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ composants (and is hence fine).

- In 1974 S. Mioduszewski used the near coherence of filters axiom (NCF) to show that $\mathbb{H}^*$ has just one composant (and is as far as possible from being w-fine).

• D. Bellamy (1978) constructed two indecomposable continua of weight ω_1; one with exactly two composants, the other with just one.

• M. Smith (1984) constructed a hereditarily indecomposable continuum of weight ω_1 with exactly two composants. (It is an open question whether a nondegenerate hereditarily indecomposable continuum can have just one composant.)

Our main theorem is the following.

Theorem 1. *Let α be an infinite cardinal.*

(1) *There is a w-fine continuum Y of weight 2^{α}.*

(2) (GCH) *There is a fine continuum Y of weight 2^{α}.*

(Moreover, in each case, Y may be taken to be hereditarily indecomposable (or not) and to be of any predetermined covering dimension.)

Our principal tool is the *ultracopower* construction.

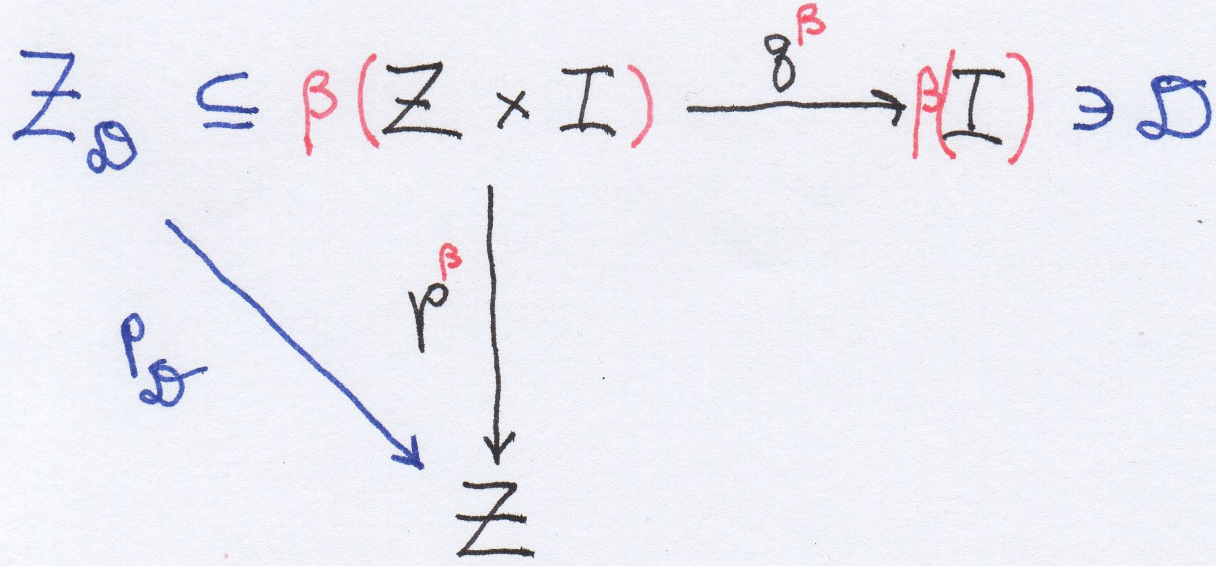
Recall that if Z is any compactum, I is a discrete infinite set, and $\mathcal{D}$ is an ultrafilter on I (i.e., $\mathcal{D} \in \beta(I)$), then the $\mathcal{D}$-ultracopower $Z_{\mathcal{D}}$ is obtained as follows:

• Let $p : Z \times I \to Z$ and $q : Z \times I \to I$ be the coordinate projections.

• Apply the Stone-Čech functor to obtain $p^{\beta} : \beta(Z \times I) \to Z$ and $q^{\beta} : \beta(Z \times I) \to \beta(I)$.

$$
\begin{array}{ccc}
\beta(Z \times I) & \xrightarrow{\;q^{\beta}\;} & \beta(I) \\
\Big\downarrow{\scriptstyle r^{\beta}} & & \\
Z & &
\end{array}
$$

- $Z_{\mathcal{D}}$ is defined to be the pre-image, under q^{β}, of the point $\mathcal{D} \in \beta(I)$. The map $p_{\mathcal{D}} : Z_{\mathcal{D}} \to Z$ is the restriction of p^{β} to $Z_{\mathcal{D}} \subseteq \beta(Z \times I)$.


Z_𝔇 ⊆ β(Z × I)
g^β
β(I) ∋ 𝔇
ρ_𝔇
ρ^β
Z

The map $p_{\mathcal{D}}$ is a continuous surjection known as the $\mathcal{D}$-*codiagonal map*, and is well known to be *weakly confluent*; i.e., subcontinua of the range are images of subcontinua of the domain.

It is a basic fact that $Z_{\mathcal{D}}$ is a continuum (totally disconnected compactum) iff the same is true of Z.

Lemma 1. *Let Z be a continuum, $\mathcal{D}$ an ultrafilter on discrete infinite set I.*

(1) *$Z_{\mathcal{D}}$ and Z share the same covering dimension.*

(2) *$Z_{\mathcal{D}}$ is (hereditarily) indecomposable iff the same is true of Z.*

The plan is to start with a nondegenerate indecomposable metrizable continuum X. Our continuum Y will be an ultracopower $X_{\mathcal{D}}$, where $\mathcal{D}$ is an ultrafilter on a set I of cardinality α. By Lemma 1, we may start with X hereditarily indecomposable–or not–and of any predetermined covering dimension. However we decide, the same will hold for Y.

This takes care of the "moreover" part of Theorem 1.

Given a set S and ultrafilter $\mathcal{D}$ on I, two I-sequences $\vec{x}$ and $\vec{y}$ in S^I are $\mathcal{D}$-**equivalent** if

$$\{i \in I : \vec{x}(i) = \vec{y}(i)\} \in \mathcal{D}.$$

The set of $\mathcal{D}$-equivalence classes is the $\mathcal{D}$-**ultrapower** of S, and is denoted $S^{\mathcal{D}}$.

Lemma 2. *If Z is a compactum, then $Z^{\mathcal{D}}$ sits naturally as a (dense) subset of $Z_{\mathcal{D}}$. If Z is a continuum and $T \subseteq Z$ is a transversal, then $T^{\mathcal{D}}$ is a transversal for $Z_{\mathcal{D}}$.*

So, given our nondegenerate indecomposable metrizable continuum X and ultrafilter $\mathcal{D} \in \beta(I)$ (where $|I| = \alpha$), let $T \subseteq X$ be an infinite transversal. By Lemma 2, $T^{\mathcal{D}}$ is a transversal for $Y = X_{\mathcal{D}}$, and it is easy to show that $w(Y)$ and $|T^{\mathcal{D}}|$ are both $\leq 2^{\alpha}$.

We're done with Theorem 1 (1) once we turn "$\leq 2^{\alpha}$" above into "$= 2^{\alpha}$". But for this we need the ultrafilter to have a special combinatorial property.

An ultrafilter $\mathcal{D}$ is **regular** if there exists a family $\mathcal{F} \subseteq \mathcal{D}$ such that:
(i) $|\mathcal{F}| = |I|$; and
(ii) for each $i \in I$, $\{F \in \mathcal{F} : i \in F\}$ is finite.

• It is well known that there are as many regular ultrafilters on a given infinite set I as there are ultrafilters; i.e., $2^{(2^{|I|})}$.

Lemma 3. *Let $\mathcal{D}$ be a regular ultrafilter on an infinite set I.*

(1) *If S is an infinite set, then $|S^{\mathcal{D}}| = |S|^{|I|}$.*

(2) *If Z is an infinite compactum, then $w(Z_{\mathcal{D}}) = w(Z)^{|I|}$.*

This completes the proof sketch of Theorem 1 (1).

The approach to proving Theorem 1 (1) yields a continuum Y, of weight 2^α, which possesses a transversal of cardinality 2^α. Note that even if we were to let the transversal T on X have cardinality $\mathfrak{c}$—rather than be just infinite—we could not get $|T^{\mathcal{D}}|$ to be any bigger. Since we have no way of getting $|Y|$ to be anything less than $2^{(2^\alpha)}$, there does not appear to be a consistent way to make Y fine.

Another approach is called for to prove Theorem 1 (2).

Our proof sketch requires a discussion of ultracopowers using good ultrafilters.

An ultrafilter $\mathcal{D}$ on an infinite set I is **good** if:

(i) $\mathcal{D}$ is countably incomplete; and

(ii) if $f : \wp_\omega(I) \to \mathcal{D}$ is *monotone* (i.e., $s \subseteq t \Rightarrow f(s) \supseteq f(t)$), there is a $g : \wp_\omega(I) \to \mathcal{D}$ such that $g(s) \subseteq f(s)$, for $s \in \wp_\omega(I)$, and g is *multiplicative* (i.e., $g(s \cup t) = g(s) \cap g(t)$, for $s, t \in \wp_\omega(I)$).

Good ultrafilters are regular, which are in turn countably incomplete, hence nonprincipal. When the index set is countable, all four concepts coincide.

H. J. Keisler originally conceived of the notion of goodness in order to achieve high degrees of saturatedness in ultraproducts of models, and proved the existence of good ultrafilters using GCH. K. Kunen later proved, in ZFC, that an infinite set has as many good ultrafilters as it has ultrafilters.

The approach we take follows the 1920 Janiszewski-Kuratowski argument involving the Baire category theorem.

For an infinite cardinal λ, a space X is λ-**Baire** if the union of any family of $\leq \lambda$ nowhere dense subsets of X has empty interior in X.

In particular, a λ-Baire space cannot be the union of $\leq \lambda$ of its nowhere dense subsets.

The Baire category theorem shows that every compactum is ω-Baire.

Lemma 4. *Let Z be a compactum, with $\mathcal{D}$ a good ultrafilter on a set of infinite cardinality α. Then $Z_{\mathcal{D}}$ is α^{+}-Baire.*

So let X be any nondegenerate indecomposable metrizable continuum, with $\mathcal{D}$ a good ultrafilter on a set of infinite cardinality α. Then, by Lemmas 1 and 3, $Y = X_{\mathcal{D}}$ is indecomposable and has weight 2^{α}. Also, by Lemma 4, we know Y is α^{+}-Baire.

The metrizable version of the following is well known; the proof in that case extends easily.

Lemma 5. *If Z is a continuum of weight λ, then every composant of Z is a union of $\leq \lambda$ proper subcontinua of Z.*

Our continuum $Y = X_{\mathcal{D}}$ has weight 2^{α}, and each of its composants is a union of $\leq 2^{\alpha}$ proper subcontinua, by Lemma 5. Also Y is α^{+}-Baire, by Lemma 4.

We now invoke the instance of GCH where $2^{\alpha} = \alpha^{+}$, and assume—for the sake of contradiction—that Y has $\leq \alpha^{+}$ composants. Then, noting that proper subcontinua of indecomposable continua are nowhere dense, we may write Y as a union of $\leq \alpha^{+}$ nowhere dense subsets, an impossibility.

So now we know that Y has at least α^{++} composants. Since its weight is α^{+}, its cardinality must be $\leq 2^{(\alpha^{+})}$. Invoking the instance of GCH where $2^{(\alpha^{+})} = \alpha^{++}$, we conclude that Y is fine.

Parting Remarks. (1) Let $\alpha = \omega$ above. Then, modulo the two GCH instances $2^\omega = \omega_1$ (i.e., CH) and $2^{\omega_1} = \omega_2$, we know that there exists a hereditarily indecomposable fine continuum Y of weight $\mathfrak{c}$. Y can also be of any desired dimension.

(2) If all we want is a fine continuum of weight $\mathfrak{c}$, we have M. E. Rudin's proof that $\mathbb{H}^*$ is fine using CH alone. This continuum does have weight $\mathfrak{c}$ and dimension one, but it is far from being hereditarily indecomposable.

(3) To get $\mathbb{H}^*$ to be fine, one does not even need the full power of CH: Rudin's proof can be made to work using MA, an axiom consistent with ZFC+$\neg$CH.

Afterword. One can indeed obtain fine continua, in ZFC, of any given size. M. Smith in 1976 (*Generating large indecomposable continua*) proved that for any infinite cardial α, there is a continuum M with $\geq 2^{\alpha}$ composants. The construction involves an inverse limit using α as the (directed) index set. One may easily infer from this construction that $w(M) \leq \alpha$; hence M is a fine continuum of weight α. This continuum is not hereditarily indecomposable, however, as it contains arcs. Most likely this defect can be overcome.

THANK YOU!